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On subspaces of Orlicz sequence spaces

by

KARL LINDBERG* (Detroit, Mich.)

Abstract. The main results concern separable Orlicz sequence spaces, l_M, which are studied as Banach spaces with a symmetric basis. Sufficient conditions are given for the existence of a contractive projection in l_M onto a subspace which is isomorphic to l_p for some p in an interval determined by l_M; for p not in this interval, there is no subspace of l_M isomorphic to l_p. Let X be an infinite dimensional subspace of l_M; then X contains a subspace isomorphic to an Orlicz sequence space; if X has an unconditional basis, there is a contractive projection in X onto a subspace of X isomorphic to an Orlicz sequence space; if X has a symmetric basis, X is isomorphic to an Orlicz sequence space. Sufficient conditions are given for two Orlicz sequence spaces to have no isomorphic infinite dimensional subspaces.

This research is motivated by the following question: Does a Banach space with a symmetric basis have a complemented subspace isomorphic to either l_p, $1 \leqslant p < \infty$, or c_0? A class of Banach spaces with symmetric bases is the class of separable Orlicz sequence spaces. In this paper we study various properties of Orlicz sequence spaces and of their subspaces. We show that an affirmative answer can be given to the above question for certain types of Orlicz sequence spaces.

This answer is provided by Theorem 4.2 and Theorem 4.5. These theorems show that if an Orlicz sequence space X satisfies a certain condition, then for some $1 \leqslant p < \infty$, there is a block basic sequence with constant coefficients with respect to a symmetric basis of X such that this block basic sequence is equivalent to the unit vectors basis of l_p. By a result in [8] there is a bounded projection in X onto the span of such a block basic sequence, and in the usual norm on the Orlicz sequence space the norm of this projection is one.

We now give a more detailed account of the results and organization of this paper.

In Section 1 we give some definitions and basic properties of Banach spaces with various types of bases.

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Orlicz sequence spaces are introduced in Section 2 and some basic properties of these spaces are given. Most of the results presented are not new and appear in various places as stated here or in some closely related form. Orlicz functions are defined and the meaning of equivalence of Orlicz functions is given. When an Orlicz function $M(x)$ generates a separable Orlicz sequence space l_M, the unit vectors form a symmetric basis for l_M. If $M'(x)$ is another Orlicz function, it is shown that $M'(x)$ is equivalent to $M(x)$ iff the unit vectors basis of l_M and $l_{M'}$ are equivalent. An interval $[a, b]$, $1 \leqslant a \leqslant b < \infty$, is associated with each separable Orlicz sequence space. Spaces with equivalent unit vectors bases have identical intervals associated with them. The interval associated with l_p, $1 \leqslant p < \infty$ is the point p. The concept of an interval associated with a separable Orlicz sequence space is new and it is useful in the study of these spaces. An example of this is Proposition 2.19 which shows that a separable Orlicz sequence space is reflexive iff the associated interval does not contain 1.

Section 3 deals with subspaces of Orlicz sequence spaces. Theorem 3.3 shows that if the intervals associated with two spaces are disjoint, then these spaces have no isomorphic infinite dimensional subspaces. This generalizes the result that if $p \neq q$, then l_p and l_q have no isomorphic infinite dimensional subspaces. Theorem 3.7 shows that each infinite dimensional subspace of a separable Orlicz sequence space contains a subspace isomorphic to an Orlicz sequence space. Among the corollaries of this theorem is Corollary 3.9 which shows that a symmetric basis of a subspace of a separable Orlicz sequence space is equivalent to the unit vectors basis of some Orlicz sequence space. Proposition 3.5 gives some necessary and sufficient conditions for Orlicz sequence spaces to contain subspaces isomorphic to c_0, l_1 or l_∞.

In Section 4 we deal with the existence of contractive projections in separable Orlicz sequence spaces onto subspaces which are isomorphic to l_p for some p, $1 \leqslant p < \infty$. If l_M is a non-reflexive, separable Orlicz sequence space, then results of ([1], C. 7) and ([3], Thm. 4) easily show that there is an isomorph of l_1 complemented in l_M. Theorem 4.2 gives additional information. This theorem proves that if l_M is a non-reflexive separable Orlicz sequence space, then, with respect to the unit vectors basis of l_M, there is a block basic sequence with constant coefficients which is equivalent to the unit vectors basis of l_1; therefore, there is a contractive projection in l_M onto a subspace isomorphic to l_1. Theorem 4.5 gives a sufficient condition for an Orlicz sequence space l_M to have a contractive projection onto a subspace isomorphic to l_p for some p in the interval associated with l_M. By Theorem 3.3, for q not in this interval, l_M contains no subspace isomorphic to l_q. Corollary 4.8 shows that if l_M satisfies a condition apparently stronger than that of Theorem 4.5, then for each p in the interval associated with l_M there is a contractive projection onto a subspace isomorphic to l_p. Corollary 4.9 shows that for each a, b such that $1 < a \leqslant b < \infty$, there is an Orlicz sequence space with associated interval $[a, b]$; this space satisfies the condition of Corollary 4.8. Using results of [2], we show that the spaces of Corollary 4.9 with intervals $[a, b]$, $1 < a < b \leqslant 2$, are isomorphic to subspaces of L_1. Section 4 is concluded with some remarks concerning open problems related to this research.

1. Definitions and basic facts. Let X and Y be Banach spaces. An *isomorphism* from X into Y is a one-to-one linear map from X into Y which is bounded and has a bounded inverse. If there is an isomorphism of X onto Y, then X and Y are *isomorphic*. A *projection* P in X is a bounded linear map from X into X such that $P^2 = P$. The range of a projection in X is called a *complemented* subspace of X. A *contractive projection*, P, in X is a projection in X such that $\|P\|_X \leqslant 1$. X^* denotes the Banach space of bounded linear functionals on X. Only closed subspaces are considered in this paper; and hence subspace is to mean closed subspace.

Let $\{x_i\}_{i=1}^\infty$ be a sequence in X. $\{x_i\}_{i=1}^\infty$ is a basis for X if every x in X can be uniquely represented in the form $x = \sum_{i=1}^\infty a_i x_i$ where $\{a_i\}_{i=1}^\infty$ is a sequence of scalars and where the sum converges in the norm on X. If $\{x_i\}_{i=1}^\infty$ is a basis for X and $\{y_i\}_{i=1}^\infty$ is a basis for Y, then $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ are *equivalent* if for any sequence of scalars, $\{a_i\}_{i=1}^\infty$, $\sum_i a_i x_i$ converges in X iff $\sum_i a_i y_i$ converges in Y. This is equivalent to the existence of an isomorphism, T, from X onto Y such that $T(x_i) = y_i$ for all i. A basis $\{x_i\}_{i=1}^\infty$ is called *normalized* if $\|x_i\| = 1$ for all i. $\{x_i\}_{i=1}^\infty$ is called *semi-normalized* if there are constants K_1 and K_2 such that for every i, $0 < K_1 \leqslant \|x_i\| \leqslant K_2 < \infty$. The sequence $\{f_i\}_{i=1}^\infty$ in X^* such that $f_i(x_j) = \delta_{ij}$ is called the sequence *biorthogonal* to $\{x_i\}_{i=1}^\infty$. A sequence $\{z_i\}_{i=1}^\infty$ in X is a *basic sequence* if it is a basis for the subspace which it spans in X. A sequence $\{z_n\}_{n=1}^\infty$ is a *block basis with respect to the basis* $\{x_i\}_{i=1}^\infty$ if for every n, $z_n = \sum_{i=p_n+1}^{p_{n+1}} a_i x_i$ where $\{p_n\}_{n=1}^\infty$ is an increasing sequence of non-negative integers. Every block basis is a basic sequence and hence may be called a *block basic sequence*. A sequence $\{z_n\}_{n=1}^\infty$ is a *block basic sequence with constant coefficients* if $z_n = \sum_{i=p_n+1}^{p_{n+1}} a_n x_i$.

Let $\{c_i\}_{i=1}^\infty$ be a sequence in X and let $x = \sum_i c_i$. If for each permutation, p, of the positive integers, $\sum_i c_{p(i)}$ converges to x, then $\sum_i c_i$ is said to *converge unconditionally*. A basis $\{x_i\}_{i=1}^\infty$ for X is an *unconditional basis* if for all x in X, $x = \sum_i a_i x_i$ converges unconditionally. A basis $\{x_i\}_{i=1}^\infty$ is called *subsequence invariant* if $\{x_i\}_{i=1}^\infty$ is equivalent to each of its infinite subsequences. A basis $\{x_i\}_{i=1}^\infty$ is called *symmetric* if for all permutations, p, of the positive integers, $\{x_i\}_{i=1}^\infty$ is equivalent to $\{x_{p(i)}\}_{i=1}^\infty$. In [10] it is shown that symmetric bases are unconditional and subsequence invariant.

The following proposition gives an important property of spaces with unconditional bases. It follows immediately from ([3], p. 73).

PROPOSITION 1.1. *If X is a Banach space with norm $\|\cdot\|$ and with an unconditional basis $\{x_i\}_{i=1}^\infty$, then there is another norm on X, $|||\cdot|||$, such that $|||\cdot|||$ is equivalent to $\|\cdot\|$ and such that there is a contractive projection in X with the norm $|||\cdot|||$ onto each subspace spanned by a subsequence of the $\{x_i\}_{i=1}^\infty$.*

The following proposition gives results similar to the above for a space with a symmetric basis.

PROPOSITION 1.2. *Let X be a space with norm $\|\cdot\|$ and a symmetric basis $\{x_i\}_{i=1}^\infty$. There is a norm on X, $|||\cdot|||$, equivalent to $\|\cdot\|$ such that for all $x = \sum_i a_i x_i$, all sequences $\{\varepsilon_i\}_{i=1}^\infty$ with $\varepsilon_i = \pm 1$, and all permutations, p, of the positive integers, $|||x||| = |||\sum_i \varepsilon_i a_{p(i)} x_i|||$. Also, for sequences of scalars, $\{b_i\}_{i=1}^\infty$ and $\{a_i\}_{i=1}^\infty$, $|||\sum_i b_i x_i||| \leqslant |||\sum_i a_i x_i|||$ whenever, for some permutation p, $|b_i| \leqslant |a_{p(i)}|$ for all i, and $\sum_i a_i x_i$ converges.*

Proof. These facts follow from results in [10].

If X is a space with a symmetric basis, $\{x_i\}_{i=1}^\infty$, then a norm on X with the above properties is called a *symmetric basis norm with respect to* $\{x_i\}_{i=1}^\infty$.

PROPOSITION 1.3. *If X is a space with a symmetric basis $\{x_i\}_{i=1}^\infty$, then, in X with a symmetric basis norm with respect to $\{x_i\}_{i=1}^\infty$, there is a contractive projection onto each subspace of X spanned by a block basic sequence of $\{x_i\}_{i=1}^\infty$ with constant coefficients.*

Proof. This is Lemma 4 of [8].

For $1 \leqslant p < \infty$, L_p denotes $L_p[0,1]$ with Lebesgue measure. l_p denotes the space of p-summable real sequences with norm $\|\{x_i\}_{i=1}^\infty\|_p = (\sum_i |x_i|^p)^{1/p}$. The space c_0 is the space of sequences converging to zero with the supremum norm and l_∞ is the space of bounded sequences with the supremum norm. For each i, e_i is the sequence whose ith coordinate is one and whose jth coordinate is zero for all $j \neq i$. The set $\{e_i\}_{i=1}^\infty$ is the set of *unit vectors*. The unit vectors form a symmetric basis for c_0 and l_p, $1 \leqslant p < \infty$. This basis is called the *unit vectors basis*. The l_p norm is a symmetric basis norm with respect to the unit vectors basis.

