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Sequences in locally convex spaces

by

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Abstract. An example is given of a non-semibornological C-sequential space. It is remarked that the fact that the dual of a Mazur space is strongly complete improves a recent result on spaces with a basis.

We give an example, Example 3, to disprove the assertion made in [7] p. 52, fortunately without proof, that a C-sequential locally convex space must be semibornological. In the course of further discussion of sequences, we are also able to give a strong improvement of a result of O. T. Jones [1], using a slight generalization of a theorem in [3].

By E we designate a fixed locally convex topological vector space, [3], Chapter 4; [6], Chapter 12; E' is its dual, E^s is the set of sequentially continuous linear functionals, i.e. linear f: $E \to$ complex numbers satisfying $f(x_n) \to 0$ whenever $\{x_n\}$ is a null sequence in E; E^b is the set of bounded linear functionals i.e. such that $f[B]$ is bounded whenever B is. We call E a *Mazur space* whenever $E^s = E'$, and *semibornological* whenever $E^b = E'$. Clearly $E' \subset E^s \subset E^b$ ([7], p. 52) so that semibornological implies Mazur. A set U is called a *sequential neighborhood of* 0 if every null sequence belongs to U eventually, i.e. $x_n \to 0$ implies that $x_n \in U$ for all sufficiently large n; E is called *C-sequential* if every convex sequential neighborhood of 0 is a neighborhood of 0.

By w^* we mean $\sigma(E', E)$, the weak star topology on E'. A linear functional f on $(E, T)'$ is called *aw^* continuous* if $f|M$ is w^* continuous for every T-equicontinuous set $M \subset E'$.

LEMMA 1. *E is C-sequential if and only if every absolutely convex sequential neighborhood of* 0 *is a neighborhood of* 0.

To prove sufficiency, let U be a convex sequential neighborhood of 0. Let $V = U \cap (-U)$ and $W = V \cap (iV)$. For $|t| \leqslant \frac{1}{2}$, $t = r + is$, we have $tW \subset rW + isW \subset \frac{1}{2}W + i\frac{1}{2}W$ (since W is convex and contains 0) $\subset \frac{1}{2}U + \frac{1}{2}U = U$. Let H be the convex hull of $\bigcup \{tW: |t| \leqslant \frac{1}{2}\}$. The above remarks imply that $H \subset U$. Now H is absolutely convex and, since $H \supset \frac{1}{2}W$, it is a sequential neighborhood of 0, hence a neighborhood of 0, and so U is also.

Now if T is the topology of E, Webb [5] denotes by T^+ the largest locally convex topology for E which has the same convergent sequences as T. A base for the T^+ neighborhoods of 0 is the set of all absolutely convex sequential neighborhoods of 0.

LEMMA 2. *T^+ is C-sequential.*

This is immediate from Lemma 1.

LEMMA 3. *Let (E, T) be a locally convex space such that (E', w^*) is a Mazur space. The w^{*+} is compatible with the duality (E', E) i.e. $(E', w^{*+})' = E$.*

First $w^{*+} \supset w^*$ so that the dual includes E. Next let f be w^{*+} continuous on E'. Then f is w^* sequentially continuous, hence w^* continuous, so that $f \in E$.

The next result is essentially [3], 21.9 (5).

LEMMA 4. *Let E be separable and complete. Then (E', w^*) is a Mazur space.*

Let f be w^* sequentially continuous and M a T-equicontinuous set in E'. Then (M, w^*) is metrizable, ([3], 21.3 (4)) so $f|M$ is continuous i.e. f is aw^* continuous. By [2], 16.9, f is w^* continuous.

EXAMPLE 1. "Separable" cannot be dropped in Lemma 4 even when E is a Banach space, [4].

EXAMPLE 2. "Complete" cannot be dropped in Lemma 4 even when E is a separable normed space. Let (E, T) be barreled, [7], p. 53, or more generally sequentially barreled in the sense of [5]. Then every aw^* continuous linear functional is w^* sequentially continuous since every w^* convergent sequence is T-equicontinuous, but need not be w^* continuous, [2], 16.9.

The next result is seentially [2] 20A. By τ we mean the Mackey topology $\tau(E', E)$.

LEMMA 5. *Let E be a Banach space such that (E', τ) is bornological. Then E is reflexive.*

For τ has the same bounded sets as the norm topology on E' hence is larger. Thus these topologies are equal. (See [6], 7.6, Theorem 1; 10.5, Problem 23.)

EXAMPLE 3. *A C-sequential locally convex space which is not semibornological.* Let E be a separable non-reflexive Banach space and $X = (E', w^{*+})$. By Lemma 2, X is C-sequential. Now suppose that X is semibornological. By Lemmas 3, 4, (E', τ) is semibornological since all compatible topolgies have the same bounded sets, [6], 12.3, Theorem 1. By [3] 28.1 (3), or [2], 19.4, (E', τ) is bornological, contradicting Lemma 5.

THEOREM 1. *Let E be a Mazur space. Then E' is strongly complete.*

This follows, with slight modifications, from the proof given in [3], 21.6 (4) which assumes E metrizable. Note that this also implies the same result if E is bornological, which is given, with a different proof in [3], 28.5 (1). Note also that we get the stronger result that E' is complete with the topology of uniform convergence on null sequences in E.

Theorem 1 is a significant generalization of Theorem 6 of [1]. This result deals with spaces with a Schauder basis $\{x^n\}$ and biorthogonal $\{f_n\}$. Jones proves that if every linear functional g satisfying $g(x) = \sum f_n(x) g(x^n)$ for all x is continuous, then E' is strongly complete. Since the given condition implies that E is a Mazur space, Theorem 1 applies, as well as the stronger result mentioned in its proof. The next result, with Theorem 1, has, as a special case, the remark of [1] that the strong dual of a barreled space with Schauder basis is complete. It is well known that this is false without the basis assumption; hence also, a barreled space need not be a Mazur space.

THEOREM 2. *Let E be a barreled space with a Schauder basis. Then E is a Mazur space.*

If g is sequentially continuous we have $g(x) = \sum f_n(x) g(x_n)$ and so g is continuous by the Banach–Steinhaus closure theorem, [6], Section 12.3, Theorem 5.

In [7], pp. 50–61, are given some connections among the various properties listed above. In addition we note that a Mazur space need not be semibornological as is shown by Lemmas 4, 5 and the fact that "semibornological" is a duality invariant.

References

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