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Generalized convolutions II

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Abstract. The purpose of this paper is to prove the uniqueness theorem for a representation of the characteristic function of infinitely divisible measures in a generalized convolution algebra. This result is used to investigate stable and self-decomposable measures.

1. Introduction. For the terminology and notation used here, see [3]. In particular, $\mathfrak{P}$ denotes the class of all probability measures defined on Borel subsets of the positive half-line. Further, E_a $(a \geqslant 0)$ denotes the probability measure concentrated at the point a. For any positive number a the transformation T_a of $\mathfrak{P}$ onto itself is defined by means of the formula $(T_aP)(\mathscr{E}) = P(a^{-1}\mathscr{E})$ where $P \in \mathfrak{P}$, $\mathscr{E}$ is a Borel set and $a^{-1}\mathscr{E} = \{a^{-1}x\colon x \in \mathscr{E}\}$. The transformation T_0 is defined by assuming $T_0P = E_0$ for all $P \in \mathfrak{P}$.

A commutative and associative $\mathfrak{P}$-valued binary operation $\circ$ defined on P is called a *generalized convolution* if it satisfies the following conditions:

(i) the measure E_0 is a unit element, i.e. $E_0 \circ P = P$ for all $P \in \mathfrak{P}$;

(ii) $(aP + bQ)\circ R = a(P\circ R) + b(Q \circ R)$, whenever $P, Q, R \in \mathfrak{P}$ and $a \geqslant 0$, $b \geqslant 0$, $a+b = 1$;

(iii) $(T_aP)\circ(T_aQ) = T_a(P\circ Q)$ for any $P, Q \in \mathfrak{P}$ and $a > 0$;

(iv) if $P_n \to P$, then $P_n \circ Q \to P \circ Q$ for all $Q \in \mathfrak{P}$ where the convergence is the weak convergence of probability measures;

(v) there exists a sequence $c_1, c_2, \ldots$ of positive numbers such that the sequence $T_{c_n}E_1^{\circ n}$ weakly converges to a measure different from E_0.

The power $E_a^{\circ n}$ is taken here in the sense of the operation $\circ$. The class $\mathfrak{P}$ with a generalized convolution $\circ$ is called a *generalized convolution algebra* and denoted by $(\mathfrak{P}, \circ)$. Algebras admitting a non-trivial homomorphism into the real field are called *regular*. We say that an algebra $(\mathfrak{P}, \circ)$ admits a *characteristic function* if there exists one-to-one correspondence $P \leftrightarrow \Phi_P$ between probability measures P from $\mathfrak{P}$ and real-valued functions Φ_P defined on the positive half-line such that $\Phi_{aP+bQ} = a\Phi_P + b\Phi_Q$ $(a \geqslant 0, b \geqslant 0, a+b = 1)$, $\Phi_{P\circ Q} = \Phi_P\Phi_Q$, $\Phi_{T_aP}(t) = \Phi_P(at)$ $(a \geqslant 0, t \geqslant 0)$ and the uniform convergence in every finite interval of

Φ_{P_n} is equivalent to the weak convergence of P_n. The function Φ_P is called the characteristic function of the probability measure P in the algebra $(\mathfrak{P}, \circ)$. It plays the same fundamental role in generalized convolution algebra as in ordinary convolution algebra, i.e. in classical problems concerning the addition of independent random variables.

It is proved in [3] (Theorem 6) that an algebra admits a characteristic function if and only if it is regular. Moreover, each characteristic function is an integral transform

$$\Phi_P(t) = \int_0^\infty \Omega(tx)P(dx), \tag{1}$$

where the kernel Ω satisfies the inequality $\Omega(x) < 1$ in a neighborhood of the origin and

$$\lim_{x\to 0} \frac{1-\Omega(tx)}{1-\Omega(x)} = t^\varkappa, \tag{2}$$

uniformly in every finite interval. The positive constant $\varkappa$ does not depend upon the choice of a characteristic function and is called a *characteristic exponent* of the algebra in question. Moreover, there exists a probability measure M called a *characteristic measure* of the algebra for which

$$\Phi_M(t) = \exp(-t^\varkappa) \tag{3}$$

([3], Theorem 7).

Throughout this paper we assume that the algebra $(\mathfrak{P}, \circ)$ is regular and Φ_P is a fixed characteristic function in $(\mathfrak{P}, \circ)$.

2. Infinitely divisible measures. A measure $P \in P$ is said to be *infinitely divisible* if for every positive integer n there exists a measure $P_n \in \mathfrak{P}$ such that $P = P_n^{\circ n}$. The class of infinitely divisible measures coincides with the class of all limit distributions of sequences $P_{n1} \circ P_{n2} \circ \ldots \circ P_{nk_n}$, where P_{nk} $(k = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$ are uniformly asymptotically neglegible (see [3], Theorem 12).