PROPOSITION 1.4. *If X is isomorphic to either l_p, $1 \leqslant p < \infty$, or c_0, every symmetric basis for X is equivalent to the unit vectors basis of l_p or c_0.*

Proof. For l_1 and c_0 the results follow from the main result of [8]. For $1 < p < \infty$, the result follows from the fact that for each p, l_p is isomorphic to a complemented subspace of L_p and the fact ([6], Cor. 13), that for $1 < p < \infty$, if X is a complemented subspace of L_p with a symmetric basis, then this basis is equivalent to the unit vectors basis of l_p or l_2.

If $\{x_i\}_{i=1}^\infty$ be a basis for X, then $\{x_i\}_{i=1}^\infty$ is *boundedly complete* if for all sequences $\{a_i\}_{i=1}^\infty$ such that for some $K > 0$, $\|\sum_{i=1}^n a_i x_i\| < K$ for all n, $\sum_i a_i x_i$ converges in X. A basis, $\{x_i\}_{i=1}^\infty$, for X is *shrinking* if the sequence biorthogonal to $\{x_i\}_{i=1}^\infty$ is a basis for X^*. The following two propositions are due to R. C. James; the proofs can be found in [3].

PROPOSITION 1.5. *If $\{x_i\}_{i=1}^\infty$ is a basis for X, then X is reflexive iff $\{x_i\}_{i=1}^\infty$ is both shrinking and boundedly complete.*

PROPOSITION 1.6. *If $\{x_i\}_{i=1}^\infty$ is an unconditional basis for X, then $\{x_i\}_{i=1}^\infty$ is boundedly complete iff X has no subspace isomorphic to c_0. Also, $\{x_i\}_{i=1}^\infty$ is shrinking iff X has no subspace isomorphic to l_1.*

The following proposition is an easy consequence of Proposition 1.5.

PROPOSITION 1.7. *If $\{x_i\}_{i=1}^\infty$ is a semi-normalized basic sequence in a reflexive space, then $\{x_i\}_{i=1}^\infty$ converges weakly to zero.*

Using some results of [1] it is possible to state Proposition 1.6 as follows:

PROPOSITION 1.8. *If $\{x_i\}_{i=1}^\infty$ is an unconditional basis for X, then $\{x_i\}_{i=1}^\infty$ ss boundedly complete iff X has no complemented subspace isomorphic to c_0. Alio $\{x_i\}_{i=1}^\infty$ is shrinking iff X has no complemented subspace isomorphic to l_1.*

2. Basic properties of Orlicz sequence spaces.

DEFINITION 2.1. An *Orlicz function* $M(x)$ is a continuous, non-negative, convex, even function such that $M(0) = 0$ and for some $a \neq 0$, $M(a) \neq 0$. If $M(x)$ is an Orlicz function, l_M is the Banach space of real sequences, $\{x_i\}_{i=1}^\infty$, such that for some $r > 0$, $\sum_i M\left(\frac{x_i}{r}\right) < \infty$ with the norm $\|\{x_i\}_{i=1}^\infty\|_M = \inf\left\{r > 0 : \sum_i M\left(\frac{x_i}{r}\right) \leqslant 1\right\}$. The space l_M with norm $\|\cdot\|_M$ is called an *Orlicz sequence space*.

By ([7], Thm. 1.1), an Orlicz function $M(x)$ has the representation $M(x) = \int_0^{|x|} p(t)\,dt$, where $p(t)$, the right-derivative of $M(x)$, is a non-decreasing, right-continuous, non-negative function defined on the non-negative reals. We call $\int_0^{|x|} p(t)\,dt$ the *representation of* $M(x)$. Using this representation it is possible to place each Orlicz function, $M(x) = \int_0^{|x|} p(t)\,dt$, into one of three distinct groups according to the behavior of $p(t)$: There is an $a>0$ such that $p(0)=a$; there is an $x_0>0$ such that for $0\leqslant x\leqslant x_0$, $p(x)=0$; $p(0)=0$ and for $x>0$, $p(x)>0$. It is shown in Proposition 2.14 that Orlicz functions of the first two groups generate l_1 and l_∞, respectively. Functions of the third group are called M-functions:

Definition 2.2. An *M-function*, $M(x)$, is an Orlicz function which has the representation $M(x)=\int_0^{|x|}p(t)\,dt$, where $p(t)$ is a real-valued, right-continuous, non-decreasing function defined on the non-negative reals such that $p(0)=0$ and $p(t)>0$ for $t>0$.

Definition 2.3. Let $M(x)$ be an M-function with representation $M(x)=\int_0^{|x|}p(t)\,dt$. Define $q(s)=\sup_{p(t)\leqslant s} t$. Then $q(s)$ is a right-continuous, non-decreasing function defined on the non-negative reals such that $q(0)=0$ and $q(s)>0$ for $s>0$. $N(x)=\int_0^{|x|}q(s)\,ds$ is an M-function and it is called the *M-function complementary to* $M(x)$. It is clear that $M(x)$ is also complementary to $N(x)$. Hence $M(x)$ and $N(x)$ are called *complementary M-functions*.

If $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then $\frac{1}{p}x^p$ and $\frac{1}{q}x^q$ are complementary M-functions.

A sequence space $\bar{l}_M$ can now be defined as the space of real sequences $\{x_i\}_{i=1}^\infty$ such that $|||\{x_i\}_{i=1}^\infty|||_M=\sup_{\sum N(y_i)\leqslant 1}(\sum x_iy_i)<\infty$. The space $\bar{l}_M$ with norm $|||\cdot|||_M$ is a Banach space over the reals. This is stated by Orlicz in [9].

The following two proposition show that $(l_M,\|\cdot\|_M)$ and $(\bar{l}_M,|||\cdot|||_M)$ are isomorphic Banach spaces and give some useful results concerning M-functions and their complementary functions. The proofs of these propositions can be obtained by slightly modifying the proofs in [7] for analogous statements in the case of Orlicz function spaces.

Proposition 2.4. *Let $M(x)$ be an M-function and let $\{x_i\}_{i=1}^\infty$ be a real sequence. Then $\{x_i\}_{i=1}^\infty$ is in l_M iff $\{x_i\}_{i=1}^\infty$ is in $\bar{l}_M$. And,*

$$\|\{x_i\}_{i=1}^\infty\|_M\leqslant|||\{x_i\}_{i=1}^\infty|||_M\leqslant 2\,\|\{x_i\}_{i=1}^\infty\|_M.$$

Proposition 2.5. *Let $M(x)$ and $N(x)$ be complementary M-functions. Let $M(x)=\int_0^{|x|}p(t)\,dt$ be the representation of $M(x)$. Then,*

(a) *For all $x,y\geqslant 0$, $xy\leqslant M(x)+N(x)$.*

(b) *For all $x\geqslant 0$, $xp(x)=M(x)+N(x)$.*

(c) *For all $\{x_i\}_{i=1}^\infty$ in l_M,*

$$\sum_i x_iy_i\leqslant|||\{x_i\}_{i=1}^\infty|||_M\quad \textit{if}\ \sum N(y_i)\leqslant 1,$$

and

$$\sum_i x_iy_i\leqslant|||\{x_i\}_{i=1}^\infty|||_M\cdot\sum N(y_i)\quad \textit{if}\ \sum N(y_i)\leqslant 1.$$

Definition 2.6. Let $M(x)$ be an Orlicz function. Define h_M as the set of real sequences $\{x_i\}_{i=1}^\infty$ such that for all $r>0$, $\sum M(x_i/r)<\infty$.

The space h_M was introduced by Gribanov [4]. Since h_M is a subset of l_M it can be given the norm $\|\cdot\|_M$ of l_M.

Proposition 2.7. *Let $M(x)$ be an Orlicz function. Then h_M is a closed subspace of l_M and h_M has a symmetric basis consisting of the unit vectors basis. Furthermore, the norm $\|\cdot\|_M$ restricted to h_M is a symmetric basis norm.*

Proof. Suppose X_n are elements of h_M and X_n converges to X_0 in the norm $\|\cdot\|_M$. It is necessary to show that if $X_0=\{x_i^0\}_{i=1}^\infty$ then $\sum_i M\left(\frac{x_i^0}{r}\right)<\infty$ for all $r>0$. But since $\frac{1}{r}X_n$ converges to $\frac{1}{r}X_0$ for each $r>0$ and $\frac{1}{r}X_n$ is also in h_M, it is only necessary to show $\sum_i M(x_i^0)<\infty$. For each n, let $X_n=\{x_i^n\}_{i=1}^\infty$. Choosing N such that for $n\geqslant N$, $\|X_n-X_0\|_M\leqslant\frac{1}{2}$, it follows that

$$\sum_i M(2|x_i^n-x_i^0|)\leqslant\sum_i M\left(\frac{|x_i^n-x_i^0|}{\|X_n-X_0\|_M}\right)=1\qquad\text{for all } n\geqslant N.$$

Therefore, using convexity of $M(x)$,

$$\sum_i M(x_i^0)=\sum_i M\left(\frac{2x_i^n-2(x_i^n-x_i^0)}{2}\right)$$
$$\leqslant\tfrac{1}{2}\sum_i M(2(x_i^n))+\tfrac{1}{2}\sum_i M(2|x_i^n-x_i^0|)<\infty.$$

Hence h_M is closed in l_M.

Let $\{e_i\}_{i=1}^\infty$ be the set of unit vectors. This is a basis for h_M if for each $\{x_i\}_{i=1}^\infty$ in h_M, $\sum_i x_ie_i$ converges in the norm $\|\cdot\|_M$ to $\{x_i\}_{i=1}^\infty$. Given $\{x_i\}_{i=1}^\infty$ in h_M and $0<\varepsilon<1$, choose an integer N such that $\sum_{i=N}^\infty M\left(\frac{x_i}{\varepsilon}\right)\leqslant 1$. This can be done since $\{x_i\}_{i=1}^\infty$ is in h_M. Now for $n\geqslant N$,

$$\left\|\{x_i\}_{i=1}^\infty-\sum_{i=1}^n x_ie_i\right\|=\inf\left\{r>0\,\middle|\,\sum_{i=n+1}^\infty M\left(\frac{x_i}{r}\right)\leqslant 1\right\}$$
$$\leqslant\inf\left\{r>0\,\middle|\,\sum_{i=N}^\infty M\left(\frac{x_i}{r}\right)\leqslant 1\right\}\leqslant\varepsilon.$$

Hence $\{e_i\}_{i=1}^\infty$ forms a basis for h_M.

That this basis is a symmetric basis and that the norm $\|\cdot\|_M$ is a symmetric basis norm follows by observing that for any sequence $\{\varepsilon_i\}_{i=1}^{\infty}$, $|\varepsilon_i| = 1$, and any permutation of the positive integers, p,

$$\|\{x_i\}_{i=1}^{\infty}\|_M = \Big\| \sum \varepsilon_i x_{p(i)} e_i \Big\|_M .$$

The last equality follows from the fact that

$$\sum_i M\left(\frac{\varepsilon_i x_{p(i)}}{r}\right) = \sum_i M\left(\frac{x_{p(i)}}{r}\right) = \sum_i M\left(\frac{x_i}{r}\right).$$

There is an important class of M-functions such that the unit vectors form a basis for the Orlicz space generated by these M-functions.

