Added in proof. J. Lindenstrauss and L. Tzafiri have shown that every l_M contains a subspace isomorphic to some l_p; that there is a reflexive l_M having at least two non-equivalent symmetric bases; and that there is a reflexive l_M which does not contain any l_p as a complemented subspace. These results and others related to the topics of this paper appear in Israel J. Math., *On Orlicz sequence spaces*, 10 (1971), p. 379, and Israel J. Math., *On Orlicz sequence spaces II*, 11 (1972), p. 355.

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Differentiable mappings on topological vector spaces

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Abstract. This paper is concerned with the connections between differentiable maps, their derivatives, and the properties: strong continuity, collective precompactness and collective boundedness. The mappings considered are between topological vector spaces. Typical problems in which we are interested are the following: suppose a differentiable map f is strongly continuous. What can we say about the derivatives $f'(x)$, at each x, or the derivative f'? If f' is precompact, does f inherit this property?

1. Preliminaries. This section contains some basic definitions, conventions on terminology, and three versions of the mean value theorem, which will be needed in many of the later proofs.

We begin with the definitions of Gâteaux and Fréchet differentiability. These definitions in topological vector spaces are due to Averbukh and Smolyanov [6], [7]. In these definitions and throughout the paper, E and F will denote arbitrary Hausdorff topological vector spaces (separated by their duals) over the real field R. $\mathscr{L}(E,F)$ will denote the set of all continuous linear maps from E into F. $\mathscr{U}$ will denote the collection of all neighbourhoods of 0 in F, and $\mathscr{B}$ the collection of all bounded subsets of E.

DEFINITION 1.1. A mapping $f\colon E\to F$ is *Gâteaux differentiable at* $x\in E$, if there exists $u\in\mathscr{L}(E,F)$ such that, for each $h\in E$ and for each $U\in\mathscr{U}$, there exists $\delta>0$ such that $f(x+th)-f(x)-u\cdot th\in tU$, whenever $|t|\leqslant\delta$.

DEFINITION 1.2. A mapping $f\colon E\to F$ is *Fréchet differentiable at* $x\in E$, if there exists $u\in\mathscr{L}(E,F)$ such that, for each $B\in\mathscr{B}$ and for each $U\in\mathscr{U}$, there exists $\delta>0$ such that $f(x+th)-f(x)-u\cdot th\in tU$, whenever $h\in B$ and $|t|\leqslant\delta$.

In each case, the continuous linear mapping u is determined uniquely and is denoted by $f'(x)$. It is called the *Gâteaux* (resp., *Fréchet*) *derivative of f at x*. f is *Gâteaux* (resp., *Fréchet*) *differentiable*, if it is Gâteaux (resp., Fréchet) differentiable at each $x\in E$.

We will also need the topological vector space version of uniform diffentiability as defined by Vainberg ([16], p. 45).

DEFINITION 1.3. Suppose $f\colon E\to F$ is Fréchet differentiable at each point of some set $\omega\subset E$. Then f is *uniformly* (*Fréchet*) *differentiable on* ω, if for each $B \in \mathscr{B}$ and for each $U \in \mathscr{U}$, there exists $\delta > 0$ such that $f(x+th) - f(x) - f'(x)\cdot th \in tU$, whenever $|t| \leqslant \delta$, $h \in B$ and $x \in \omega$.

Three versions of the mean value theorem will be needed. The proofs of the first and second versions (Theorems 1.4 and 1.5) may be easily obtained by following the Banach space case in Vainberg ([16], pp. 36–37), and so are omitted. The third version (Theorem 1.6) is due to Averbukh and Smolyanov ([6], p. 219). The notation $C\Gamma(A)$ will denote the closed convex hull of the set A. E^* will denote the dual $\mathscr{L}(E, R)$ of E.

THEOREM 1.4. *Let $f\colon E \to F$ be Gâteaux differentiable at each point of some convex set $\omega \subset E$. Then, for each $e \in F^*$ and for each $x, x+h \in \omega$, there exists $\zeta \in (0,1)$ such that $\langle f(x+h)-f(x), e\rangle = \langle f'(x+\zeta h)\cdot h, e\rangle$.*

THEOREM 1.5. *Let $f\colon E \to F$ be Gâteaux differentiable at each point of some convex set $\omega \subset E$, and suppose F is a locally convex space. Then for each continuous seminorm p on F and for each $x, x+h \in \omega$, there exists $\zeta \in (0,1)$ such that $p[f(x+h)-f(x)] \leqslant p[f'(x+\zeta h)\cdot h]$.*

THEOREM 1.6. *Let $f\colon E \to F$ be Gâteaux differentiable at each point of some convex set $\omega \subset E$, and suppose F is a locally convex space. Then if $x, x+h \in \omega$, we have $f(x+h)-f(x) \in C\Gamma\{f'(x+\zeta h)\cdot h \mid \zeta \in [0,1]\}$.*

The theory of topological vector spaces which will be needed can be found, for example, in Schaefer [15] and Robertson and Robertson [14]. The latter book will be regarded as our standard reference, and we adopt their terminology, except we will use the more common term "locally convex space" instead of their "convex space".

We will always regard the set $\mathscr{L}(E, F)$ as having the topology of uniform convergence on bounded subsets of E ([15], p. 79), under which it becomes a topological vector space. (B, V) will denote a basic neighbourhood of 0 in this topology. Thus $(B, V) = \{u \in \mathscr{L}(E, F)\colon u(B) \subset V\}$, where $B \in \mathscr{B}$ and $V \in \mathscr{U}$.

Finally, we require some simple properties of nets. These may be found in Kelley [10]. Nets will be denoted by $(x_\alpha, \alpha \in A)$. Thus A is a directed set. $(x'_\gamma, \gamma \in \Gamma)$ will denote a subnet of $(x_\alpha, \alpha \in A)$. The phrase "$(x_\alpha, \alpha \in A)$ is a bounded net in E" will mean that $\{x_\alpha\}_{\alpha \in A} \in \mathscr{B}$.