Taking an arbitrary number $x_0 > 0$ such that $\Omega(x) < 1$ whenever $0 < x \leqslant x_0$, we put

$$\omega(x) = \begin{cases} 1-\Omega(x) & \text{if} \quad 0 \leqslant x \leqslant x_0, \\ 1-\Omega(x_0) & \text{if} \quad x > x_0. \end{cases} \tag{4}$$

In [3] (Theorem 13) I proved that the class of characteristic functions of infinitely divisible measures $P \in \mathfrak{P}$ coincides with the class of functions

$$\Phi_P(t) = \exp \int_0^\infty \frac{\Omega(tx)-1}{\omega(x)} m(dx), \tag{5}$$

when m runs over all finite Borel measures on the positive half-line and the integrand is defined as its limiting value $-t^\varkappa$ when $x = 0$.

The aim of the present paper is to prove that the representation (5) is unique, i.e. that the function Φ_P determines the measure m. The uniqueness of the representation (5) will lead to some results concerning stable and self-decomposable probability measures in the algebra $(\mathfrak{P}, \circ)$.

THEOREM 1. *The representation* (5) *of the characteristic function of infinitely divisible measures is unique.*

Proof. Suppose that Φ_P is given by formula (5). We introduce an auxiliary finite measure m_0 defined on the positive half-line by means of the formula

$$m_0(\mathscr{E}) = \int_{\mathscr{E}} \big(1-\exp(-v^\varkappa)\big)\Big(1-\int_0^1 \Omega(uv)\,du\Big)\frac{m(dv)}{\omega(v)}. \tag{6}$$

We note that, by Theorems 1,5 and 6 in [3], the inequality $\int_0^1 \Omega(uv)\,du < 1$ is true. Consequently, the density function in (6) is positive for $v > 0$. Moreover, by (2) and (4), this density function is bounded which implies the finiteness of m_0.

First we shall prove that the function Φ_P determines the measure m on the open half-line $(0, \infty)$. Of course, to prove this it suffices to prove that Φ_P determines the measure m_0. Let us introduce the notation

$$I(t) = \int_0^\infty \big(1-\Omega(tv)\big)\Big(1-\int_0^1 \Omega(uv)\,du\Big)\frac{m(dv)}{\omega(v)}.$$

Taking into account the formula

$$\Omega(tv)\Omega(uv) = \Phi_{E_t}(v)\Phi_{E_u}(v) = \Phi_{E_t \circ E_u}(v) = \int_0^\infty \Omega(yv)(E_t \circ E_u)(dy)$$

by a simple computation we get the equation

$$I(t) = -\log\Phi_P(t) - \int_0^1 \log\Phi_P(u)\,du + \int_0^1\int_0^\infty \log\Phi_P(v)(E_t\circ E_u)(dv)\,du. \tag{7}$$

Further, integrating with respect to the characteristic measure M of the algebra we get, by virtue of (1) and (3), the formula

$$\int_0^\infty I(ty)M(dy) = \int_0^\infty \big(1-\exp(-t^\varkappa v^\varkappa)\big)\Big(1-\int_0^1 \Omega(uv)\,du\Big)\frac{m(dv)}{\omega(v)}.$$

Thus, by (6),

$$\int_0^\infty \exp(-t^\varkappa v^\varkappa)\,m_0(dv) = \int_0^\infty I\big((t^\varkappa+1)^{1/\varkappa}y\big)M(dy) - \int_0^\infty I(ty)M(dy).$$

Hence and from (7) it follows that the function Φ_P determines the modified Laplace transform of the measure m_0. This proves that the measure m_0 and, consequently, the measure m is uniquely determined by Φ_P on the open half-line $(0, \infty)$.

It remains to prove that $m(\{0\})$ is also determined by Φ_P. But this is a direct consequence of the formula

$$m(\{0\})t^{\varkappa} = -\log\Phi_P(t) + \int\limits_{(0,\infty)} \big(\Omega(tx)-1\big)\frac{m(dx)}{\omega(x)}$$

which completes the proof.

3. Stable measures. A measure $P \in \mathfrak{P}$ is said to be *stable* if for any pair a, b of positive numbers there exists a positive number c such that $T_aP\circ T_bP = T_cP$. The class of stable measures coincides with the class of all limit distributions of sequences $T_{c_n}P^{\circ n}$ where $c_n > 0$ $(n = 1, 2, \ldots)$ and $P \in \mathfrak{P}$ (see [3], Theorem 15). A description of the characteristic function of stable measures was given by Theorem 16 in [3]. By the uniqueness of the representation (5) we are now in a position to establish a simpler description of these functions. We start with a Lemma.