Definition 2.8. An M-function $M(x)$ *satisfies the Δ_2 condition for small x* if for all $Q > 0$ there are $K > 0$ and $x_0 > 0$ such that $M(Qx) \leqslant KM(x)$ for all $0 \leqslant x \leqslant x_0$. Using convexity of $M(x)$, it follows that the above is equivalent to the existence of K and x_0 such that $M(2x) \leqslant KM(x)$ for all $0 \leqslant x \leqslant x_0$.

Proposition 2.9. *Let $M(x)$ be an M-function with representation $M(x) = \int_0^{|x|} p(t)\,dt$. $M(x)$ satisfies the Δ_2 condition for small x iff there are $K > 0$ and $x_0 > 0$ such that $1 \leqslant \dfrac{xp(x)}{M(x)} \leqslant K$ for all $0 < x < x_0$.*

Proof. This follows from the proof of ([7], Thm. 4.1) with slight changes.

Proposition 2.10. *Let $M(x)$ be an M-function. The following are equivalent:*

(a) *$M(x)$ satisfies the Δ_2 condition for small x.*

(b) *$l_M = h_M$.*

(c) *l_M is separable.*

(d) *l_M has a symmetric basis.*

Proof. The equivalence of the first three statements is stated by Gribanov [4]. That each of the first three statements is equivalent to the last follows using Proposition 2.7.

Proposition 2.11. *If $M(x)$ and $N(x)$ are complementary M-functions then h_M^* is isomorphic to l_N.*

Remark. The above proposition is stated in [4]. There is an isomorphism, T, from l_N onto h_M^* such that for $y = \{y_i\}_{i=1}^{\infty}$ in l_N, $T(y)\,(\{x_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} x_i y_i$. If l_N is separable, in which case l_N^* is isomorphic to l_M, h_M^{**} is isomorphic to l_M in such a way that the identity injection of h_M into l_M yields the canonical embedding of h_M into h_M^{**}.

Since we are interested in those properties of the unit vectors basis which hold for all equivalent unit vectors bases, it is important to know which Orlicz functions generate spaces with equivalent unit vectors bases. Thus we have the following definition:

Definition 2.12. Two functions $M_1(x)$ and $M_2(x)$ are equivalent if there are positive constants A, B, a, b and x_0, such that for all $0 \leqslant x \leqslant x_0$,

$$AM_2(ax) \leqslant M_1(x) \leqslant BM_2(bx).$$

Proposition 2.13. *Let $M_1(x)$ and $M_2(x)$ be Orlicz functions. If $M_1(x)$ is equivalent to $M_2(x)$, then l_{M_1} and l_{M_2} are isomorphic. If l_{M_1} and l_{M_2} are separable, then $M_1(x)$ is equivalent to $M_2(x)$ iff the unit vectors bases of l_{M_1} and l_{M_2} are equivalent.*

Proof. The proofs of these statements can be derived, with slight modification, from the proofs of analogous statements for Orlicz function spaces in ([7], pp. 112–113).

Remark. By Proposition 1.4 and the above proposition, we have that if $M(x)$ is an Orlicz function, then l_M is isomorphic to l_p iff $M(x)$ is equivalent to x^p. In particular if l_M is isomorphic to l_1, then for positive constants A, B and x_0, $Ax \leqslant M(x) \leqslant Bx$ for $0 \leqslant x \leqslant x_0$. If $M(x) = \int_0^{|x|} p(t)\,dt$ is the representation of $M(x)$, we have that $p(0) = a > 0$, and $M(x)$ is not an M-function. Conversely, if $p(0) = a > 0$, then, for $M(x) = \int_0^{|x|} p(t)\,dt$, $\lim_{x\to 0+} M(x)/x = a$ and there are positive constants A, B and x_0 such that $Ax \leqslant M(x) \leqslant Bx$ for $0 \leqslant x \leqslant x_0$. In this case l_M is isomorphic to l_1. We have proved the first assertion in the following proposition.

Proposition 2.14. *Let $M(x)$ be an Orlicz function with representation $M(x) = \int_0^{|x|} p(t)\,dt$. Then $p(0) = a > 0$ iff l_M is isomorphic to l_1. Also, if for some $x_0 > 0$ $p(x) = 0$ for $0 \leqslant x \leqslant x_0$, then l_M is isomorphic to l_∞ and the unit vectors basis of h_M is equivalent to the unit vectors basis of c_0.*

Proof. The first assertion is proved in the remark above. Suppose $p(x) = 0$ for $0 \leqslant x \leqslant x_0$. For all sequences $\{x_i\}_{i=1}^{\infty}$, $\{x_i\}_{i=1}^{\infty}$ is bounded iff for some $K > 0$, $\dfrac{|x_i|}{K} \leqslant x_0$ for all i; and this occurs iff $\sum_i M\left(\dfrac{x_i}{r}\right) < \infty$ for some $r > 0$. Since for some $c > x_0$, $M(c) > 0$, there is some $x_1 > 0$ such that $M(x_1) \geqslant 1$. For any bounded sequence $\{x_i\}_{i=1}^{\infty}$, let $N = \|\{x_i\}_{i=1}^{\infty}\|_\infty$. Since $\sum_i M\left(\dfrac{x_i}{N}\,x_1\right) \geqslant 1$ and $\sum_i M\left(\dfrac{x_i}{N}\,x_0\right) = 0$, it follows that $\dfrac{1}{x_1}\|\{x_i\}_{i=1}^{\infty}\|_\infty \leqslant \|\{x_i\}_{i=1}^{\infty}\|_M \leqslant \dfrac{1}{x_0}\|\{x_i\}_{i=1}^{\infty}\|_\infty$. Hence l_M is isomorphic to l_∞. That h_M is isomorphic to c_0 follows from the observation that $\sum_{i=1}^{\infty} M\left(\frac{x_i}{r}\right)$ converges for all $r > 0$ iff $\lim x_i = 0$.

The following is an important property of M-functions which generate separable spaces.

PROPOSITION 2.15. *Let $M(x) = \int_0^{|x|} p(t)\,dt$ be an M-function which satisfies the Δ_2 condition for small x. Then there is an M-function $M_1(x)$ which is equivalent to $M(x)$ such that $M_1(x)$ has the representation $M_1(x) = \int_0^{|x|} p_1(t)\,dt$, where $p_1(t)$ is a continuous, strictly increasing function with $p_1(0) = 0$.*

Proof. By Proposition 2.9 there are positive constants, K and x_0, such that $1 \leqslant \frac{xp(x)}{M(x)} \leqslant K$ for all $0 < x \leqslant x_0$. Hence $\frac{M(x)}{x} \leqslant p(x) \leqslant K\frac{M(x)}{x}$ for all $0 < x \leqslant x_0$, and $\int_0^{|x|} \frac{M(t)}{t}\,dt \leqslant M(x) \leqslant K\int_0^{|x|} \frac{M(t)}{t}\,dt$ for all $0 \leqslant x \leqslant x_0$. By ([7], p. 8), $\frac{M(x)}{x}$ is strictly increasing. Setting $p_1(x) = \frac{M(x)}{x}$, $x > 0$, $p_1(0) = 0$, we have that $M_1(x) = \int_0^{|x|} p_1(t)\,dt$ has the desired properties.

The importance of the derivative of an M-function being strictly increasing and continuous is seen in the following proposition which is proved in the same manner as ([7], Thm. 4.3).

PROPOSITION 2.16. *Let $M(x)$ and $N(x)$ be complementary M-functions. Suppose $M(x)$ has the representation $M(x) = \int_0^{|x|} p(t)\,dt$ where $p(t)$ is continuous and strictly increasing with $p(0) = 0$. Then $N(x)$ satisfies the Δ_2 condition for small x iff there are constants $K > 1$ and $x_0 > 0$ such that for $0 < x < x_0$, $\frac{xp(x)}{M(x)} \geqslant K$.*

The following definition is partially motivated by the above results.

DEFINITION 2.17. Let $M(x)$ be an M-function which satisfies the Δ_2 condition for small x. If $M(x)$ has the representation $\int_0^{|x|} p(t)\,dt$ where $p(t)$ is continuous and strictly increasing, define $f_M(x) = \frac{xp(x)}{M(x)}$ for $x > 0$. If $M(x)$ does not have such a representation, then, by Proposition 2.15, $M_1(x) = \int_0^{|x|} \frac{M(t)}{t}\,dt$ does have the desired representation and $M_1(x)$ is equivalent to $M(x)$. In this case define $f_M(x)$ to be $f_{M_1}(x)$. The function $f_M(x)$ is called the *function associated with* $M(x)$ and it is continuous for $x > 0$. Let $a_M = \varliminf_{x\to 0} f_M(x)$ and $b_M = \varlimsup_{x\to 0} f_M(x)$. The interval $[a_M, b_M]$ is called the *interval associated with* $M(x)$. By Proposition 2.9, $1 \leqslant a_M \leqslant b_M < \infty$.

In this paper we are concerned with certain properties which, if possessed by a separable Orlicz sequence space l_M, are shared by all spaces l_M with unit vectors bases equivalent to the unit vectors basis of l_M. As equivalent Orlicz functions generate spaces with equivalent unit vectors bases, we are mainly interested in properties of equivalence classes of Orlicz functions and not with properties peculiar to individual members of a class. The following definition is motivated by this concern.

DEFINITION 2.18. Let l_M be a separable Orlicz sequence space generated by the Orlicz function $M(x)$. Define $[a, b]_{l_M}$, the *interval associated with* l_M, as follows: $[a, b]_{l_M} = 1$, if l_M is isomorphic to l_1; otherwise, $M(x)$ is an M-function and $[a, b]_{l_M}$ is defined as the intersection of all intervals, $[a_{M'}, b_{M'}]$, which are associated with the M-functions, $M'(x)$, equivalent to $M(x)$.

In the next section it is shown that for all separable spaces, l_M, $[a, b]_{l_M}$ is a non-empty interval. For any two Orlicz sequence spaces with equivalent unit vectors bases, the intervals associated with these spaces are identical. The association of an interval with an Orlicz sequence space is a useful concept. Its usefulness is shown in the next two sections, but it can be illustrated in the following proposition.

PROPOSITION 2.19. *Let l_M be a separable Orlicz sequence space. Then l_M is reflexive iff $[a, b]_{l_M}$ does not contain 1.*

Proof. If l_M is isomorphic to l_1, $[a, b]_{l_M} = 1$ and l_1 is not reflexive. We may then assume that $M(x)$ is an M-function with a strictly increasing, continuous derivative. Let $N(x)$ be the M-function complementary to $M(x)$. Since h_N^* is isomorphic to l_M and l_M^* is isomorphic to l_N, using the remark following Proposition 2.11, we have l_M is reflexive iff $h_N = l_N$. This occurs iff $N(x)$ satisfies the Δ_2 condition for small x, which, by Proposition 2.16, occurs iff $a_M > 1$. Therefore if l_M is reflexive, 1 is not contained in $[a, b]_{l_M}$. If l_M is not reflexive, then for any $M'(x)$ such that $l_{M'}$ is isomorphic to l_M, $a_{M'} = 1$ by Proposition 2.16; and hence $[a, b]_{l_M}$ contains 1.