Sections 2, 3 and 4 contain the results. Section 5 contains some examples. We also present three diagrams which illustrate the main results. In these diagrams an arrow represents implication. The numbers next to the arrows indicate the theorem which states the precise result.

2. Strong continuity. In this section, we will investigate the connection between strong continuity and differentiability. As a bonus for working in topological vector spaces, instead of normed spaces, we obtain some corollaries on weak continuity as well.

Previous results in this direction have been obtained in normed spaces by Palmer [13] and Vainberg [16]. Palmer has shown that under

certain conditions, the strong continuity of a differentiable map f implies the strong continuity of the derivative f'. Vainberg has shown the close connections between strongly continuous and compact differentiable maps ([16], pp. 14, 17, 47–50).

We begin with some definitions. Throughout, $x_\alpha \to x$ will denote convergence of the net $(x_\alpha, \alpha \in A)$ to x, and $x_\alpha \rightharpoonup x$ will denote convergence in the weak topology.

DEFINITION 2.1. A mapping $f\colon E \to F$ is *strongly continuous*, if $x_\alpha \rightharpoonup x$, where $(x_\alpha, \alpha \in A)$ is a bounded net in E and $x \in E$, implies $f(x_\alpha) \to f(x)$.

In view of the fact that a weakly convergent sequence in a Banach space is bounded, the strong continuity defined here is a natural generalisation of the definition used by, say, Vainberg ([16], p. 10) in a Banach space.

DEFINITION 2.2. A mapping $f\colon E \to F$ is *weakly continuous*, if $x_\alpha \rightharpoonup x$, where $(x_\alpha, \alpha \in A)$ is a bounded net in E and $x \in E$, implies $f(x_\alpha) \rightharpoonup f(x)$.

In the next two definitions, $f\colon E \to F$ is Gâteaux differentiable. The mapping $f'\colon E \to \mathscr{L}(E, F)$ is defined by $x \mapsto f'(x)$.

DEFINITION 2.3. The mapping $f'\colon E \to \mathscr{L}(E, F)$ is *jointly strongly continuous*, if for each $x \in E$, $y \in E$ and bounded nets $(x_\alpha, \alpha \in A)$, $(y_\alpha, \alpha \in A)$ in E, such that $x_\alpha \rightharpoonup x$ and $y_\alpha \rightharpoonup y$, then $f'(x_\alpha)\cdot y_\alpha \to f'(x)\cdot y$.

DEFINITION 2.4. The mapping $f'\colon E \to \mathscr{L}(E, F)$ is *jointly weakly continuous*, if for each $x \in E$, $y \in E$ and bounded nets $(x_\alpha, \alpha \in A)$, $(y_\alpha, \alpha \in A)$ in E, such that $x_\alpha \rightharpoonup x$ and $y_\alpha \rightharpoonup y$, then $f'(x_\alpha)\cdot y_\alpha \rightharpoonup f'(x)\cdot y$.

The most obvious question concerning strong continuity is: What does the strong continuity of f imply about its derivatives?

THEOREM 2.5. *Let $f\colon E \to F$ be Fréchet differentiable at $x \in E$. If f is strongly continuous, then $f'(x)\colon E \to F$ is strongly continuous.*

Proof. Let $x_0 \in E$ and $(x_\alpha, \alpha \in A)$ be a bounded net in E such that $x_\alpha \rightharpoonup x_0$. Suppose $U \in \mathscr{U}$. Choose a balanced $V \in \mathscr{U}$ such that $V+V+V \subset U$. Suppose $\{x_\alpha\}_{\alpha \in A} \cup \{x_0\} = B \in \mathscr{B}$.

Since f is Fréchet differentiable at x, there exists $\delta > 0$ such that $f(x+\delta h) - f(x) - f'(x)\cdot \delta h \in \delta V$, whenever $h \in B$. Further, $x + \delta x_\alpha \rightharpoonup x + \delta x_0$. Hence, $f(x+\delta x_\alpha) \to f(x+\delta x_0)$, and so there exists $\beta \in A$ such that $f(x+\delta x_\alpha) - f(x+\delta x_0) \in \delta V$, whenever $\alpha \geqslant \beta$.

Now

$$\begin{aligned} f'(x)\cdot \delta x_\alpha - f'(x)\cdot \delta x_0 &= [f'(x)\cdot \delta x_\alpha + f(x) - f(x+\delta x_\alpha)] + [f(x+\delta x_0) - f(x) - f'(x)\cdot \delta x_0] + \\ &\quad + [f(x+\delta x_\alpha) - f(x+\delta x_0)] \in \delta V + \delta V + \delta V \subset \delta U, \end{aligned}$$

whenever $\alpha \geqslant \beta$. That is $f'(x)\cdot x_\alpha - f'(x)\cdot x_0 \in U$, whenever $\alpha \geqslant \beta$. Thus $f'(x)$ is strongly continuous.

We would like to know the connection between the strong continuity of f, and the joint strong continuity of f'. It turns out that under a uniform differentiability condition on f, they are equivalent.

THEOREM 2.6. *Let $f\colon E\to F$ be Gâteaux differentiable, where F is a locally convex space. Then $f'\colon E\to\mathscr{L}(E,F)$ is jointly strongly continuous implies f is strongly continuous. If f is uniformly differentiable on bounded subsets of E, the converse holds.*

Proof. Let $(x_a,\ a\in A)$ be a bounded net in E, with $x_a\to x$, say. Let p be a continuous seminorm on F. For each $a\in A$, there exists $\zeta_a\in(0,1)$ such that $p[f(x_a)-f(x)]\leqslant p[f'(x+\zeta_a(x_a-x))\cdot(x_a-x)]$.