LEMMA 1.

$$\overline{\lim_{x\to 0}}\frac{\omega(x)}{x^{\varkappa}} < \infty.$$

Proof. Suppose the contrary, i.e.

$$\lim_{n\to\infty}\frac{\omega(x_n)}{x_n^{\varkappa}} = \infty$$

for a sequence $\{x_n\}$ tending to 0. Since, by (2) and (4),

$$\lim_{n\to\infty}\frac{1-\Omega(xx_n)}{\omega(x_n)} = x^{\varkappa}$$

we have, for every positive number x, the formula

$$\lim_{n\to\infty}\frac{1-\Omega(xx_n)}{x_n^{\varkappa}} = \infty.$$

Obviously, the characteristic measure M of the algebra is not concentrated at the origin. Consequently, the Fatou Lemma yields the equation

$$\lim_{n\to\infty}\int\limits_0^\infty \frac{1-\Omega(xx_n)}{x_n^{\varkappa}}M(dx) = \infty.$$

On the other hand, by (3),

$$\int\limits_0^\infty \frac{1-\Omega(xx_n)}{x_n^{\varkappa}}M(dx) = \frac{1-\exp(-x_n^{\varkappa})}{x_n^{\varkappa}} \to 1$$

which implies a contradiction. The Lemma is thus proved.

THEOREM 2. *The class of characteristic functions of stable measures in $(\mathfrak{P}, \circ)$ coincides with the class of functions*

$$\Phi_P(t) = \exp(-ct^p),$$

where $c \geqslant 0$ and $0 < p \leqslant \varkappa$; $\varkappa$ being the characteristic exponent of the algebra in question.

Proof. By Theorem 16 in [3] it suffices to prove that the integral $\int\limits_0^1 \frac{\omega(x)}{x^{1+p}}dx$ is finite if and only if $p < \varkappa$. The finiteness of this integral for $p < \varkappa$ is a direct consequence of Lemma 1. It remains to prove that

$$\int\limits_0^1 \frac{\omega(x)}{x^{1+\varkappa}}dx = \infty.$$

Contrary to this let us assume that the last integral is finite. Then the measure $m_\varkappa$ defined by the formula

$$m_\varkappa(E) = b\int\limits_E \frac{\omega(x)}{x^{1+\varkappa}}dx,$$

where $b^{-1} = \int\limits_0^\infty \frac{1-\Omega(x)}{x^{1+\varkappa}}dx$ is finite. Moreover,

$$\int\limits_0^\infty \frac{\Omega(tx)-1}{\omega(x)}m_\varkappa(dx) = -t^{\varkappa}.$$

Thus, by (3), $m_\varkappa$ is the representing measure in (5) corresponding to the measure M. On the other hand, the unit measure E_0 has the same property which contradicts the Theorem 1. Theorem 2 is thus proved.

4. Self-decomposable measures. A measure $P \in \mathfrak{P}$ is said to be *self-decomposable* if for every number c satisfying the condition $0 < c < 1$ there exists a measure $Q_c \in \mathfrak{P}$ such that $P = T_cP\circ Q_c$.

The following Lemmas is used in the sequel. They are a generalization of Lemmas which are well-known for ordinary convolution algebra.

LEMMA 2. *The characteristic function of a self-decomposable measure does not vanish.*

Proof. Suppose the contrary and assume that $\Phi_P(a) = 0$ and $\Phi_P(t) \neq 0$ whenever $0 \leqslant t < a$. We note that for each number c satisfying the condition $0 < c < 1$ the formula

$$\Phi_P(t) = \Phi_P(ct)\Phi_{Q_c}(t) \tag{8}$$

is true. Hence we get the relation

$$\lim_{c\to 1} \Phi_{Q_c}(t) = 1$$

in the interval $0 \leqslant t < c$. Applying the Compactness Lemma ([3], 230) we can choose then a sequence $\{c_n\}$ $(0 < c_n < 1)$ tending to 1 such that the sequence of measures $\{Q_{c_n}\}$ is weakly convergent to a measure from $\mathfrak{P}$. Thus

$$\lim_{n\to\infty} \Phi_{Q_{c_n}}(t) = 1$$

uniformly in the interval $0 \leqslant t \leqslant a$ and, consequently, $\Phi_{Q_{c_n}}(a) \neq 0$ for sufficiently large indices n. Hence and from (8) it follows that $\Phi_P(c_n a) = 0$ for sufficiently large n which yields a contradiction. The Lemma is thus proved.

LEMMA 3. *Let P be a self-decomposable measure. Then for each c $(0 < c < 1)$ the associated measure Q_c is infinitely divisible. Further, setting*

$$P_1 = P, \quad P_n = T_n Q_{\frac{n-1}{n}} \quad (n = 2, 3, \ldots) \tag{9}$$

we have

$$P = T_{n^{-1}}(P_1 \circ P_2 \circ \ldots \circ P_n) \quad (n = 1, 2, \ldots). \tag{10}$$

Moreover, the measures $T_{n^{-1}}P_k$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ are uniformly asymptotically negligible.