3. Subspaces of Orlicz sequence spaces. The following lemma and proposition are used to show that for a separable Orlicz sequence space, l_M, $[a, b]_{l_M}$ is a non-empty interval.

LEMMA 3.1. *Let $M(x)$ be an M-function which satisfies the Δ_2 condition for small x. Let $[a_M, b_M]$ be the interval associated with $M(x)$. Further, let*

$\{p_k\}_{k=1}^\infty$ be a strictly increasing sequence of non-negative integers and let $\{B_k\}_{k=1}^\infty$ be a semi-normalized block basic sequence with respect to the unit vectors basis of l_M, where $B_k = \sum_{i=p_k+1}^{p_{k+1}} t_i e_i$. Then for all $\varepsilon > 0$, if $\sum_k a_k B_k$ converges in l_M, then $\sum_k |a_k|^{b_M+\varepsilon}$ converges. And, if $\sum_k |a_k|^{a_M-\varepsilon}$ converges, then $\sum_k a_k B_k$ converges in l_M.

Proof. It can be assumed that $M(x) = \int_0^{|x|} p(t)\,dt$ where $p(t)$ is a continuous, strictly increasing function such that $p(0) = 0$. Also, since a block basic sequence with respect to an unconditional basis is unconditional, it can be assumed that $\|B_k\|_M = 1$ for all k, and hence $M(t_i) \leqslant 1$ for all i. Since for all $K > 0$ and $a > 0$, $KM(ax)$ is an equivalent M-function with the same interval, $[a_M, b_M]$, as $M(x)$, it can be assumed, given $\varepsilon > 0$, that $M(1) = 1$ and $a_M - \varepsilon \leqslant \frac{xp(x)}{M(x)} \leqslant b_M + \varepsilon$ for all $0 < x \leqslant 1$. Hence for each i and all $0 < x \leqslant 1$

$$(a_M - \varepsilon) \int_{|t_i|x}^{|t_i|} \frac{1}{t}\,dt \leqslant \int_{|t_i|x}^{|t_i|} \frac{p(t)}{M(t)}\,dt \leqslant (b_M + \varepsilon) \int_{|t_i|x}^{|t_i|} \frac{1}{t}\,dt.$$

Or, for all $t_i \neq 0$ and $0 \leqslant x \leqslant 1$,

$$x^{(a_M-\varepsilon)} \geqslant \frac{M(t_i x)}{M(t_i)} \geqslant x^{(b_M+\varepsilon)}. \tag{3.1}$$

Now let $\{a_k\}_{k=1}^\infty$ be a sequence of real numbers. It can be assumed that $0 \leqslant |a_k| \leqslant 1$. Then $\sum_k a_k B_k$ converges in l_M iff $\sum_k \sum_{i=p_k+1}^{p_{k+1}} M(t_i a_k)$ converges. But this converges iff $\sum_k \sum_{i=p_k+1}^{p_{k+1}} \frac{M(t_i a_k)}{M(t_i)} M(t_i)$ converges, for $t_i \neq 0$. But, using (3.1) and the fact that for all k, $\sum_{i=p_k+1}^{p_{k+1}} M(t_i) = 1$, which follows from the assumption that $M(1) = 1$ and $\|B_k\|_M = 1$, the following holds for $|t_i| \neq 0$:

$$\sum_k |a_k|^{a_M-\varepsilon} \geqslant \sum_k \sum_{i=p_k+1}^{p_{k+1}} \frac{M(t_i a_k)}{M(t_i)} M(t_i) \geqslant \sum_k |a_k|^{b_M+\varepsilon}.$$

The conclusion of the lemma follows immediately.

PROPOSITION 3.2. *Let $M(x)$ and $M'(x)$ be two M-functions which satisfy the Δ_2 condition for small x. Suppose $[a_M, b_M]$ and $[a_{M'}, b_{M'}]$, the intervals associated with $M(x)$ and $M'(x)$ respectively, are disjoint. Then l_M and $l_{M'}$ have no isomorphic infinite dimensional subspaces.*

Proof. Since the intervals are disjoint it can be assumed that $1 \leqslant a_M \leqslant b_M < a_{M'} \leqslant b_{M'} < \infty$, and hence $l_{M'}$ is reflexive by Proposition 2.19. Suppose X is an infinite dimensional subspace of l_M isomorphic to Y, a subspace of $l_{M'}$, where T is the isomorphism $T(X) = Y$. By ([1], C.2), there is a basic sequence $\{x_k\}_{k=1}^\infty$ in X equivalent to a normalized block basic sequence $\{B_k\}_{k=1}^\infty$ with respect to the unit vectors basis of l_M. Now $\{T(x_k)\}_{k=1}^\infty$ is a semi-normalized basic sequence in Y. And since $l_{M'}$ is reflexive, the basic sequence $\{T(x_k)\}_{k=1}^\infty$ converges weakly to zero by Proposition 1.7. Hence by ([1], C.1) there is a subsequence, call it $\{T(x_k)\}_{k=1}^\infty$, which is equivalent to a normalized block basic sequence with respect to the unit vectors basis of $l_{M'}$. Choosing $\varepsilon > 0$ and $\{a_k\}_{k=1}^\infty$, $0 \leqslant |a_k| \leqslant 1$, such that $b_M + \varepsilon < a_{M'} - \varepsilon$ and $\sum_k |a_k|^{b_M+\varepsilon}$ diverges and $\sum_k |a_k|^{a_{M'}-\varepsilon}$ converges, it follows, by Lemma 3.1, that $\sum_k a_k x_k$ does not converge in X while $\sum a_k T(x_k)$ converges in Y. Hence X and Y cannot be isomorphic; and l_M and $l_{M'}$ have no isomorphic infinite dimensional subspaces.

It now follows that for l_M, a separable Orlicz sequence space, $[a, b]_{l_M}$ is a non-empty interval. If l_M is isomorphic to l_1, $[a, b]_{l_M} = [1]$. Otherwise, l_M is generated by an M-function, $M(x)$. If $M'(x)$ is equivalent to $M(x)$, l_M and $l_{M'}$ are isomorphic; hence by Proposition 3.2, the intervals associated with $M(x)$ and $M'(x)$ are not disjoint. It follows that $[a, b]_{l_M}$ is equal to the interval $[a, b]$ where $a = \sup\{a_{M'}\colon M'(x)$ is equivalent to $M(x)\}$ and $b = \inf\{b_{M'}\colon M'(x)$ is equivalent to $M(x)\}$ where for each $M'(x)$, $[a_{M'}, b_{M'}]$ is the interval associated with $M'(x)$.

The following theorem is a restatement of Proposition 3.2 in terms of the intervals associated with the spaces.

THEOREM 3.3. *Let l_{M_1} and l_{M_2} be separable Orlicz sequence spaces. If $[a, b]_{l_{M_1}}$ and $[a, b]_{l_{M_2}}$ are disjoint, then l_{M_1} and l_{M_2} have no infinite dimensional isomorphic subspaces.*

Proof. If neither l_{M_1} nor l_{M_2} is isomorphic to l_1, there are M-functions $M_1'(x)$ and $M_2'(x)$ equivalent to $M_1(x)$ and $M_2(x)$ respectively, such that $[a_{M_1'}, b_{M_1'}]$ and $[a_{M_2'}, b_{M_2'}]$ are disjoint. In this case the conclusion of the theorem follows from Proposition 3.2. If, say, l_{M_1} is isomorphic to l_1, then there is an M-function $M_2'(x)$, equivalent to $M_2(x)$, such that $[a_{M_2'}, b_{M_2'}]$ does not contain 1. Now, since every normalized block basic sequence with respect to the unit vectors basis of l_1 is equivalent to the unit vectors basis of l_1, the reasoning of the proof of Proposition 3.2 yields the conclusion of the theorem in this case.

Any two Orlicz sequence spaces with equivalent unit vectors bases have the same interval associated with them. The converse to this does not hold as can be seen in the following example.

EXAMPLE 3.4. Consider the M-function $M(x)$ which on some interval $[0, x_0]$ is equal to $x^p \ln\left(\frac{1}{x}\right)$ with $p > 1$. On some interval $(0, x_1]$, $f_M(x)$, the function associated with $M(x)$, equals

$$\frac{x\left(x^p \ln\left(\frac{1}{x}\right)\right)'}{\left(x^p \ln\left(\frac{1}{x}\right)\right)} = p - \frac{1}{\ln\left(\frac{1}{x}\right)}.$$

Hence, the interval associated with $M(x)$ is the interval consisting of the point p. Consider the M-function x^p. The interval associated with x^p is also the single point p. It can easily be seen that x^p is not equivalent to $M(x)$. Hence, the unit vectors basis of l_M and l_p are not equivalent. But since all symmetric bases of l_p are equivalent to its unit vectors basis by Proposition 1.4, it follows that l_M is not isomorphic to l_p.

It should be noted that by Corollary 4.7 the Orlicz space of Example 3.4 does contain a subspace isomorphic to l_p and does not show the converse to Theorem 3.3 is false. It is not known whether this converse is false.

The following proposition initiates a study of specific subspaces of l_M.

PROPOSITION 3.5. *Let l_M be an Orlicz sequence space.*

(a) *If l_M is separable, then l_M^* is separable iff l_M contains no complemented subspace isomorphic to l_1.*

(b) *If $M(x)$ and $N(x)$ are complementary M-functions, then l_N is separable iff h_M contains no complemented subspace isomorphic to l_1.*

(c) *The space l_M is separable iff l_M contains no subspace isomorphic to l_∞ iff h_M contains no complemented subspace isomorphic to c_0.*

Proof. If $M(x)$ is not an M-function, l_M is isomorphic to either l_1 or l_∞, by Proposition 2.14. In this case (a) and (c) are immediate. Hence, assume $M(x)$ is an M-function complementary to $N(x)$. From the definition of h_M we have that the unit vectors basis of h_M is boundedly complete iff $h_M = l_M$, which, by Proposition 2.10, occurs iff l_M is separable. Also, using the remark following Proposition 2.11, we have that the basic sequence in h_M^* biorthogonal to the unit vectors of h_M is equivalent to the unit vectors basis of h_N; and, therefore, by Proposition 2.10 and Proposition 2.11, the unit vectors basis of h_M is shrinking iff l_N is separable. Now using Proposition 1.8 we easily obtain the desired results.