Now $x+\zeta_a(x_a-x)\rightharpoonup x$, and so $f'(x+\zeta_a(x_a-x))\cdot(x_a-x)\to 0$. Thus $p[f(x_a)-f(x)]\to 0$, and f is strongly continuous.

Conversely, suppose f is uniformly differentiable on bounded subsets of E, and f is strongly continuous. Let $(x_a,\ a\in A)$, $(y_a,\ a\in A)$ be bounded nets in E with $x_a\rightharpoonup x$ and $y_a\rightharpoonup y$. Let $U\in\mathscr{U}$ and choose a balanced $V\in\mathscr{U}$ such that $V+V+V+V\subset U$. Suppose $\{y_a\}_{a\in A}\cup\{y\}=B_1\in\mathscr{B}$, and $\{x_a\}_{a\in A}\cup\{x\}=B_2\in\mathscr{B}$.

Now since f is uniformly differentiable on B_2, there exists $\delta>0$ such that $f(x+\delta h)-f(x)-f'(x)\cdot\delta h\in\delta V$ whenever $h\in B_1$ and $x\in B_2$. Further $f(x_a)\to f(x)$ and $f(x_a+\delta y_a)\to f(x+\delta y)$.

Thus

$$\begin{aligned}f'(x_a)\cdot\delta y_a-f'(x)\cdot\delta y &= [f'(x_a)\cdot\delta y_a+f(x_a)-f(x_a+\delta y_a)]+\\ &\quad+[f(x+\delta y)-f(x)-f'(x)\cdot\delta y]+\\ &\quad+[f(x)-f(x_a)]+[f(x_a+\delta y_a)-f(x+\delta y)]\\ &\in\delta V+\delta V+\delta V+\delta V\subset\delta U,\end{aligned}$$

eventually. Thus $f'(x)\cdot y_a\to f'(x)\cdot y$, and so f' is jointly strongly continuous.

COROLLARY 2.7. *Let $f\colon E\to F$ be Gâteaux differentiable. Then $f'\colon E\to\mathscr{L}(E,F)$ is jointly weakly continuous implies f is weakly continuous. If f is uniformly differentiable on bounded subsets of E, the converse holds.*

Proof. It suffices to remark that f will remain Gâteaux differentiable and uniformly differentiable on bounded sets, if F is given the weak topology. Further, $f'\colon E\to\mathscr{L}(E,F)$ is jointly weakly continuous if and only if f' is jointly strongly continuous, when F has the weak topology.

Next we examine the relationship between the strong continuity and joint strong continuity of f'.

THEOREM 2.8. *Let $f\colon E\to F$ be Gâteaux differentiable. Then*

(I) *if bounded subsets of E are relatively weakly compact, $f'\colon E\to\mathscr{L}(E,F)$ is jointly strongly continuous implies f' is strongly continuous, and*

(II) *if $f'(x)\colon E\to F$ is strongly continuous, for each $x\in E$, f' is strongly continuous implies f' is jointly strongly continuous.*

Proof. (I) Suppose the conditions of the theorem hold, but f' is not strongly continuous. Hence there exists a bounded net $(x_a,\ a\in A)$ in E, such that $x_a\rightharpoonup x$, say, but $f'(x_a)\nrightarrow f'(x)$. Thus there exists a neighbourhood of 0, (B,V), in $\mathscr{L}(E,F)$, and a subnet $(x'_a,\ a\in A)$ of $(x_a,\ a\in A)$ such that $f'(x'_a)-f'(x)\notin(B,V)$, for each $a\in A$. Thus there exists a net $(h_a,\ a\in A)$ in B such that $f'(x'_a)\cdot h_a-f'(x)\cdot h_a\notin V$, for each $a\in A$.

Now choose a balanced $U\in\mathscr{U}$ such that $U+U\subset V$. Since B is relatively weakly compact, $(h_a,\ a\in A)$ has a weakly convergent subnet $(h'_\gamma,\ \gamma\in\Gamma)$ with $h'_\gamma\rightharpoonup h$, say. Let $(x''_\gamma,\ \gamma\in\Gamma)$ be the corresponding net of x's.

Since f' is jointly strongly continuous, $f'(x''_\gamma)\cdot h'_\gamma\to f'(x)\cdot h$ and $f'(x)\cdot h'_\gamma\to f'(x)\cdot h$. Hence, for a sufficiently large γ, we have

$$\begin{aligned}f'(x''_\gamma)\cdot h'_\gamma-f'(x)\cdot h'_\gamma &= [f'(x''_\gamma)\cdot h'_\gamma-f'(x)\cdot h]+[f'(x)\cdot h-f'(x)\cdot h'_\gamma]\\ &\in U+U\subset V.\end{aligned}$$

Thus we have a contradiction, and so f' must be strongly continuous.

(II) Let $(x_a,\ a\in A)$ and $(y_a,\ a\in A)$ be bounded nets in E, with $x_a\rightharpoonup x$ and $y_a\rightharpoonup y$, say. Let $U\in\mathscr{U}$ and choose a balanced $V\in\mathscr{U}$ such that $V+V\subset U$. Suppose $\{y_a\}_{a\in A}=B\in\mathscr{B}$. Since f' is strongly continuous, and $f'(x)$ is strongly continuous, $f'(x_a)-f'(x)\in(B,V)$ eventually and $f'(x)\cdot y_a-f'(x)\cdot y\in V$ eventually.

Thus

$$\begin{aligned}f'(x_a)\cdot y_a-f'(x)\cdot y &= [f'(x_a)\cdot y_a-f'(x)\cdot y_a]+[f'(x)\cdot y_a-f'(x)\cdot y]\\ &\in V+V\subset U\end{aligned}$$

eventually. Thus f' is jointly strongly continuous.