Proof. By Lemma 2 the characteristic function Φ_P does not vanish. Since

$$\Phi_P(t) = \Phi_P(ct)\Phi_{Q_c}(t), \tag{11}$$

we have the equations

$$\Phi_{P_1} = \Phi_P, \quad \Phi_{P_n}(t) = \frac{\Phi_P(nt)}{\Phi_P\big((n-1)t\big)} \quad (n = 2, 3, \ldots), \tag{12}$$

which imply

$$\Phi_P(nt) = \prod_{k=1}^{n} \Phi_{P_k}(t).$$

Formula (10) is a direct consequence of the last equation. Further, by (12), $\Phi_{T_{n^{-1}}P_k} \to 1$ uniformly in k $(k \leqslant n)$ which shows that the measures $T_{n^{-1}}P_k$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ are uniformly asymptotically negligible.

Given a number c $(0 < c < 1)$ we put

$$R_n = T_{n^{-1}}(P_{[cn]+1} \circ P_{[cn]+2} \circ \ldots \circ P_n) \quad (n = 1, 2, \ldots),$$

where the square brackets denote the integral part of a real number. By (11) and (12)

$$\Phi_{R_n}(t) = \frac{\Phi_P(t)}{\Phi_P\left(\frac{[cn]}{n}t\right)} \to \frac{\Phi_P(t)}{\Phi_P(ct)} = \Phi_{Q_c}(t)$$

uniformly in every finite interval. Thus $R_n \to Q_c$. Since the measures $T_{n^{-1}}P_k$ $(k = [cn]+1, [cn]+2, \ldots, n;\ n = 1, 2, \ldots)$ are uniformly asymptotically negligible, the limit measure Q_c is infinitely divisible ([3], Theorem 12) which completes the proof.

LEMMA 4. *If the measures $T_{c_n}P_k$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ are uniformly asymptotically negligible and the sequence $T_{c_n}(P_1 \circ P_2 \circ \ldots \circ P_n)$ converges to a probability measure P different from E_0, then*

$$\lim_{n\to\infty} c_n = 0 \tag{13}$$

and

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = 1. \tag{14}$$

Proof. Contrary to (13) let us suppose that there exists a subsequence $k_1 < k_2 < \ldots$ for which

$$\lim_{n\to\infty} c_{k_n}^{-1} = b < \infty.$$

Then, taking into account that the measures $T_{c_{k_n}}P_k$ $(k = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$ are uniformly asymptotically negligible, we infer that

$$P_k = T_{c_{k_n}^{-1}}(T_{c_{k_n}}P_k) \to T_b E_0 = E_0 \quad (k = 1, 2, \ldots).$$

Hence we get the equation

$$T_{c_n}(P_1 \circ P_2 \circ \ldots \circ P_n) = E_0 \quad (n = 1, 2, \ldots).$$

Consequently, $P = E_0$ which contradicts the assumption. Formula (13) is thus proved.

Let us turn next to (14). Suppose that we could find a subsequence $s_1 < s_2 < \ldots$ for which the condition

$$d = \lim_{n\to\infty} \frac{c_{s_n+1}}{c_{s_n}} \neq 1$$

is fulfilled. First we consider the case $d < \infty$. Since $T_{c_n}P_n \to E_0$, we have

$$T_{c_{s_n+1}}(P_1 \circ P_2 \circ \ldots \circ P_{s_n+1}) = T_{d_n} T_{c_{s_n}}(P_1 \circ P_2 \ldots \circ P_{s_n}) \circ T_{c_{s_n+1}} P_{s_n+1} \to T_d P,$$

where d_n denotes the quotient c_{s_n+1}/c_{s_n}. Thus $P = T_d P$ and, consequently, $P = T_{d^k}P$ $(k = 0, \pm 1, \pm 2, \ldots)$. Since 0 is a limit point of the sequence d^k $(k = 0, \pm 1, \pm 2, \ldots)$ the last equation yields $P = T_0 P = E_0$ which contradicts the assumption.

It remains the case $d = \infty$. Then, denoting c_{s_n}/c_{s_n+1} shortly by q_n we have the relation

$$T_{c_{s_n}}(P_1 \circ P_2 \circ \ldots \circ P_{s_n}) \circ T_{q_n}(T_{c_{s_n+1}} P_{s_n+1}) \to P.$$

On the other hand this sequence being equal to

$$T_{q_n}\big(T_{c_{s_n+1}}(P_1 \circ P_2 \circ \ldots \circ P_{s_n+1})\big)$$

tends to $T_0 P$, i.e. to E_0. Consequently, $P = E_0$ which contradicts the assumption. The Lemma is thus proved.