LEMMA 3.6. *Let l_M be a separable Orlicz sequence space. Let $\{B_k\}_{k=1}^\infty$ be a semi-normalized block basic sequence with respect to the unit vectors basis of l_M. Then there is a subsequence of the $\{B_k\}_{k=1}^\infty$ equivalent to the unit vectors basis of some Orlicz sequence space, l_{M_1}.*

Proof. Since all semi-normalized block basic sequences with respect to the unit vectors basis of l_1 are equivalent to the unit vectors basis of l_1, it can be assumed that $M(x)$ is an M-function with representation $M(x) = \int_0^{|x|} p(t)\,dt$ for all $0 \leqslant |x| \leqslant 1$ such that $p(t)$ is a strictly increasing, continuous function with $p(0) = 0$. In addition, it can be assumed that $\|B_k\|_M = 1$ for all k and $M(1) = 1$. And, since $M(x)$ satisfies the Δ_2 condition for small x, it can be assumed that for some $K > 0$, $1 \leqslant \frac{xp(x)}{M(x)} \leqslant K$ for all $0 < x \leqslant 1$. Now let $\{p_i\}_{i=1}^\infty$ be a strictly increasing sequence of non-negative integers and let $B_k = \sum_{i=p_k+1}^{p_{k+1}} t_i e_i$ where $\{e_i\}_{i=1}^\infty$ is the unit vectors basis of l_M. Then for any sequence $\{a_k\}_{k=1}^\infty$ such that $0 \leqslant a_k \leqslant 1$, $\sum_k a_k B_k$ converges in l_M iff $\sum_k \sum_{i=p_k+1}^{p_{k+1}} M(a_k t_i)$ converges. But the last sum converges iff $\sum_k \sum_{\substack{i=p_k+1 \\ t_i \neq 0}}^{p_{k+1}} M(a_k |t_i|)$ converges. Let $b_k(x) = \sum_{i=p_k+1}^{p_{k+1}} M(x|t_i|)$. Note that $\{b_k(x)\}_{k=1}^\infty$ is a family of convex functions which are non-negative with $b_k(1) = 1$ and $b_k(0) = 0$ for all k. Also, for all $0 < x < 1$

$$b_k'(x) = \sum_{i=p_k+1}^{p_{k+1}} |t_i| p(x|t_i|) = \sum_{\substack{i=p_k+1 \\ t_i \neq 0}}^{p_{k+1}} \frac{|t_i| p(x|t_i|)}{M(|t_i|)} M(|t_i|)$$
$$\leqslant \sum_{\substack{i=p_k+1 \\ t_i \neq 0}}^{p_{k+1}} \frac{|t_i| p(|t_i|)}{M(|t_i|)} M(|t_i|) \leqslant K \sum_{i=p_k+1}^{p_{k+1}} M(t_i) = K.$$

The last equality follows from the assumption that $\|B_k\|_M = 1$ and $M(1) = 1$. Hence on the interval $[0,1]$, $\{b_k(x)\}_{k=1}^\infty$ form an equicontinuous bounded family of continuous functions and therefore, there exists a continuous function $M_1(x)$ and a subsequence $\{b_{k_i}(x)\}_{i=1}^\infty$ such that $|M_1(x) - b_{k_i}(x)| \leqslant 1/2^i$ for all i and all x in $[0,1]$. $M_1(x)$ is a continuous, non-negative, convex function with $M_1(0) = 0$, $M_1(1) = 1$. Hence, $M_1(x)$ is an Orlicz function.

Now for sequences $\{a_i\}_{i=1}^\infty$ with $0 \leqslant a_i \leqslant 1$, and hence all sequences, $\sum_i a_i B_{k_i}$ converges in l_M iff $\sum_i b_{k_i}(a_i)$ converges, and from the choice of $\{b_{k_i}(x)\}_{i=1}^\infty$ the last sum converges iff $\sum_i M_1(a_i)$ converges. Therefore, the subsequence $\{B_{k_i}\}_{i=1}^\infty$ is equivalent to the unit vectors basis of l_{M_1}.

Lemma 3.6 has several implications concerning the subspaces of Orlicz sequence spaces. These are seen in the following theorems and its corollaries.

THEOREM 3.7. *Every infinite dimensional subspace of a separable Orlicz sequence space contains a subspace with a basis equivalent to the unit vectors basis of an Orlicz sequence space.*

Proof. Let X be an infinite dimensional subspace of l_M, a separable Orlicz sequence space. By ([1], C. 2), X contains a basic sequence $\{x_k\}_{k=1}^{\infty}$ equivalent to a normalized block basic sequence with respect to the unit vectors basis of l_M. By Lemma 3.6 there is a subsequence, $\{x_{k_i}\}_{i=1}^{\infty}$, which is equivalent to the unit vectors basis of an Orlicz sequence space, l_{M_1}.

COROLLARY 3.8. *Let $\{x_i\}_{i=1}^{\infty}$ be a semi-normalized basic sequence in l_M, a separable Orlicz sequence space. Let $\{f_n\}_{n=1}^{\infty}$ be the sequence biorthogonal to the unit vectors basis of l_M. If $\lim_k f_n(x_k) = 0$ for each n, then there is a subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ equivalent to the unit vectors basis of some Orlicz sequence space.*

Proof. By ([1], Thm. 3), the condition on $\{x_i\}_{i=1}^{\infty}$ assures the existence of a normalized block basic sequence, with respect to the unit vectors basis of l_M, which is equivalent to a subsequence of $\{x_i\}_{i=1}^{\infty}$. The corollary now follows from Lemma 3.6.

COROLLARY 3.9. *Let X be a subspace of l_M with a symmetric basis or, more generally, with a normalized unconditional, subsequence invariant basis, $\{x_i\}_{i=1}^{\infty}$. Then $\{x_i\}_{i=1}^{\infty}$ is equivalent to the unit vectors basis of some Orlicz sequence space.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be the sequence biorthogonal to the unit vectors basis of l_M. There is a $K > 0$ such that for each n and i, $|f_n(x_i)| < K$. Hence, by choosing a subsequence of $\{x_i\}_{i=1}^{\infty}$, if necessary, it can be assumed that $f_n(x_i)$ converges for each n. There is no loss in generality since $\{x_i\}_{i=1}^{\infty}$ is subsequence invariant. Now $f_n(x_{2i+1}-x_{2i})$ converges to zero for each n. Using Corollary 3.8, $\{x_{2i+1}-x_{2i}\}_{i=1}^{\infty}$ is equivalent to the unit vectors basis of some Orlicz sequence space. But, using the unconditionality of $\{x_i\}_{i=1}^{\infty}$, $\sum_i a_i(x_{2i+1}-x_{2i})$ converges iff $\sum_i a_i x_{2i+1}$ and $\sum_i a_i x_{2i}$ converge. But, since $\{x_i\}_{i=1}^{\infty}$ is subsequence invariant, the last statement is equivalent to convergence of $\sum_i a_i x_i$. Hence $\{x_i\}_{i=1}^{\infty}$ is equivalent to the unit vectors basis of some Orlicz sequence space.

COROLLARY 3.10. *Let $\{x_i\}_{i=1}^{\infty}$ be a semi-normalized unconditional basis for X, a subspace of a separable Orlicz space. Then X has a complemented subspace isomorphic to an Orlicz sequence space.*

Proof. If X is reflexive, $\{x_i\}_{i=1}^{\infty}$ converges to zero weakly by Proposition 1.7. Hence, by Corollary 3.8 there is a subsequence of $\{x_i\}_{i=1}^{\infty}$ equivalent to the unit vectors basis of some Orlicz space. Since $\{x_i\}_{i=1}^{\infty}$ is unconditional, by Proposition 1.1 there is a projection onto the space spanned by any subsequence. Hence the conclusion follows. If X is not

reflexive, by Proposition 3.5, l_M, and hence X, does not contain an isomorph of c_0. Therefore, X contains a complemented subspace isomorphic to l_1 by Proposition 1.5 and Proposition 1.8.

The remainder of this section is concerned with certain Orlicz sequence spaces some of which are subspaces of L_p for some p, $1 \leqslant p < 2$. In [2], it is shown that if $M(x)$ is an M-function such that $\frac{M(x)}{x^2}$ is equivalent to a decreasing function and for some p, $1 < p < 2$, $\frac{M(x)}{x^p}$ is equivalent to an increasing function, then the unit vectors basis of l_M is equivalent to a normalized basic sequence $\{x_n\}_{n=1}^{\infty}$ in L_p, and furthermore, $\{x_n\}_{n=1}^{\infty}$ may be chosen to be a sequence of independent, identically distributed, infinitely-divisible symmetric random variables.

If $M(x)$ is an M-function with associated interval $[a_M, b_M]$, then for all $\varepsilon > 0$, $\frac{M(x)}{x^{(b_M+\varepsilon)}}$ is a decreasing function and $\frac{M(x)}{x^{(a_M-\varepsilon)}}$ is an increasing function. This is true since, assuming $M(x)$ has a continuous, strictly increasing derivative $p(x)$, $\left(\frac{M(x)}{x^{b_M+\varepsilon}}\right)' < 0$ near zero iff $xp(x) - (b_M+\varepsilon)M(x) < 0$ near zero. But $xp(x)-(b_M+\varepsilon)M(x) < 0$ near zero iff $\frac{xp(x)}{M(x)} < b_M+\varepsilon$ near zero, which follows from the definition of b_M. Hence $\frac{M(x)}{x^{(b_M+\varepsilon)}}$ is decreasing near zero, and similarly, $\frac{M(x)}{x^{(a_M-\varepsilon)}}$ is increasing near zero.

Hence, if l_M is a reflexive Orlicz sequence space which has an associated interval $[a, b]_{l_M}$ with $1 < a \leqslant b < 2$, then l_M is isomorphic to a subspace of L_p for $1 \leqslant p < a$.

In [2] it is also shown that each reflexive subspace of L_1 which has a symmetric basis is isomorphic to l_M where $M(x)$ is an M-function such that $\frac{M(x)}{x^2}$ is equivalent to a decreasing function and for some $p > 1$, $\frac{M(x)}{x^p}$ is equivalent to an increasing function.

This section is concluded with an example of an M-function for each $1 < a < b < \infty$, which has an interval $[a, b]$ such that $\frac{M(x)}{x^a}$ is increasing and $\frac{M(x)}{x^b}$ is decreasing.

EXAMPLE 3.11. Let $M(x)$ be a function with range and domain the non-negative reals such that $M(0) = 0$ and near zero, $M(x) = x^{p+k\sin(\ln(-\ln x))}$

where $k > 0$, $p > 1+\sqrt{2}k$. $M(x) = \int_0^{|x|} p(t)\,dt$ where $p(x) = x^{p-1+k\sin(\ln(-\ln x))}\times$ $\times[p+k\sin(\ln(-\ln x))+k\cos(\ln(-\ln x))]$ for $x > 0$ and $p(0) = 0$. A calculation shows that $M''(x) = p'(x) > 0$ near zero if $p-1+k\sin(\ln(-\ln x)) + k\cos(\ln(-\ln x)) > 0$ for all x near zero, which occurs when $p > 1+\sqrt{2}k$. Hence $M(x)$ is an M-function with a continuous strictly increasing derivative $p(x)$. The function associated with $M(x)$ is

$$f_M(x) = p+k\big(\sin(\ln(-\ln x))+\cos(\ln(-\ln x))\big)$$

and the interval associated with $M(x)$ is $[p-\sqrt{2}k, p+\sqrt{2}k]$. Since $p-\sqrt{2}k > 1$, by Proposition 2.19, l_M is reflexive.

It now follows that for any $1 < a < b < \infty$, there is an M-function $M(x)$ such that the interval associated with $M(x)$ is exactly $[a, b]$. Let $p = \frac{b+a}{2}$ and $k > 0$ be such that $a = p-\sqrt{2}k$ in the example above. Now,

$$\left(\frac{M(x)}{x^a}\right)' = x^{p-1-a+k\sin(\ln(-\ln x))}\big[p-a+k\sin(\ln(-\ln x))+k\cos(\ln(-\ln x))\big] > 0$$

except for a finite number of zeros in each finite interval not containing zero. Hence $\frac{M(x)}{x^a}$ is increasing near zero. Similarly $\frac{M(x)}{x^b}$ is decreasing near zero.