COROLLARY 2.9. *Let $f\colon E\to F$ be Gâteaux differentiable. Then $f'\colon E\to\mathscr{L}(E,F)$ is strongly continuous implies f' is jointly weakly continuous.*

Finally, we present three results which are obtained by combining the results of this section. Theorem 2.11 was proved in normed spaces by Palmer ([13], p. 442).

THEOREM 2.10. *Let $f\colon E\to F$ be Gâteaux differentiable, where F is a locally convex space. If f' is strongly continuous, and $f'(x)$ is strongly continuous, for each $x\in E$, then f is strongly continuous.*

THEOREM 2.11. *Let $f\colon E\to F$ be uniformly differentiable on bounded subsets of E, where bounded subsets of E are relatively weakly compact. Then f is strongly continuous implies f' is strongly continuous.*

THEOREM 2.12. *Let $f\colon E\to F$ be uniformly differentiable on bounded subsets of E, where bounded subsets of E are relatively weakly compact and F is a locally convex space. Then f is strongly continuous if and only if f' is strongly continuous and $f'(x)$ is strongly continuous, for each $x\in E$.*

3. Collective precompactness. Collective compactness has been studied intensively in recent years, partly because of its intrinsic interest as a generalisation of compactness for a single map, and partly because of its applications. The concept was introduced by Anselone and Moore [3] in connection with the approximate solution of integral equations. Various properties of collectively compact maps have been studied by Anselone [1], [2], Anselone and Palmer [4], [5], Daniel [9] and Moore [11]. Vainberg [16] has studied the compactness of a single mapping.

We wish to study such mappings in topological vector spaces. It turns out to be more convenient to consider collectively precompact mappings, instead of collectively compact mappings. If F is a Banach space, or more generally, a quasi-complete topological vector space, the two concepts coincide. $\mathscr{F}$ will denote a family of mappings from E into F.

DEFINITION 3.1. The family $\mathscr{F}$ is (*Fréchet*) *equidifferentiable at* $x \in E$, if each $f \in \mathscr{F}$ is Fréchet differentiable at x, and given $B \in \mathscr{B}$ and $U \in \mathscr{U}$, the $\delta > 0$ chosen (in the definition of Fréchet differentiability) is independent of $f \in \mathscr{F}$.

The family $\mathscr{F}$ is (*Fréchet*) *equidifferentiable*, if it is equidifferentiable at each $x \in E$.

DEFINITION 3.2. The family $\mathscr{F}$ is *uniformly* (*Fréchet*) *equidifferentiable* on $\omega \subset E$, if each $f \in \mathscr{F}$ is uniformly differentiable on ω, and given $B \in \mathscr{B}$ and $U \in \mathscr{U}$, the $\delta > 0$ chosen (in the definition of uniform differentiability) is independent of $f \in \mathscr{F}$.

DEFINITION 3.3. The family $\mathscr{F}$ is *collectively precompact*, if for each $B \in \mathscr{B}$, $\bigcup_{f\in\mathscr{F}} f(B)$ is a precompact subset of F.

In the next definition, we suppose that each $f \in \mathscr{F}$ is Gâteaux differentiable. The family $\mathscr{F}'$ is defined by $\mathscr{F}' = \{f' \mid f \in \mathscr{F}\}$, where, for each $f \in \mathscr{F}$, f' is defined as in Section 2.

DEFINITION 3.4. The family $\mathscr{F}'$ is *collectively jointly precompact*, if given $B_1 \in \mathscr{B}$ and $B_2 \in \mathscr{B}$, $\bigcup_{f\in\mathscr{F}} f'(B_1)\cdot B_2$ is a precompact subset of F.

It is easy to see that the collective joint precompactness of $\mathscr{F}'$ is equivalent to the collective precompactness of the induced family $\{f'\colon E \times E \to F \mid f \in \mathscr{F}\}$ where each $f'\colon E \times E \to F$ is defined by $f'((x, y)) = f'_l(x)\cdot y$.

DEFINITION 3.5. $\mathscr{F}$ is *weakly equicontinuous*, if for each $x \in E$ and bounded net $(x_\alpha, \alpha \in A)$ such that $x_\alpha \rightharpoonup x$, then $f(x_\alpha) \rightharpoonup f(x)$, uniformly over $f \in \mathscr{F}$.

The phrase "$f(x_\alpha) \rightharpoonup f(x)$, uniformly over $f \in \mathscr{F}$" is to mean, given $e \in F^*$ and $\varepsilon > 0$, there exists $\beta \in A$ such that $|\langle f(x_\alpha) - f(x), e\rangle| < \varepsilon$, whenever $\alpha \geqslant \beta$ and $f \in \mathscr{F}$.

The next result is Schauder's approximation theorem, generalised to collectively precompact mappings between topological vector spaces.

We omit the proof which may be easily supplied by using the following lemma due to Nagumo [12], and following the proof of the corresponding theorem in Banach spaces, due to Daniel [9].

LEMMA 3.6. *Let E be a locally convex space, and K a precompact subset of E. Then for any neighbourhood of 0, U, there exists a finite dimensional subspace E_m in E, and a continuous map $S\colon K \to E_m$ such that*

(I) *$S(x) - x \in U$, for each $x \in K$;*

(II) *$S(K)$ is a bounded subset of E_m.*

THEOREM 3.7. *Suppose F is a locally convex space. Then $\mathscr{F}$ is collectively precompact and each $f \in \mathscr{F}$ is continuous, if and only if, for each $B \in \mathscr{B}$ and each $U \in \mathscr{U}$, there exists a family $\mathscr{F}^* = \{f^* \mid f \in \mathscr{F}\}$ of continuous maps from B into F such that*

(I) *$f^*(x) - f(x) \in U$, for each $f \in \mathscr{F}$ and each $x \in B$;*

(II) *$\bigcup_{f\in\mathscr{F}} f^*(B)$ is a bounded subset of a finite dimensional subspace F_m of F.*

It is well know that the compactness of f' implies f is weakly continuous ([16], p. 47). The next result extends this to collections of mappings.