We are now in a position to give a characterization of self-decomposable measures.

THEOREM 3. *The class of self-decomposable measures in* $(\mathfrak{P}, \circ)$ *coincides with the class of limit distributions of sequences* $T_{c_n}(P_1 \circ P_2 \circ \ldots \circ P_n)$ *where* $T_{c_n}P_k$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ *are uniformly asymptotically negligible.*

Proof. First suppose that P is a limit distribution of a sequence $T(_{c_n}P_1 \circ P_2 \circ \ldots \circ P_n)$ where $T_{c_n}P_k$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ form a triangular array of uniformly asymptotically negligible measures. Since the unit measure E_0 is obviously self-decomposable, we may assume that $P \neq E_0$. Then, by Lemma 4, for any number c $(0 < c < 1)$ we can find sequences $k_1 < k_2 < \ldots$ and $s_1 < s_2 < \ldots$ such that $s_n < k_n$ and $\lim_{n\to\infty} \frac{k_n}{s_n} = c$. Setting

$$U_n = T_{c_{k_n}}(P_1 \circ P_2 \circ \ldots \circ P_{k_n}),$$

$$V_n = T_{c_{s_n}}(P_1 \circ P_2 \circ \ldots \circ P_{s_n}),$$

$$W_n = T_{k_n}(P_{s_n+1} \circ P_{s_n+2} \circ \ldots \circ P_{k_n}),$$

we have the relations

(16) $$U_n = T_{\frac{k_n}{s_n}} V_n \circ W_n \quad (n = 1, 2, \ldots),$$

(17) $$U_n \to P, \quad T_{\frac{k_n}{s_n}} V_n \to T_c P.$$

From (16) it follows that $\Phi_{W_n}(t)$ tends to $\Phi_P(t)/\Phi_P(ct)$ in a neighborhood of the origin. Applying the Compactness Lemma ([3], p. 230) we infer that the sequence $\{W_n\}$ is compact. Let Q_c be its limit point. Then, by (16) and (17), $P = T_c P \circ Q_c$ which shows that P is a self-decomposable measure.

The converse implication is a direct consequence of Lemma 3, which completes the proof.

We proceed now to a representation problem for characteristic functions of self-decomposable measures. First we establish some properties of measures m corresponding to self-decomposable probability measures by the representation formula (5).

Let $[0, \infty]$ denote the compactified half-line. A subset of $[0, \infty]$ is said to be separated from the origin if its closure is contained in $(0, \infty]$. Let m be a finite Borel measure on $[0, \infty]$. For any Borel subset $\mathscr{E}$ of $[0, \infty]$ separated from the origin we put

$$I_m(\mathscr{E}) = \int_{\mathscr{E}} \frac{m(dx)}{\omega(x)},$$

where, according to (4), the integrand is assumed to be $\big(1 - \Omega(x_0)\big)^{-1}$ if $x = \infty$. Denote by $\mathfrak{M}$ the set of all finite Borel measures m on $[0, \infty]$ satisfying for all numbers c $(0 < c < 1)$ and all Borel subsets $\mathscr{E}$ seperated from the origin the following condition

(18) $$I_m(\mathscr{E}) - I_m(c^{-1}\mathscr{E}) \geqslant 0.$$

It is clear that the set $\mathfrak{M}$ is convex. Let $\mathfrak{K}$ be the subset of $\mathfrak{M}$ consisting of probability measures on $[0, \infty]$. The set $\mathfrak{K}$ is convex and compact.

Suppose that the measure m is concentrated on the open half-line $(0, \infty)$ and put

(19) $$J_m(x) = \int_x^\infty \frac{m(du)}{\omega(u)} \quad (x > 0).$$

Obviously, $I_m\big([a, b)\big) = J_m(a) - J_m(b)$. It is easy to see that $m \in \mathfrak{M}$ if and only if the inequality (18) holds for all c $(0 < c < 1)$ and all subsets $\mathscr{E}$ of the form $[a, b)$. Consequently, $m \in \mathfrak{M}$ if and only if for every triplet a, b, c satisfying the conditions $0 < c < 1$, $0 < a < b$ the inequality