4. Complemented subspaces isomorphic to l_p. In this section the existence of contractive projections onto subspaces of l_M isomorphic to l_p is investigated. By Proposition 1.3 such projections exist if with respect to the unit vectors basis of l_M there are block basic sequences with constant coefficients equivalent to the unit vectors basis of l_p.

The first lemma generalizes the observation that for the M-functions x^p and x^q, where $\frac{1}{p}+\frac{1}{q} = 1$, for each n, $\big\|\sum_{i=1}^n e_i\big\|_p \big\|\sum_{i=1}^n e_i\big\|_q = n$.

LEMMA 4.1. *Let $M(x)$ be an M-function with a strictly increasing continuous derivative and let $N(x)$ be the M-function complementary to $M(x)$. Then for each integer $n \geqslant 1$, $\frac{1}{2}n \leqslant \big\|\sum_{i=1}^n e_i\big\|_M \big\|\sum_{i=1}^n e_i\big\|_N \leqslant n$ where $\{e_i\}_{i=1}^\infty$ are the unit vectors.*

Proof. By Proposition 2.4, the following holds:

$$\tag{4.1} \tfrac{1}{2}\Big\|\Big\|\sum_{i=1}^n e_i\Big\|\Big\|_N \Big\|\sum_{i=1}^n e_i\Big\|_M \leqslant \Big\|\sum_{i=1}^n e_i\Big\|_N \Big\|\sum_{i=1}^n e_i\Big\|_M \leqslant \Big\|\Big\|\sum_{i=1}^n e_i\Big\|\Big\|_N \Big\|\sum_{i=1}^n e_i\Big\|_M.$$

But $\big\|\sum_{i=1}^n e_i\big\|_M = \frac{1}{M^{-1}\left(\frac{1}{n}\right)}$. And $\Big\|\Big\|\sum_{i=1}^n e_i\Big\|\Big\|_N = \sup_{\sum_i M(x_i)=1} \left(\sum_{i=1}^n x_i\right) = nM^{-1}\left(\frac{1}{n}\right)$.

The last equality can be understood by observing, with the use of Lagrange multipliers, that the maximum of $f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n x_i$ with restrictions $\sum_{i=1}^n M(x_i) = 1$ occurs where $x_1 = x_2 = \ldots = x_n = M^{-1}\left(\frac{1}{n}\right)$. Substituting $\big\|\sum_{i=1}^n e_i\big\|_M = \frac{1}{M^{-1}\left(\frac{1}{n}\right)}$ and $\Big\|\Big\|\sum_{i=1}^n e_i\Big\|\Big\|_N = nM^{-1}\left(\frac{1}{n}\right)$ into (4.1), the lemma follows.

The following theorem improves the results of Proposition 3.5.

THEOREM 4.2. *Let l_M be a separable Orlicz sequence space. Suppose l_M^* is not separable. Then there is a normalized block basic sequence with constant coefficients with respect to the unit vectors basis of l_M, equivalent to the unit vectors basis of l_1. Furthermore, if $M(x)$ is an M-function with complementary function $N(x)$, then there is a normalized block basic sequence with constant coefficients with respect to the unit vectors basis of h_N, equivalent to the unit vectors basis of c_0.*

Proof. If l_M is isomorphic to l_1, then the unit vectors bases of l_M and l_1 are equivalent by Proposition 1.4. Therefore, it can be assumed that $M(x) = \int_0^{|x|} p(t)\,dt$ where $p(0) = 0$ and $p(t)$ is continuous and strictly increasing. If $M(x)$ does not have this property, by Proposition 2.15, $M(x)$ is equivalent to an M-function which does satisfy this property. Since l_M is not reflexive, by Proposition 2.19 $\lim_{x\to 0}\frac{xp(x)}{M(x)} = 1$ and $N(x)$ does not satisfy the Δ_2 condition for small x. It is possible to choose a sequence of positive reals $\{y_n\}_{n=1}^\infty$, converging to zero such that $N(y_n) \leqslant \frac{1}{2^n}$ and $\frac{y_n N'(y_n)}{N(y_n)} \geqslant 2^n$ where $N'(x) = p^{-1}(x)$. Since for all $x \geqslant 0$, $N(2x) \geqslant xN'(x)$, it follows that

$$\tag{4.2} N(2y_n) \geqslant y_n N'(y_n) \geqslant 2^n N(y_n).$$

It is now possible to choose integers $m_n \geqslant 1$ such that

$$\tag{4.3} \frac{1}{2^n} \leqslant m_n N(y_n) \leqslant \frac{1}{2^{n-1}}.$$

Define $p_{n+1} = \sum_{i=1}^n m_i$ with $p_1 = 0$. Let $z_n = \sum_{i=p_n+1}^{p_{n+1}} e_i$ where $\{e_i\}_{i=1}^\infty$ are the unit vectors. Now, since $\sum_{i=p_n+1}^{p_{n+1}} N\left(\frac{y_n}{\frac{1}{2}}\right) = m_n N(2y_n) \geqslant m_n 2^n N(y_n) \geqslant 1$, by

(4.2) and (4.3), it follows that $\|y_n z_n\|_N \geqslant \frac{1}{2}$ for each n. Also $\sum_n y_n z_n$ is an element of l_N, since

$$\sum_n \sum_{i=p_n+1}^{p_{n+1}} N(y_n) = \sum_n m_n N(y_n) \leqslant 2.$$

Now by Proposition 2.4,

$$\Big\|\sum_n y_n z_n\Big\|_N \leqslant \Big|\Big|\Big|\sum_n y_n z_n\Big|\Big|\Big|_N = \sup_{\Sigma M(x_i)=1} \Big(\sum_n \sum_{i=p_n+1}^{p_{n+1}} y_n x_i\Big).$$

But by Proposition 2.5,

$$\sup_{\sum_i M(x_i)=1} \Big(\sum_n \sum_{i=p_n+1}^{p_{n+1}} y_n x_i\Big) \leqslant \sup_{\Sigma M(x_i)=1} \Big(\sum_i M(x_i)\Big) + \sum_n m_n N(y_n).$$

Hence, by (4.3), for each n,

$$\frac{1}{2} \leqslant \|y_n z_n\|_N \leqslant \Big\|\sum_n y_n z_n\Big\|_N \leqslant 3. \tag{4.4}$$

Identifying l_M^* with l_N, let f be the linear functional in l_M^* corresponding to the element $\sum_n y_n z_n$ in l_N. From (4.4) it follows that $\frac{1}{2} \leqslant \|f\|_N \leqslant 3$. Consider the normalized block basic sequence $\left\{\frac{z_n}{\|z_n\|_M}\right\}_{n=1}^{\infty}$ with respect to the unit vectors basis of l_M. Now from Lemma 4.1 and (4.4) it follows that

$$f\left(\frac{z_n}{\|z_n\|_M}\right) = \frac{m_n y_n}{\|z_n\|_M} \geqslant \|y_n z_n\|_N \geqslant \frac{1}{2}. \tag{4.5}$$

For all sequences $\{t_n\}_{n=1}^{\infty}$ and all positive integers q, using (4.5),

$$\sum_{n=1}^{q} |t_n| \geqslant \Big\|\sum_{n=1}^{q} |t_n| \frac{z_n}{\|z\|_M}\Big\|_M \geqslant \frac{f}{\|f\|_N}\Big(\sum_{n=1}^{q} |t_n| \frac{z_n}{\|z_n\|_M}\Big)$$
$$\geqslant \frac{1}{3}\sum_{n=1}^{q} |t_n| f\left(\frac{z_n}{\|z_n\|_M}\right) \geqslant \frac{1}{6}\sum_{n=1}^{q} |t_n|.$$

Hence $\left\{\frac{z_n}{\|z_n\|_M}\right\}_{n=1}^{\infty}$ is equivalent to the unit vectors basis of l_1.

Also $\{y_n z_n\}_{n=1}^{\infty}$ which by (4.4) is a semi-normalized block basic sequence with respect to the unit vectors basis of h_N, is equivalent to the unit vectors basis of c_0. This is true since for every sequence $\{a_n\}_{n=1}^{\infty}$, for each n and integer $q \geqslant 1$, using (4.4),

$$\frac{1}{2}|a_n| \leqslant \|a_n y_n z_n\|_N \leqslant \Big\|\sum_{n=1}^{q} a_n y_n z_n\Big\|_N$$
$$\leqslant \sup_n |a_n| \Big\|\sum_{n=1}^{q} y_n z_n\Big\|_N \leqslant 3 \sup_n |a_n|.$$

Hence, for each $q \geqslant 1$,

$$\frac{1}{2}\sup_n |a_n| \leqslant \Big\|\sum_{n=1}^{q} a_n y_n z_n\Big\|_N \leqslant 3 \sup_n |a_n|.$$

Therefore, $\sum_n a_n y_n z_n$ converges iff $\{a_n\}_{n=1}^{\infty}$ converges to zero. Then $\{y_n z_n\}_{n=1}^{\infty}$ and $\left\{\frac{z_n}{\|z_n\|_N}\right\}_{n=1}^{\infty}$ are equivalent to the unit vectors basis of c_0.

The next theorem gives a sufficient condition for the existence of contractive projections onto a subspace of l_M isomorphic to l_p. The following proposition and lemma are needed in the proof of Theorem 4.5.

PROPOSITION 4.3. *Let $M(x)$ be an M-function such that for all x, $M(x) = \int_0^{|x|} p(t)\,dt$ where $p(0) = 0$ and $p(t)$ is strictly increasing and continuous. Suppose $M(x)$ satisfies the Δ_2 condition for small x. Then for all positive sequences $\{a_n\}_{n=1}^{\infty}$ converging to zero such that $\lim_n \frac{a_n}{a_{n+1}} = 1$, there are constants $K_1 > 0$, $K_2 > 0$, such that for all n, $K_1 \leqslant \frac{M^{-1}(a_n)}{M^{-1}(a_{n+1})} \leqslant K_2$ where $M^{-1}(x)$ is the inverse function of $M(x)$.*

Proof. Let $[a, b]$ be the interval associated with $M(x)$. Then for all $1 \geqslant x > 0$, there is an $\varepsilon > 0$ such that $a - \varepsilon \leqslant \frac{M^{-1}(x)p(M^{-1}(x))}{M(M^{-1}(x))} \leqslant b + \varepsilon$. Noting that $(M^{-1}(x))' = \frac{1}{p(M^{-1}(x))}$, the following holds for all $1 \geqslant x > 0$:

$$a - \varepsilon \leqslant \frac{M^{-1}(x)}{x(M^{-1}(x))'} \leqslant b + \varepsilon.$$

Hence, fixing n and assuming that $a_{n+1} \leqslant a_n \leqslant 1$, we have

$$\frac{1}{a-\varepsilon}\int_{a_{n+1}}^{a_n} \frac{1}{x}\,dx \geqslant \int_{a_{n+1}}^{a_n} \frac{(M^{-1}(x))'}{M^{-1}(x)}\,dx \geqslant \frac{1}{b+\varepsilon}\int_{a_{n+1}}^{a_n} \frac{1}{x}\,dx.$$

Integrating, we have

$$\frac{1}{a-\varepsilon}\ln\left(\frac{a_n}{a_{n+1}}\right) \geqslant \ln\left(\frac{M^{-1}(a_n)}{M^{-1}(a_{n+1})}\right) \geqslant \frac{1}{b+\varepsilon}\ln\left(\frac{a_n}{a_{n+1}}\right).$$

If $a_{n+1} \geqslant a_n$, the inequalities are, of course, reversed. Since $\lim_n \frac{a_n}{a_{n+1}} = 1$, $\lim_n \frac{M^{-1}(a_n)}{M^{-1}(a_{n+1})} = 1$ and therefore K_1 and K_2 can be chosen as stated.