THEOREM 3.8. *Let $\mathscr{F}$ be a family of Gâteaux differentiable maps from E into F. Then $\mathscr{F}'$ is collectively precompact implies $\mathscr{F}$ is weakly equicontinuous.*

Proof. Suppose the conditions of the theorem hold, but $\mathscr{F}$ is not weakly equicontinuous. Thus there is a bounded net $(x_\alpha, \alpha \in A)$ in E, such that $x_\alpha \rightharpoonup x$, say, but $f(x_\alpha) \not\rightharpoonup f(x)$, uniformly over $f \in \mathscr{F}$. Hence there exists $e \in F^*$, $\varepsilon > 0$, a subnet $(x'_\alpha, \alpha \in A)$ and a net $(f_\alpha, \alpha \in A)$ in $\mathscr{F}$ such that $|\langle f_\alpha(x'_\alpha) - f_\alpha(x), e\rangle| > \varepsilon$, for each $\alpha \in A$.

By the mean value theorem, for each $\alpha \in A$, there exists $\zeta_\alpha \in (0, 1)$ such that

$$\langle f_\alpha(x'_\alpha) - f_\alpha(x), e\rangle = \langle f'_\alpha\big(x + \zeta_\alpha(x'_\alpha - x)\big)\cdot(x'_\alpha - x), e\rangle.$$

Put $z_\alpha = f'_\alpha\big(x + \zeta_\alpha(x'_\alpha - x)\big)$, for each $\alpha \in A$.

Since e is continuous, there exists $U \in \mathscr{U}$ such that $|\langle y, e\rangle| \leqslant \varepsilon/2$, whenever $y \in U$.

Let B denote the balanced hull of $\{x_\alpha\}_{\alpha\in A} - x$. Then $B \in \mathscr{B}$. Now since $\mathscr{F}'$ is collectively precompact, $\bigcup_{f\in\mathscr{F}} f'(x + B)$ is a precompact subset of F. Thus $(z_\alpha, \alpha \in A)$ has a Cauchy subnet $(z'_\lambda, \lambda \in \Lambda)$ small of order (B, U). Choose any fixed z'_λ, and denote it by z. Thus, if $(x''_\lambda, \lambda \in \Lambda)$ is the subnet of $(x'_\alpha, \alpha \in A)$ corresponding to $(z'_\lambda, \lambda \in \Lambda)$, we have $|\langle z(x''_\lambda - x), e\rangle| \to 0$.

Thus, for a sufficiently large λ, we have

$$|\langle f_\lambda(x''_\lambda) - f_\lambda(x), e\rangle| = |\langle z'_\lambda(x''_\lambda - x), e\rangle|$$
$$\leqslant |\langle (z'_\lambda - z)(x''_\lambda - x), e\rangle| + |\langle z(x''_\lambda - x), e\rangle| \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus we have a contradiction, and so $\mathcal{F}$ must be weakly equicontinuous.

We now extend a result of Moore ([11], p. 66) to topological vector spaces.

THEOREM 3.9. *Let $\mathcal{F}$ be equidifferentiable at $x \in E$. Then $\mathcal{F}$ is collectively precompact implies $\{f'(x)\}_{f\in\mathcal{F}}$ is collectively precompact.*

Proof. Let $B \in \mathcal{B}$ and $U \in \mathcal{U}$. Since $\mathcal{F}$ is equidifferentiable at $x \in E$, there exists $\delta > 0$ such that $(1/\delta)[f(x+\delta h)-f(x)]-f'(x)\cdot h \in U$, whenever $h \in B$ and $f \in \mathcal{F}$. Now $(1/\delta)\bigcup_{f\in\mathcal{F}}[f(x+\delta B)-f(x)]$ is precompact, since $\mathcal{F}$ is collectively precompact and so $\bigcup_{f\in\mathcal{F}} f'(x)\cdot B$ is also precompact.

Next we examine the connection between the collective precompactness of $\mathcal{F}'$, and the collective joint precompactness of $\mathcal{F}'$.

THEOREM 3.10. *Suppose each $f \in \mathcal{F}$ is Gâteaux differentiable. Then $\mathcal{F}'$ is collectively precompact, and $f'(x)$ is precompact, for each $f \in \mathcal{F}$, and each $x \in E$, implies that $\mathcal{F}'$ is collectively jointly precompact.*

Proof. Let $B_1 \in \mathcal{B}$, $B_2 \in \mathcal{B}$ and $U \in \mathcal{U}$. Choose a balanced $V \in \mathcal{U}$ such that $V+V \subset U$. Now $\bigcup_{f\in\mathcal{F}} f'(B_1)$ is a precompact subset of $\mathcal{L}(E, F)$. Hence there exists a finite set $\{f_1'(x_1), \ldots, f_n'(x_n)\}$ in $\bigcup_{f\in\mathcal{F}} f'(B_1)$ such that

$$\bigcup_{i=1}^{n} [f_i'(x_i)+(B_2, V)] \supset \bigcup_{f\in\mathcal{F}} f'(B_1).$$

For each $i = 1, 2, \ldots, n$, $f_i'(x_i)\cdot B_2$ is a precompact subset of F. Hence there exists a finite set $\{y_{ij}\}_{i=1}^{n}{}_{j=1}^{m}$ in B_2 such that, for each i,

$$\bigcup_{j=1}^{m} [f_i'(x_i)\cdot y_{ij}+V] \supset f_i'(x_i)\cdot B_2.$$

We show the finite set $\{f_i'(x_i)\cdot y_{ij}\}_{i=1}^{n}{}_{j=1}^{m}$ is a U-net for $\bigcup_{f\in\mathcal{F}} f'(B_1)\cdot B_2$.