(20) $$J_m(a) - J_m(b) - J_m\left(\frac{a}{c}\right) + J_m\left(\frac{b}{c}\right) \geqslant 0$$

is true. Introducing the notation

(21) $$F(x) = J_m(e^x) \quad (-\infty < x < \infty)$$

and substituting $a = e^{x-h}$, $b = e^x$, $c = e^{-h}$ $(-\infty < x < \infty,\ 0 < h < \infty)$ into (20) we get the inequality

$$F(x) \leqslant \tfrac{1}{2}\big(F(x-h) + F(x+h)\big).$$

Thus the function F is convex on the real line. Moreover, by (19), it is also monotone non-increasing with $F(\infty)=0$. Consequently, it can be represented in the form

$$F(x)=\int_x^\infty q_m(u)\,du,$$

where q_m is monotone non-increasing and non-negative. Further, by (19) and (21),

$$(22)\qquad m(\mathscr{E})=\int_E \omega(x)q_m(\log x)\frac{dx}{x}.$$

Conversely, if q is monotone non-increasing and non-negative function and

$$\int_0^\infty \omega(x)q(\log x)\frac{dx}{x}<\infty$$

then the measure m defined by means of the formula

$$m(\mathscr{E})=\int_E \omega(x)q(\log x)\frac{dx}{x}$$

belongs to $\mathfrak{M}$. Indeed, then $J_m(x)=\int_x^\infty q(\log u)\frac{du}{u}$ and the inequality (20) is evident. Moreover $q_m=q$ at all continuity points.

We may assume that the function q_m is continuous from the right. In this case q_m is uniquely determined by the measure m. Thus we have proved the following Lemma.

Lemma 5. *Equation* (22) *establishes a one-to-one correspondence between measures m from $\mathfrak{M}$ concentrated on the open half-line $(0,\infty)$ and non-negative monotone non-increasing continuous from the right functions q_m on the real line satisfying the condition*

$$\int_0^\infty \omega(x)q_m(\log x)\frac{dx}{x}<\infty.$$

Further, the measures m from $\mathfrak{K}$ corresponds to functions q_m satisfying the condition

$$(23)\qquad \int_0^\infty \omega(x)q_m(\log x)\frac{dx}{x}=1.$$

We define a family m_x $(x\in[0,\infty])$ of probability measures on $[0,\infty]$ as follows: $m_0=E_0$, $m_\infty=E_\infty$ and

$$(24)\qquad m_x(\mathscr{E})=a(x)\int_{\mathscr{E}\cap[0,x]}\omega(u)\frac{du}{u}\qquad(0<x<\infty),$$

where $a(x)^{-1}=\int_0^x\frac{\omega(u)}{u}du$. We note that, by Lemma 1, $a(x)$ is finite for all x. It is obvious that m_0 and m_∞ belong to $\mathfrak{K}$. Since the measures m_x $(0<x<\infty)$ are concentrated on the open half-line $(0,\infty)$ and

$$(25)\qquad q_{m_x}(u)=\begin{cases}a(x) & \text{if } u<\log x\\ 0 & \text{if } u\geqslant\log x\end{cases}$$

we infer, by Lemma 5, that $m_x\in\mathfrak{K}$ too.

Lemma 6. *The set $\{m_x\colon x\in[0,\infty]\}$ coincides with the set of extreme points of $\mathfrak{K}$.*

Proof. Let $m,w_1,w_2\in\mathfrak{K}$. It is evident that $m=cw_1+(1-c)w_2$ if and only if $q_m=cq_{w_1}+(1-c)q_{w_2}$. Thus a measure m from $\mathfrak{K}$ is an extreme point of $\mathfrak{K}$ if and only q_m can not be decomposed into a convex combination of two different q-functions satisfying condition (23). It is very easy to verify that for $x\in(0,\infty)$ the function q_{m_x} is not a convex combination of two different q-functions with property (23). Consequently, the measures m_x $\big(x\in(0,\infty)\big)$ are extreme points of the set $\mathfrak{K}$. Further, m_0 and m_∞ are extreme points too.

On the other hand the only q-functions which can not be decomposed into a convex combination of two different q-functions satisfying condition (23) are the functions of the form $q_m(u)=b$ whenever $u<y$ and $q_m(u)=0$ in the remaining case. By (23) we have the relation $b=a(e^y)$. Thus $m=m_{e^y}$. Consequently, each extreme point of $\mathfrak{K}$ concentrated on the open half-line coincides with one of the measures m_x $\big(x\in(0,\infty)\big)$. It is clear that a measure belongs to $\mathfrak{M}$ if and only all its restriction to $(0,\infty)$, $\{0\}$ and $\{\infty\}$ respectively belong to $\mathfrak{M}$. Hence it follows that the extreme points of $\mathfrak{K}$ which are not concentrated on the open half-line $(0,\infty)$ are supported by the one-point sets $\{0\}$ and $\{\infty\}$ respectively. Consequently, they coincide with one of the measures m_0 and m_∞. The Lemma is thus proved.

One can easily prove that the mapping $x\to m_x$ is a homeomorphism between $[0,\infty]$ and the set of extreme points of $\mathfrak{K}$. Once the extreme points of $\mathfrak{K}$ are found we can apply a Theorem by Choquet ([1]). Since each element of $\mathfrak{M}$ is of the form cw, where $c\geqslant 0$ and $w\in\mathfrak{K}$ we then get the following Lemma.