LEMMA 4.4. *Let $M(x)$ be an M-function with a strictly increasing continuous derivative such that l_M is separable. If $\{x_k\}_{k=1}^{\infty}$ is a sequence of positive reals converging to zero, then, with respect to the unit vectors basis of l_M, there is a normalized block basic sequence $\{B_k\}_{k=1}^{\infty}$ with constant coefficients such that for all sequences $\{a_k\}_{k=1}^{\infty}$, $\sum_k \frac{M(a_k x_k)}{M(x_k)}$ converges iff $\sum_k a_k B_k$ converges in l_M.*

Proof. Assume $M(1)=1$. Choose positive integers n_k such that for each k, $\frac{1}{1+n_k} \leqslant M(x_k) \leqslant \frac{1}{n_k}$. Hence,

$$M^{-1}\left(\frac{1}{1+n_k}\right) \leqslant x_k \leqslant M^{-1}\left(\frac{1}{n_k}\right), \tag{4.6}$$

$$\tfrac{1}{2} \leqslant \frac{n_k}{1+n_k} \leqslant n_k M(x_k) \leqslant 1, \tag{4.7}$$

and

$$1 \leqslant (1+n_k)M(x_k) \leqslant \frac{1+n_k}{n_k} \leqslant 2. \tag{4.8}$$

Therefore, for all x, using (4.6),

$$\tfrac{1}{2}(1+n_k)M\left(xM^{-1}\left(\frac{1}{1+n_k}\right)\right) \leqslant \tfrac{1}{2}(1+n_k)M(x_k x). \tag{4.9}$$

Now since

$$\tfrac{1}{2}(1+n_k)M(x_k x) = \tfrac{1}{2}(1+n_k)M(x_k)\frac{M(x_k x)}{M(x_k)},$$

using (4.8), (4.9) becomes,

$$\tfrac{1}{2}(1+n_k)M\left(xM^{-1}\left(\frac{1}{1+n_k}\right)\right) \leqslant \frac{M(xx_k)}{M(x_k)}. \tag{4.10}$$

But by (4.6),

$$\frac{M(xx_k)}{M(x_k)} \leqslant \frac{n_k}{n_k M(x_k)} M\left(xM^{-1}\left(\frac{1}{n_k}\right)\right)$$

and hence using this and (4.7), (4.10) becomes

$$\tfrac{1}{2}(1+n_k)M\left(xM^{-1}\left(\frac{1}{1+n_k}\right)\right) \leqslant \frac{M(xx_k)}{M(x_k)} \leqslant 2n_k M\left(xM^{-1}\left(\frac{1}{n_k}\right)\right). \tag{4.11}$$

Since $\lim_k \dfrac{\frac{1}{n_k}}{\frac{1}{1+n_k}} = 1$, by Proposition 4.3, there is a $K>0$ such that for all k,

$$1 \leqslant \frac{M^{-1}\left(\frac{1}{n_k}\right)}{M^{-1}\left(\frac{1}{1+n_k}\right)} \leqslant K. \tag{4.12}$$

Since l_M is separable, $M(x)$ satisfies the Δ_2 conditions and there is a $C>0$ and $x_0>0$ such that for all $0 \leqslant x \leqslant x_0$, $M(Kx) \leqslant CM(x)$. Now by the assumption that $M(1)=1$, for all $0 \leqslant x \leqslant x_0$, $0 \leqslant xM^{-1}\left(\frac{1}{1+n_k}\right) \leqslant x_0$; and therefore for all $0 \leqslant x \leqslant x_0$,

$$(1+n_k)M\left(KxM^{-1}\left(\frac{1}{1+n_k}\right)\right) \leqslant C(1+n_k)M\left(xM^{-1}\left(\frac{1}{1+n_k}\right)\right). \tag{4.13}$$

Now, using (4.12), it follows that

$$\begin{aligned} n_k M\left(xM^{-1}\left(\frac{1}{n_k}\right)\right) &\leqslant (1+n_k)M\left(xM^{-1}\left(\frac{1}{1+n_k}\right)\frac{M^{-1}\left(\frac{1}{n_k}\right)}{M^{-1}\left(\frac{1}{1+n_k}\right)}\right) \\ &\leqslant (1+n_k)M\left(KxM^{-1}\left(\frac{1}{1+n_k}\right)\right). \end{aligned} \tag{4.14}$$

From (4.11), (4.13) and (4.14), we have that for all $0 \leqslant x \leqslant x_0$ and for all k,

$$\frac{1}{2C} n_k M\left(xM^{-1}\left(\frac{1}{n_k}\right)\right) \leqslant \frac{M(xx_k)}{M(x_k)} \leqslant 2n_k M\left(xM^{-1}\left(\frac{1}{n_k}\right)\right). \tag{4.15}$$

Hence, for any sequence $\{a_k\}_{k=1}^{\infty}$,

$$\sum_k \frac{M(a_k x_k)}{M(x_k)} \text{ converges iff } \sum_k n_k M\left(a_k M^{-1}\left(\frac{1}{n_k}\right)\right) \text{ converges.} \tag{4.16}$$

Letting $p_{n+1} = \sum_{i=1}^{n} n_i$, $p_1 = 0$ and $z_k = \sum_{i=p_k+1}^{p_{k+1}} e_i$, where $\{e_i\}_{i=1}^{\infty}$ is the unit vectors basis of l_M, $\|z_k\|_M = \dfrac{1}{M^{-1}\left(\dfrac{1}{n_k}\right)}$. If $B_k = \dfrac{z_k}{\|z_k\|_M}$ for each k, then for all sequences, $\{a_k\}_{k=1}^{\infty}$,

$$(4.17)\qquad \sum_k a_k B_k \text{ converges in } l_M \text{ iff } \sum_k n_k M\left(a_k M^{-1}\left(\frac{1}{n_k}\right)\right) \text{ converges.}$$

But this along with (4.16) implies the desired results.

Remark. Since for all k, $n_k M\left(xM^{-1}\left(\dfrac{1}{n_k}\right)\right) = \dfrac{M\left(xM^{-1}\left(\dfrac{1}{n_k}\right)\right)}{M\left(M^{-1}\left(\dfrac{1}{n_k}\right)\right)}$, it follows from (4.17) that for any normalized block basic sequence with constant coefficients, $\{B_k\}_{k=1}^{\infty}$, there is a sequence $\{x_k\}_{k=1}^{\infty}$ such that $\sum_k a_k B_k$ converges in l_M iff $\sum_k \dfrac{M(a_k x_k)}{M(x_k)}$ converges. For each k, let $x_k = M^{-1}\left(\dfrac{1}{n_k}\right)$ where n_k is the number of unit vectors in B_k with nonzero coefficients.

THEOREM 4.5. *Let $M_1(x)$ be an M-function such that l_{M_1} is separable with associated interval $[a, b]$. Suppose for some M-function $M(x)$ equivalent to $M_1(x)$, $f_M(x)$, the function associated with $M(x)$, satisfies the following condition: There exists a sequence $\{x_n\}_{n=1}^{\infty}$ of positive reals, converging to zero such that for each $\delta > 0$, the sequence of functions, indexed by n, $|f_M(x_n) - -f_M(\xi x_n)|$ converges to zero uniformly for ξ in the closed interval $[\delta, 1]$. Then with respect to the unit vectors basis of l_M there is a normalized block basic sequence with constant coefficients which is equivalent to the unit vectors basis of l_p for some $a \leqslant p \leqslant b$.*

Proof. From the definition of $f_M(x)$ it can be seen that there is no loss of generality by assuming that $M(x) = \int_0^{|x|} p(t)\,dt$ where $p(0) = 0$ and $p(t)$ is continuous and strictly increasing. Let $[a_M, b_M]$ be the interval associated with $M(x)$. Hence, $a_M \leqslant a \leqslant b \leqslant b_M$. It can be assumed that $a_M > 1$ since if $a_M = 1$, the conclusion of the theorem follows from Theorem 4.2 with $p = 1$. Choose a sequence of positive reals, $\{K_k\}_{k=1}^{\infty}$, converging to zero, such that $\sum_k K_k \leqslant 1$. Now since $a_M \leqslant \varliminf_n f_M(x_n) \leqslant \varlimsup_n f_M(x_n) \leqslant b_M$, it can also be assumed, choosing a subsequence if necessary, that $f_M(x_n)$ converges to p for some $a_M \leqslant p \leqslant b_M$. Now using ([5], Thm. 1)

we may choose $\varepsilon_k > 0$ such that $1 \leqslant p - \varepsilon_k$ and for all sequences $\{a_k\}_{k=1}^{\infty}$

$$\sum_k (a_k)^{p-\varepsilon_k} \quad \text{converges iff} \quad \sum_k (a_k)^{p+\varepsilon_k} \quad \text{converges.}$$

Now choose a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that for $K_k \leqslant \xi \leqslant 1$, $p - \varepsilon_k \leqslant f_M(\xi x_{n_k}) \leqslant p + \varepsilon_k$. From the definition of $f_M(x)$, for each k and all ξ such that $K_k \leqslant \xi \leqslant 1$,

$$(p - \varepsilon_k) \int_{\xi x_{n_k}}^{x_{n_k}} \frac{1}{t}\,dt \leqslant \int_{\xi x_{n_k}}^{x_{n_k}} \frac{p(t)}{M(t)}\,dt \leqslant (p + \varepsilon_k) \int_{\xi x_{n_k}}^{x_{n_k}} \frac{1}{t}\,dt.$$

Integrating, we have

$$(p - \varepsilon_k) \ln\left(\frac{1}{\xi}\right) \leqslant \ln \frac{M(x_{n_k})}{M(\xi x_{n_k})} \leqslant (p + \varepsilon_k) \ln\left(\frac{1}{\xi}\right)$$

for all k and $K_k \leqslant \xi \leqslant 1$. Hence for all k and $K_k \leqslant \xi \leqslant 1$

$$\xi^{p-\varepsilon_k} \geqslant \frac{M(\xi x_{n_k})}{M(x_{n_k})} \geqslant \xi^{p+\varepsilon_k}.$$

From this it follows that for any sequence $\{a_k\}_{k=1}^{\infty}$ such that $K_k \leqslant |a_k| \leqslant 1$,

$$(4.18)\qquad \sum_k \frac{M(a_k x_{n_k})}{M(x_{n_k})} \quad \text{converges iff} \quad \sum_k |a_k|^p \text{ converges.}$$

But then (4.18) holds for all sequences $\{a_k\}_{k=1}^{\infty}$, since in order that either sum in (4.18) converges it is necessary that for only a finite number of k's $|a_k| > 1$, and for $0 \leqslant |a_k| \leqslant K_k$ both sums in (4.18) are not greater than one. This follows from the fact that for $0 \leqslant |a_k| \leqslant K_k$

$$\sum_k \frac{M(a_k x_{n_k})}{M(x_{n_k})} \leqslant \sum_k |a_k| \frac{M(x_{n_k})}{M(x_{n_k})} \leqslant \sum_k K_k \leqslant 1.$$

and also,

$$\sum_k |a_k|^p \leqslant \sum_k |K_k| \leqslant 1,$$

Now Lemma 4.4 and (4.18) show that with respect to the unit vectors basis of l_M there is a normalized block basic sequence with constant coefficients equivalent to the unit vectors basis of l_p where $a_M \leqslant p \leqslant b_M$. But since the interval associated with l_M is $[a, b]$, $a \leqslant p \leqslant b$ by Theorem 3.3. And since $M_1(x)$ is equivalent to $M(x)$, the conclusion follows for l_{M_1}.