Suppose $f'(x)\cdot y \in \bigcup_{f\in\mathcal{F}} f'(B_1)\cdot B_2$. First choose i such that $f'(x)-f_i'(x_i) \in (B_2, V)$. Then choose j such that $f_i'(x_i)\cdot y - f_i'(x_i)\cdot y_{ij} \in V$. Then

$$f'(x)\cdot y - f_i'(x_i)\cdot y_{ij} = [f'(x)\cdot y - f_i'(x_i)\cdot y]+[f_i'(x_i)\cdot y - f_i'(x_i)\cdot y_{ij}] \in V+V \subset U.$$

Thus $\bigcup_{f\in\mathcal{F}} f'(B_1)\cdot B_2$ is precompact.

We would like to find some conditions under which $\mathcal{F}$ is collectively precompact. However, as an example in Section 5 will show, $\mathcal{F}'$ collectively jointly precompact does not imply this, nor does the stronger condition of $\mathcal{F}'$ collectively precompact, and $\{f'(x)\}_{f\in\mathcal{F}}$ collectively precompact, for each $x \in E$. However, in case $\mathcal{F}$ consists of only finitely many maps, the first condition is sufficient. The proof of Theorem 3.11 is due to Sadayuki Yamamuro.

THEOREM 3.11. *Suppose $\mathcal{F}$ is a finite family of Gâteaux differentiable maps and F is a locally convex space. Then $\mathcal{F}'$ is collectively jointly precompact implies $\mathcal{F}$ is collectively precompact.*

Proof. Obviously, it suffices to only consider the case when $\mathcal{F}$ consists of a single map f, say. Let $B \in \mathcal{B}$. If $x \in B$, then by the mean value theorem (1.6),

$$f(x)-f(0) \in C\Gamma\{f'(\zeta x)\cdot x \mid \zeta \in [0, 1]\}.$$

Let B_1 denote the balanced hull of B. Then, by [14], pp. 25, 50. $C\Gamma(f'(B_1)\cdot B)$ is precompact, since f' is jointly precompact. Thus since $f(B) \subset C\Gamma(f'(B_1)\cdot B)+f(0)$, $f(B)$ is precompact.

Finally, we examine a converse to Theorem 3.11.

THEOREM 3.12. *Suppose $\mathcal{F}$ is uniformly equidifferentiable on bounded subsets of E. Then $\mathcal{F}$ is collectively precompact implies $\mathcal{F}'$ is collectively jointly precompact.*

Proof. Let $B_1 \in \mathcal{B}$, $B_2 \in \mathcal{B}$ and $U \in \mathcal{U}$. Now $\mathcal{F}$ is uniformly equidifferentiable on B_1. Thus there exists $\delta > 0$ such that $(1/\delta)[f(x+\delta h)-f(x)]-f'(x)\cdot h \in U$, whenever $h \in B_2$, $x \in B_1$, $f \in \mathcal{F}$. Now $(1/\delta)\bigcup_{f\in\mathcal{F}}[f(B_1+\delta B_2)-f(B_1)]$ is precompact, since $\mathcal{F}$ is collectively precompact. Hence $\bigcup_{f\in\mathcal{F}} f'(B_1)\cdot B_2$ is also precompact.

4. Collective boundedness. In this section, we investigate possible analogies between collectively precompact and collectively bounded mappings. It turns out that collectively bounded mappings are better behaved than in the precompact case. For example, the concepts of collective boundedness and collective joint boundedness for $\mathcal{F}'$ coincide. We begin with two definitions.

DEFINITION 4.1. The family $\mathcal{F}$ is *collectively bounded* if for each $B \in \mathcal{B}$, $\bigcup_{f\in\mathcal{F}} f(B)$ is a bounded subset of F.

In the next definition, we suppose that $\mathcal{F}$ is a collection of Gâteaux differentiable maps. $\mathcal{F}'$ is defined as in Section 3.

DEFINITION 4.2. The family $\mathcal{F}'$ is *collectively jointly bounded*, if for each $B_1 \in \mathcal{B}$ and $B_2 \in \mathcal{B}$, $\bigcup_{f\in\mathcal{F}} f'(B_1)\cdot B_2$ is a bounded subset of F.

It is easy to see that the collective joint boundedness of $\mathcal{F}'$ is equivalent to the collective boundedness of the induced family $\{f'\colon E\times E \to F \mid f \in \mathcal{F}\}$ defined as in Section 3.

THEOREM 4.3. *Let $\mathcal{F}$ be equidifferentiable at $x \in E$. Then $\mathcal{F}$ is collectively bounded implies $\{f'(x)\}_{f\in\mathcal{F}}$ is collectively bounded.*

Proof. Let $B \in \mathcal{B}$ and $U \in \mathcal{U}$. Choose a balanced $V \in \mathcal{U}$ such that $V+V \subset U$. Choose $\delta > 0$ such that $(1/\delta)[f(x+\delta h)-f(x)]-f'(x)\cdot h \in V$, whenever $f \in \mathcal{F}$ and $h \in B$.

Now $(1/\delta)\bigcup_{f\in\mathscr{F}}[f(x+\delta B)-f(x)]$ is a bounded subset of F. Thus there exists $\lambda\in R$ such that $(1/\delta)\bigcup_{f\in\mathscr{F}}[f(x+\delta B)-f(x)]\subset\lambda V$. We may assume $\lambda\geqslant 1$.

Then, for all $h\in B$ and $f\in\mathscr{F}$, we have

$$f'(x)\cdot h = \{f'(x)\cdot h-(1/\delta)\,[f(x+\delta h)-f(x)]\}+(1/\delta)\,[f(x+\delta h)-f(x)]$$
$$\in V+\lambda V\subset\lambda V+\lambda V\subset\lambda U.$$

Thus $\bigcup_{f\in\mathscr{F}}f'(x)\cdot B\subset\lambda U$, and so $\bigcup_{f\in\mathscr{F}}f'(x)\cdot B$ is bounded.