Lemma 7. *A measure m belongs to $\mathfrak{M}$ if and only if there exists a finite Borel measure p on $[0,\infty]$ such that*

$$\int_{[0,\infty]} f(u)m(du)=\int_{[0,\infty]}\int_{[0,\infty]} f(u)m_x(du)p(dx)$$

for all continuous functions f on $[0,\infty]$.

COROLLARY. *A measure m concentrated on $[0, \infty)$ belongs to $\mathfrak{M}$ if and only if there exists a finite Borel measure p on $[0, \infty)$ such that*

$$\int_0^\infty g(u)m(du) = \int_0^\infty \int_0^\infty g(u)m_x(du)p(dx)$$

for all continuous bounded functions g on $[0, \infty)$.

Now we shall give a description of measures associated by representation formula (5) to self-decomposable probability measures. We note that by Theorem 3 of the present paper and Theorem 12 in [3] self-decomposable measures are infinitely divisible.

LEMMA 8. *A measure m concentrated on $[0, \infty)$ is a representing measure in* (5) *of a self-decomposable probability measure if and only if $m \in \mathfrak{M}$.*

Proof. Suppose that the characteristic function of a probability measure P is given by formula (5). Taking into account Lemma 3 we infer that P is self-decomposable if and only if the quotient Φ_P/Φ_{T_cP} for every number c satisfying the condition $0 < c < 1$ is the characteristic function of an infinitely divisible measure. Since

$$\Phi_P(t)/\Phi_{T_cP}(t) = \Phi_P(t)/\Phi_P(ct) = \exp\int_0^\infty \frac{\Omega(tx)-1}{\omega(x)}\left(m(dx)-\frac{\omega(x)}{\omega(c^{-1}x)}m(c^{-1}dx)\right)$$

we infer, by Theorem 1, that the measure

$$r_c(\mathscr{E}) = \int_{\mathscr{E}} m(dx) - \int_{\mathscr{E}} \frac{\omega(x)}{\omega(c^{-1}x)}m(c^{-1}dx)$$

for every c $(0 < c < 1)$ is non-negative. Of course, the last condition is equivalent to the condition

$$\int_{\mathscr{E}} \frac{r_c(dx)}{\omega(x)} \geqslant 0$$

for every c $(0 < c < 1)$ and every Borel set $\mathscr{E}$ separated from the origin. But the left-hand side of the last inequality is equal to $I_m(\mathscr{E}) - I_m(c^{-1}\mathscr{E})$. Consequently, P is self-decomposable if and only if $m \in \mathfrak{M}$ which completes the proof of the Lemma.

Lemma 8, Corollary to Lemma 7 and representation formula (5) yield the following Theorem.

THEOREM 4. *The class of characteristic functions of self-decomposable measures in $(\mathfrak{P}, \circ)$ coincides with the class of all functions of the form*

$$\Phi_P(t) = \exp\int_0^\infty \int_0^x \frac{\Omega(tu)-1}{u}du\left(\int_0^x \frac{\omega(v)}{v}dv\right)^{-1} p(dx),$$

where p is a finite Borel measure on $[0, \infty)$.

5. An example. As an example of a generalized convolution we quote the $(1, r)$-convolutions $(1 \leqslant r < \infty)$ considered by J. F. Kingman in [2] (see also [3], p. 218). The (1,1)-convolution is defined by means of the formula

$$\int_0^\infty f(x)(P\circ Q)(dx) = \tfrac{1}{2}\int_0^\infty\int_0^\infty \big(f(x+y)+f(|x-y|)\big)P(dx)Q(dy)$$

where f runs over all bounded continuous functions on $[0, \infty)$. The $(1, r)$-convolution for $r > 1$ is defined by the formula

$$\int_0^\infty f(x)(P\circ Q)(dx) = \frac{\Gamma\left(\frac{r}{2}\right)}{\Gamma\left(\frac{r-1}{2}\right)}\int_0^\infty\int_0^\infty\int_{-1}^1 f(x^2+y^2+2xyz)(1-z^2)^{\frac{r-3}{2}}dzP(dx)Q(dy).$$

All $(1, r)$-convolution algebras are regular. As a characteristic function in these algebras we can take the integral transformation

$$\Phi_P(t) = \int_0^\infty \Gamma\left(\frac{r}{2}\right)\left(\frac{2}{tx}\right)^{\frac{r}{2}-1} J_{\frac{r}{2}-1}(tx)P(dx) \tag{26}$$

where J_k is the Bessel function.