COROLLARY 4.6. *Let l_M be separable with M-function $M(x)$. If for some $M_1(x)$ equivalent to $M(x)$ the interval associated with $M_1(x)$ is a point, p, then the conclusion of Theorem 4.5 holds for l_p and for no l_q, $q \neq p$.*

Proof. Observe $\lim_{x\to 0} f_{M_1}(x) = p$ which assures the existence of $\{x_n\}_{n=1}^{\infty}$ as in Theorem 4.5. Since $[a, b]_{l_M} = \{p\}$, l_q, $q \neq p$, is not isomorphic to a subspace of l_M by Theorem 3.3.

COROLLARY 4.7. *For each $p > 1$, there is a reflexive Orlicz sequence space X not isomorphic to l_p such that there is a contractive projection in X onto a subspace isomorphic to l_p. For $p \neq q$, X contains no subspace isomorphic to l_q.*

Proof. Example 3.4 gives an example of an Orlicz sequence space with M-function, $M(x) = x^p \ln\left(\frac{1}{x}\right)$ near zero. The function associated with $M(x)$ is $f_M(x) = p - \frac{1}{\ln\left(\frac{1}{x}\right)}$ and $\lim_{x\to 0} f_M(x) = p$. Now use Corollary 4.6.

Remark. If $[a, b]_{l_M} = \{p\}$, it is not known whether l_p is isomorphic to a subspace of l_M. However, by Theorem 3.3, for $p \neq q$, there is no subspace of l_M isomorphic to l_q.

COROLLARY 4.8. *Let l_M be a separable Orlicz space with M-function $M(x)$. Suppose the interval associated with l_M is $[a, b]$. Also suppose for some $M_1(x)$ equivalent to $M(x)$, for each $\delta > 0$, $|f_{M_1}(x) - f_{M_1}(\xi x)|$ converges to zero uniformly for $\delta \leqslant \xi \leqslant 1$. Then the conclusion of Theorem 4.5 holds for all p, $a \leqslant p \leqslant b$.*

Proof. $a_{M_1} = \varliminf_{x\to 0} f_{M_1}(x) \leqslant a \leqslant b \leqslant \varlimsup_{x\to 0} f_{M_1}(x) = b_{M_1}$. Since $f_{M_1}(x)$ is continuous, for each $a_{M_1} \leqslant p \leqslant b_{M_1}$, it is possible to choose $\{x_n\}_{n=1}^{\infty}$ such that $\lim_n f_{M_1}(x_n) = p$. Now the proof of Theorem 4.5 shows that its conclusion holds for this p since $\{x_n\}_{n=1}^{\infty}$ is as in Theorem 4.5. But then by Theorem 3.3, $a \leqslant p \leqslant b$. Hence $a_{M_1} = a$, $b_{M_1} = b$.

COROLLARY 4.9. *For all $1 < a \leqslant b < \infty$, there exists a reflexive Banach space X with a symmetric basis such that for all p, $a \leqslant p \leqslant b$, there is a contractive projection in X onto a subspace isomorphic to l_p.*

Proof. For all $1 < a \leqslant b < \infty$, Example 3.11 gives an example of an Orlicz sequence space, l_M, such that the function

$$f_M(x) = p + k\big(\sin(\ln(-\ln x)) + \cos(\ln(-\ln x))\big),$$

where $a = p - k\sqrt{2}, b = p + k\sqrt{2}$. Observe that for any $0 < \delta < 1$ $|f_M(x) - f_M(\xi x)|$ converges to zero as x converges to zero, uniformly for $\delta \leqslant \xi \leqslant 1$. This is easily seen by noticing that

$$|\ln(-\ln x) - \ln(-\ln(\xi x))| = \left|\ln\left(\frac{\ln x}{\ln(\xi) + \ln(x)}\right)\right|$$

which converges to zero uniformly for $\delta \leqslant \xi \leqslant 1$. Now apply Corollary 4.8.

COROLLARY 4.10. *For each $1 < a < 2$, there is a reflexive subspace X of L_1 with a symmetric basis such that for all p in $[a, 2]$ there is a complemented subspace of X isomorphic to l_p.*

Proof. The space l_M in Corollary 4.9 for the interval $[a, 2]$ is shown in Example 3.11 to have the property that $\frac{M(x)}{x^a}$ is increasing near zero and $\frac{M(x)}{x^2}$ is decreasing near zero. Hence, by [2], as indicated in the discussion preceding Example 3.11, there is an isomorphic embedding of l_M into L_1.

Concluding remarks: Open questions. The main motivation for this research is the question: Does every separable Orlicz sequence space l_M contain a complemented subspace isomorphic to l_p, for some $p, 1 \leqslant p < \infty$? We associate with l_M an interval I. Theorem 3.3 shows that l_M contains no subspace isomorphic to l_p for p not in this interval. Theorem 4.5 shows that if one member of the equivalence class of M-functions which generate l_M satisfies a given condition then l_M contains a subspace isomorphic to l_p for some p in I. Therefore an affirmative answer can be given to the above question if the following question has an affirmative answer: Is every M-function which satisfies the Δ_2 condition for small x equivalent to an M-function which satisfies the condition of Theorem 4.5? We may ask the same question regarding the condition of Corollary 4.8. If this question has an affirmative answer, then l_M has a complemented subspace isomorphic to l_p iff p is in I, the interval associated with l_M. This motivates the more general question: Is it possible to assign to each separable Orlicz sequence space some interval such that the space has a complemented subspace isomorphic to l_p iff p is in the assigned interval?

A different group of questions concerns symmetric bases for l_M. By Proposition 1.4, l_p, $1 \leqslant p < \infty$, has a unique symmetric basis. Is this true for separable Orlicz sequence spaces? Is every symmetric basis for l_M equivalent to the unit vectors basis of l_M? Corollary 3.9 shows that any symmetric basis of l_M is equivalent to the unit vectors basis of some Orlicz space, l_{M_1}. Therefore the above question reduces to the question: Is $M(x)$ equivalent to $M_1(x)$? Another way to phrase this question is: If l_M is isomorphic to l_{M_1}, is $M(x)$ equivalent to $M_1(x)$? Of course, the converse is true by Proposition 2.13.

Added in proof. J. Lindenstrauss and L. Tzafiri have shown that every l_M contains a subspace isomorphic to some l_p; that there is a reflexive l_M having at least two non-equivalent symmetric bases; and that there is a reflexive l_M which does not contain any l_p as a complemented subspace. These results and others related to the topics of this paper appear in Israel J. Math., *On Orlicz sequence spaces*, 10 (1971), p. 379, and Israel J. Math., *On Orlicz sequence spaces II*, 11 (1972), p. 355.

References

[1] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), pp. 151–164.

[2] J. Bretagnolle and D. Dacunha-Castelle, *Applications de l'etude de certaines forms lineaires aleatoires au plongement d'espaces de Banach dans de Banach dans des espaces L^p*, Ann. Scient. Ec. Norm. Sup., 4e serie, t. 2 (1969), pp. 432–480.

[3] M. M. Day, *Normed Linear Spaces*, Berlin 1958.

[4] Y. Gribanov, *On the theory of l_M-spaces* (Russian), Uc. zap. Kazansk. un-ta, 117 (1957), pp. 62–65.

[5] I. Halperin and H. Nakano, *Generalized l_p spaces and the Schur property*, Journ. Math. Soc. Japan, 5 (1953), pp. 49–58.

[6] M. I. Kadec and A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. 21 (1962), pp. 161–176.

[7] M. A. Krasnosel'skii and Y. R. Rutickii, *Convex Functions and Orlicz Spaces*, Groningen, 1961.

[8] J. Lindenstrauss and M. Zippin, *Banach spaces with a unique unconditional basis*, Jour. Funct. Anal. 3 (1969), pp. 115–125.

[9] W. Orlicz, *Über Raüme* (L^M), Bull. Inter. de l'Acad. Pol., Serie A, Kraków (1936), pp. 93–107.

[10] I. Singer, *Some characterizations of symmetric bases in Banach spaces*, Bull. Acad. Pol. Sci. 10 (1962), pp. 185–192.

WAYNE STATE UNIVERSITY
DETROIT, MICHIGAN

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Differentiable mappings on topological vector spaces

by

JOHN LLOYD (Canberra, Australia)

Abstract. This paper is concerned with the connections between differentiable maps, their derivatives, and the properties: strong continuity, collective precompactness and collective boundedness. The mappings considered are between topological vector spaces. Typical problems in which we are interested are the following: suppose a differentiable map f is strongly continuous. What can we say about the derivatives $f'(x)$, at each x, or the derivative f'? If f' is precompact, does f inherit this property?

1. Preliminaries. This section contains some basic definitions, conventions on terminology, and three versions of the mean value theorem, which will be needed in many of the later proofs.

We begin with the definitions of Gâteaux and Fréchet differentiability. These definitions in topological vector spaces are due to Averbukh and Smolyanov [6], [7]. In these definitions and throughout the paper, E and F will denote arbitrary Hausdorff topological vector spaces (separated by their duals) over the real field R. $\mathscr{L}(E, F)$ will denote the set of all continuous linear maps from E into F. $\mathscr{U}$ will denote the collection of all neighbourhoods of 0 in F, and $\mathscr{B}$ the collection of all bounded subsets of E.

Definition 1.1. A mapping $f\colon E \to F$ is *Gâteaux differentiable at* $x \in E$, if there exists $u \in \mathscr{L}(E, F)$ such that, for each $h \in E$ and for each $U \in \mathscr{U}$, there exists $\delta > 0$ such that $f(x+th)-f(x)-u\cdot th \in tU$, whenever $|t| \leqslant \delta$.

Definition 1.2. A mapping $f\colon E \to F$ is *Fréchet differentiable at* $x \in E$, if there exists $u \in \mathscr{L}(E, F)$ such that, for each $B \in \mathscr{B}$ and for each $U \in \mathscr{U}$, there exists $\delta > 0$ such that $f(x+th)-f(x)-u\cdot th \in tU$, whenever $h \in B$ and $|t| \leqslant \delta$.

In each case, the continuous linear mapping u is determined uniquely and is denoted by $f'(x)$. It is called the *Gâteaux* (resp., *Fréchet*) *derivative of f at x*. f is *Gâteaux* (resp., *Fréchet*) *differentiable*, if it is Gâteaux (resp., Fréchet) differentiable at each $x \in E$.

We will also need the topological vector space version of uniform diffentiability as defined by Vainberg ([16], p. 45).

Definition 1.3. Suppose $f\colon E \to F$ is Fréchet differentiable at each point of some set $\omega \subset E$. Then f is *uniformly* (*Fréchet*) *differentiable on* ω,