THEOREM 4.4. *Suppose $\mathscr{F}$ is a finite family of Gâteaux differentiable maps and F is a locally convex space. Then $\mathscr{F}'$ is collectively jointly bounded implies $\mathscr{F}$ is collectively bounded.*

Proof. This result is a corollary of Theorem 3.11 since in a locally convex space the bounded sets are precisely the weakly precompact sets ([14], pp. 50, 67). We remark here that if we assume F is a locally convex space in each of the theorems in this section (except Theorem 4.6), they become corollaries of the corresponding results for collective precompactness in Section 3.

THEOREM 4.5. *Suppose $\mathscr{F}$ is uniformly equidifferentiable on bounded subsets of E. Then $\mathscr{F}$ is collectively bounded implies $\mathscr{F}'$ is collectively jointly bounded.*

Proof. Let $B_1\in\mathscr{B}$, $B_2\in\mathscr{B}$ and $U\in\mathscr{U}$. Choose a balanced $V\in\mathscr{U}$ such that $V+V\subset U$. Now $\mathscr{F}$ is uniformly equidifferentiable on B_1, and so there exists $\delta>0$ such that $(1/\delta)\,[f(x+\delta h)-f(x)]-f'(x)\cdot h\in V$, whenever $x\in B_1$, $h\in B_2$ and $f\in\mathscr{F}$. But $(1/\delta)\bigcup_{f\in\mathscr{F}}[f(B_1+\delta B_2)-f(B_1)]$ is a bounded subset of F. Thus there exists $\lambda\geqslant 1$ such that

$$(1/\delta)\bigcup_{f\in\mathscr{F}}[f(B_1+\delta B_2)-f(B_1)]\subset\lambda V.$$

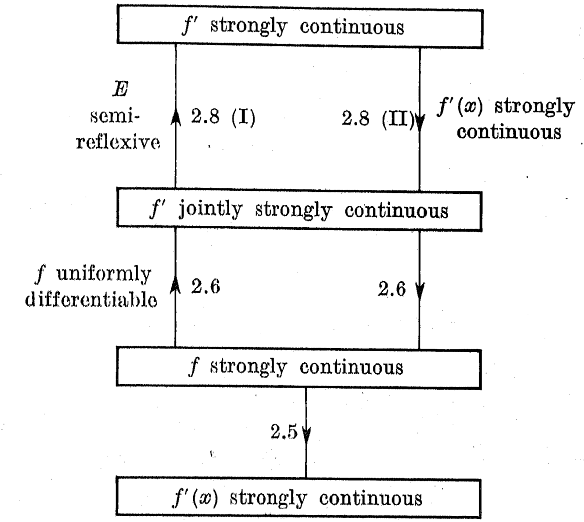
Then, for each $x\in B_1$, $h\in B_2$ and $f\in\mathscr{F}$, we have

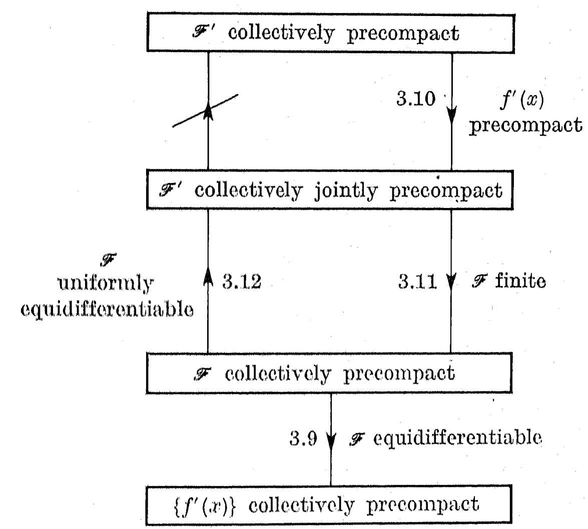
$$f'(x)\cdot h = \{f'(x)\cdot h-(1/\delta)\,[f(x+\delta h)-f(x)]\}+(1/\delta)\,[f(x+\delta h)-f(x)]$$
$$\in V+\lambda V\subset\lambda U.$$

Thus $\bigcup_{f\in\mathscr{F}}f'(B_1)\cdot B_2$ is bounded.

THEOREM 4.6. *Let $\mathscr{F}$ be a family of Gâteaux differentiable maps from E into F. Then $\mathscr{F}'$ is collectively bounded if and only if $\mathscr{F}'$ is collectively jointly bounded.*

Proof. Let $B_1\in\mathscr{B}$, $B_2\in\mathscr{B}$ and $U\in\mathscr{U}$. Then $\bigcup_{f\in\mathscr{F}}f'(B_1)\cdot B_2\subset\lambda U$ if and only if $\bigcup_{f\in\mathscr{F}}f'(B_1)\subset\lambda(B_2,U)$, where $\lambda\in R$. The result follows.





$$
\begin{tikzcd}
\boxed{\mathscr{F}' \text{ collectively bounded}} \arrow[d, shift left=3, "4.6"] \\
\boxed{\mathscr{F}' \text{ collectively jointly bounded}} \arrow[u, shift left=3, "4.6"] \arrow[d, shift left=3, "4.4", "\mathscr{F}\text{ finite}"'] \\
\boxed{\mathscr{F} \text{ collectively bounded}} \arrow[u, shift left=3, "4.5", "\mathscr{F}\text{ uniformly equidifferentiable}"'] \arrow[d, "4.3"', "\mathscr{F}\text{ equidifferentiable}"] \\
\boxed{\{f'(x)\} \text{ collectively bounded}}
\end{tikzcd}
$$

5. Examples. The purpose of this section is to present some examples related to the results which have been obtained. For convenience, we define the following mappings:

(a) $f\colon l^2 \to R$, where $f(x) = (x, x)$. f is Fréchet differentiable, and $f'(x)\colon l^2 \to R$ is defined by $f'(x)\cdot y = 2(x, y)$.