The $(1, r)$-convolution is closely connected with a random walk problem in Euclidean r-space. Namely, consider a random walk in r-space given by

$$S_n = X_1+X_2+\ldots+X_n \quad (n = 1, 2, \ldots)$$

where $X_1, X_2, \ldots$ are independent random r-vectors having spherical symmetric distribution. The probability distribution of the length $|S_n|$ is determined by that of the length $|X_1|, |X_2|, \ldots, |X_n|$ (see [2]). More precisely, the probability distribution of $|S_n|$ is the $(1, r)$-convolution of the probability distributions of $|X_1|, |X_2|, \ldots, |X_n|$. The asymptotic behaviour of $|S_n|$ $(n = 1, 2, \ldots)$ can be described in terms of the limit distribution of the sequence $c_n|S_n|$ $(n = 1, 2, \ldots)$ where c_n are suitable chosen positive numbers. It is clear that the class of all possible limit distributions coincides with the class of all self-decomposable probability distributions in the $(1, r)$-convolution algebra. Since

$$\int_0^x \frac{\omega(u)}{u}du \sim \log(1+x^2)$$

on the whole positive half-line, we get, by virtue of Theorem 4, the following statement:

The class of all possible limit distributions of sequences $c_n|S_n|$, where $c_n > 0$ and $S_n = X_1+X_2+\ldots+X_n$ $(n = 1, 2, \ldots)$ X_k being independent random r-vectors with spherical symmetric distribution coincides with the class of all probability distributions P on $[0, \infty)$ whose integral transform (26) is of the form

$$\Phi_P(t) = \exp \int_0^\infty \int_x^x \frac{\Gamma\left(\frac{r}{2}\right)\left(\frac{2}{tu}\right)^{\frac{r}{2}-1} J_{\frac{r}{2}-1}(tu)-1}{u}\, du \frac{m(dx)}{\log(1+x^2)},$$

where m is a finite Borel measure on $[0, \infty)$.

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On generalized variations (II)

by

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Abstract. A φ-function is a non-decreasing function, continuous for $u \geqslant 0$, $\varphi(u) = 0$ only for $u = 0$ and $\lim f(u) = \infty$ when $u \to \infty$. For a function x with domain $[a, b]$ put

$$V_\varphi(x) = \sup \sum_{\nu=1}^{n} \varphi(|x(t_\nu)-x(t_{\nu-1})|),$$

supremum is taken over all partitions of $[a, b]$, $\mathscr{V}^{*\varphi}$ denotes the class of all functions x defined on $[a, b]$ for which $x(a) = 0$ and $V_\varphi(\lambda x) < \infty$ for certain $\lambda > 0$, and $\mathscr{C}\mathscr{V}^{*\varphi}$ denotes the class of all functions continuous on $[a, b]$ belonging to $\mathscr{V}^{*\varphi}$. Among all φ-functions the log-convex φ-functions are distinguished i.e. ones satisfying the condition

$$\varphi(u^\alpha v^\beta) \leqslant \alpha\varphi(u)+\beta\varphi(v) \quad \text{for } u, v > 0,\ \alpha, \beta \geqslant 0, \alpha+\beta = 1.$$

There are presented two proofs of L. C. Young's Theorem that if φ and $\varphi^{\sim}$ are log-convex φ-functions satisfying the following L. C. Young's condition

$$(*) \qquad \sum_{\nu=1}^{\infty} \varphi_{-1}(1/\nu)\varphi^{\sim}_{-1}(1/\nu) < \infty$$

where φ_{-1} and $\varphi^{\sim}_{-1}$ are the inverse functions to φ and $\varphi^{\sim}$ respectively then the integral $\int_a^b x(t)\,dy(t)$ for functions $x \in \mathscr{C}\mathscr{V}^{*\varphi}$ and $y \in \mathscr{V}^{*\varphi^{\sim}}$ exists in the sense of Riemann–Stieltjes. Estimations of this integral with the use of series in $(*)$ are given. On the same assumptions is proved the theorem on passing to the limit under the sign of RS-integral, in particular – the analogue of Helly's theorem. It is shown also that if φ and $\varphi^{\sim}$ are convex φ-functions satisfying the certain conditions for which L. C. Young's condition $(*)$ does not hold then there are functions $x \in \mathscr{C}\mathscr{V}^{*\varphi}$ and $y \in \mathscr{C}\mathscr{V}^{*\varphi^{\sim}}$ such that their RS-integral does not exist. These results proved for scalar functions are generalized for functions with values in Banach spaces.

0. Introduction. The present paper can be regarded as a second part of paper [9] which, under the same title, appeared in Studia Math. in 1959 (results of [9] were earlier announced in [8]). In the present paper the notations essentially differ from those employed in [9] i.e. in all places where in [9] and other papers dealing with the theory of Orlicz spaces symbols M, N etc. were used we now write $\varphi, \psi, \ldots$ The purpose