(b) For each positive integer n, we define $g_n\colon R \to R$, by $g_n(x) = \sin nx$, for each $x \in R$.

(c) $h\colon l^2 \to l^2$, where $h(x) = (x_n^2)$, $x = (x_n)$. h is Fréchet differentiable, and $h'(x)\colon l^2 \to l^2$ is defined by

$$h'(x)\cdot y = 2(x_n y_n), \qquad y = (y_n).$$

(d) $\psi\colon R \to R$, where $\psi(x) = x^2 \sin(1/x)$, $(x \neq 0)$ and $\psi(0) = 0$. Then $\psi'(x) = 2x\sin(1/x) - \cos(1/x)$, $(x \neq 0)$ and $\psi'(0) = 0$.

(e) $\varphi\colon R \to R$, where $\varphi(x) = x^2\sin(1/x^2)$, $(x \neq 0)$ and $\varphi(0) = 0$. Then $\varphi'(x) = 2x\sin(1/x^2) - (2/x)\cos(1/x^2)$, $(x \neq 0)$ and $\varphi'(0) = 0$.

(f) $\pi\colon l^2 \to l^2$, where $\pi(x) = x$, for each $x \in l^2$.

(g) For each positive integer n, we define $\mu_n\colon R \to R$, where $\mu_n(x) = n$, for each $x \in R$.

Then, we have the following examples:

(I) The converse of Theorem 2.5 does not hold, since $h'(x)$ is strongly continuous, for each $x \in l^2$, but h is not strongly continuous.

(II) We cannot drop the assumption of uniform differentiability on bounded sets in Theorem 2.6 and Corollary 2.7, since the map ψ is strongly continuous, but ψ' is not jointly strongly continuous. ψ is also weakly continuous, but ψ' is not jointly weakly continuous.

(III) The condition: $f'(x)$ is strongly continuous, for each $x \in E$, is needed in Theorem 2.8 (II), since the map π' is strongly continuous, but π' is not jointly strongly continuous.

(IV) The converse of Corollary 2.9 does not hold, since h' is jointly weakly continuous, but not strongly continuous.

(V) The converse of Theorem 3.8 does not hold, since φ is weakly continuous, but φ' is not precompact.

(VI) The converse of Theorem 3.9 does not hold, since $h'(x)$ is precompact, for each $x \in l^2$, but h is not precompact. This example is due to Yamamuro ([17], p. 131) and Bonic ([8], p. 392).

(VII) In Theorem 3.12 the assumption that $\mathscr{F}$ be uniformly equidifferentiable on bounded sets cannot be omitted, since φ is precompact, but φ' is not jointly precompact. This example also shows that uniform equidifferentiability cannot be dropped in Theorem 4.5.

(VIII) The map f is precompact and uniformly differentiable on l^2, but f' is not precompact.

(IX) The condition that $f'(x)$ be precompact, for each $x \in E$ and each $f \in \mathscr{F}$, cannot be dropped in Theorem 3.10, since π' is precompact, but not jointly precompact.

(X) The collection $\mathscr{F}' = \{\mu_n'\}$ is collectively precompact, and $\{\mu_n'(x)\}$ is collectively precompact, for each $x \in R$ but $\mathscr{F} = \{\mu_n\}$ is not collectively precompact. Thus Theorem 3.11 does not hold for infinite $\mathscr{F}$. This example also shows that Theorem 4.4 does not hold for infinite $\mathscr{F}$, and the converse to Theorem 4.3 does not hold.

(XI) The assumption of equidifferentiability cannot be omitted from the Theorems 3.9 and 4.3, since the collection $\mathscr{F} = \{g_n\}$ is collectively bounded and collectively precompact, but $\{g_n'(0)\}$ is neither collectively bounded nor collectively precompact.

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Optimal control by means of switchings

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Abstract. A stochastic control model is considered. A general theorem about the optimal strategy and the maximal reward is proved. In two special cases the optimal solutions are found effectively.

Introduction. Let $\{X^d\}_{d\in D}$ be a finite family of Markov processes and let non-negative functions f, c_d $(d \in D)$, be defined on the state space E. At time $t=0$, when being in a state $x \in E$ we choose a process X^{d_1}. The cost arising from this choice is equal to $c_{d_1}(x)$. We observe the process X^{d_1} and at the stopping time τ_1 we choose a process X^{d_2}. Our reward at any time $t<\tau_1$ is equal to

$$\int_0^t f(x_s^{d_1})\,ds - c_{d_1}(x).$$

Next we observe the process X^{d_2} and at the stopping time $\tau_2 \geqslant \tau_1$ we choose a process X^{d_3}. At time $t \in [\tau_1, \tau_2)$ our reward is equal to

$$\Big(\int_0^{\tau_1} f(x_s^{d_1})\,ds - c_{d_1}(x)\Big) + \Big(\int_{\tau_1}^{t} f(x_s^{d_2})\,ds - c_{d_2}(x_{\tau_1}^{d_1})\Big).$$

Suppose that we can repeat these selections N times. What is the maximal total expected reward? Which strategies should be chosen to maximize the total expected reward starting in some state $x \in E$?

In this paper we prove a theorem which gives an answer to these questions. In two special cases we find effectively the maximal reward and optimal strategies.

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The optimality theorem. To precise the above problems we shall formulate the described situation in terms of controlled Markov chains. To do this we assume that the Markov processes X^d are equal to $(\Omega, \boldsymbol{M}, \boldsymbol{M}_t, X_t, \Theta_t, P_x^d)$ where only P_x^d depends on d (see [1], p. 20).