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Semi-spectral integrals and related mappings

by

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Abstract. The present paper deals with mappings $X(\cdot) \to \int XdF$. X stands here for a bounded operator function on a Hilbert space and F for a semi-spectral measure. We prove some L^2 type estimates for $\int XdF$. Next we show that the above mapping is completely contractive provided F is normalized. Similar properties hold true for integrals $\int XGd\mu$ with operator density G. One always assumes that $X(\ )$ intertwines measures or densities respectively. The last part of the paper deals with dilation properties of the mapping $X(\cdot) \to \int XdF$.

The present paper deals with operators which intertwine semi-spectral measures. There are also considered some completely contractive mappings related to semi-spectral integrals of operator valued functions. Among others we prove a generalization of the inequality obtained by S. Parrott in [9]. We also present some general properties of operator intertwining completely positive maps and derive therefore several properties of mappings induced by semi-spectral integrals. As to dilation theory we refer here to [1], [5].

1. Throughout the present paper H stands for a complex Hilbert space. $L(H)$ denotes the algebra of all bounded linear operators in H. I is the identity operator in H. Let A be a C^*-algebra and M_n the C^*-algebra of all complex $n \times n$ matrices. The tensor product $A \otimes M_n$ of all $n \times n$ matrices over A is a $*$-algebra. It is also C^*-algebra because there is a unique C^*-norm on this $*$-algebra (see [1] for details). For symmetric subspace $S \subset A$ we define $S \otimes M_n = \{V \mid V = (v_{ij}) \in A \otimes M_n,\ v_{ij} \in S\}$. Suppose we are given the linear map $\varphi\colon B \to L(H)$ of the symmetric subspace B of A. We define a linear map

$$\varphi_n\colon B \otimes M_n \to L(H) \otimes M_n \qquad (n \geqslant 1)$$

by applying φ element by element, to each matrix over B. The following definition appears in [1]: We say that φ is completely contractive (positive) if for every $n \geqslant 1$, φ_n is contractive (positive). Suppose we are given a σ-field $\mathscr{B}$ of subsets of the space Ω and a positive, finite measure μ on this field. In what follows we are interested merely in integrating of bounded operator valued functions.

DEFINITION 1. The bounded operator valued function $X\colon \Omega \to L(H)$ is called *simple* if $X(w) = \sum_{i=1}^{\infty} X_i \psi_{\sigma_i}(w)$ where $X_i \in L(H)$ and $\sigma_1, \ldots, \sigma_n, \ldots$ is a partition of Ω i.e. $\sigma_i \cap \sigma_j = \emptyset$ $(i \neq j)$ and $\Omega = \bigcup_i \sigma_i$.

DEFINITION 2. We say that the bounded function $X\colon \Omega \to L(H)$ is *$\mathscr{B}$-measurable,* if there exists a sequence $X_n(\cdot)$ of simple functions such that $\sup_{w \in \Omega \setminus \gamma} \|X(w) - X_n(w)\| \underset{n\to\infty}{\to} 0$, where $\mu(\gamma) = 0$.

Since μ is finite every $\mathscr{B}$-measurable function is integrable in the sense of Bochner (see [3] — for definition). More precisely, if $X(\cdot)$ is simple and $X(w) = \sum_{i=1}^{\infty} X_i \psi_{\sigma_i}(w)$ then $\int_\Omega X(w)\,d\mu = \sum_{i=1}^{\infty} X_i \mu(\sigma_i)$ by definition. Now if X_n is a sequence of simple functions such that $\sup_{\Omega\setminus\Omega_0} \|X_n(w) - X(w)\| \to 0$, $n \to \infty$ $(\mu(\Omega_0) = 0)$ then $\int_\Omega X(w)\,d\mu = \lim_{n\to\infty} \int_\Omega X_n(w)\,d\mu$ — see [3] for details. Note that a product of $\mathscr{B}$-measurable functions is also integrable, since it is $\mathscr{B}$-measurable.

Let $X(\cdot)$ and $G(\cdot)$ be $\mathscr{B}$-measurable operator valued functions and $0 \leqslant G(w) \leqslant I$ for $w \in \Omega$. It follows that the function $X(w)G(w)$ is integrable. Let $Y_i\colon \Omega \to L(H_i)$ $i = 1, \ldots, k$ be a set of functions. Then the vectorial function $\vec{Y}(w) = (Y_1(w), \ldots, Y_k(w)) \in L(H_1) \times \ldots \times L(H_k)$ is $\mathscr{B}$-measurable if and only if every Y_i is $\mathscr{B}$-measurable. This shows that the following proposition holds true.

PROPOSITION 1.1. *Suppose we are given the set of integrals* $\int_\Omega X_{ik}(w)G(w)\,d\mu$ $(i, k = 1, \ldots, m)$. *Then for every* $\varepsilon > 0$ *there exists a partition* $\{\sigma_i\}$ *of* Ω *such that*

$$\Big\| \sum_{j=1}^{\infty} B_j^{ik} \mu(\sigma_j) - \int_\Omega X_{ik}(w) G(w)\,d\mu \Big\| < \varepsilon, \quad \text{for } i, k = 1, \ldots, m,$$

where $B_j^{ik} = X_{ik}(w_j) G(w_j)$ *and* $w_j \in \sigma_j$.

Let us consider two operator valued $\mathscr{B}$-measurable functions $G_i(w)$ $(w \in \Omega)$ such that $0 \leqslant G_i(w) \leqslant I$. Denote by $\mathscr{T}$ the set of all $\mathscr{B}$-measurable functions which intertwine $G_i(w)$ i.e.

$$G_2(w) X(w) = X(w) G_1(w) \tag{1}$$

μ-almost everywhere. $\mathscr{T}$ becomes the Banach space with the following norm $\|X(\cdot)\| = \sup_w \|X(w)\|$. The spectral theorem yields that $X(\cdot) \in \mathscr{T}$ intertwines $G_i^{1/2}(\cdot)$ that is $G_2^{1/2}(w)X(w) = X(w)G_1^{1/2}(w)$ μ-almost everywhere. We can prove now the following

THEOREM 1.1. *Let* $\mu(\Omega) = 1$ *and let* $\mathscr{T} \ni X(\cdot) \to \int_\Omega G_2(w) X(w)\,d\mu$

$= \int_\Omega X(w) G_1(w)\,d\mu \in L(H)$ *be the linear map of* $\mathscr{T}$ *into* $L(H)$. *For every* $n = 1, 2, \ldots$, *the following inequality holds true:*

$$\Big\| \Big(\int_\Omega G_2(w) X_{ik}(w)\,d\mu \Big) \Big\|_{L(H\oplus\ldots\oplus H)} \leqslant \sup_w \big\| \big(X_{ik}(w)\big) \big\|_{L(H\oplus\ldots\oplus H)} \qquad i, k = 1, 2, \ldots, n.$$

Proof. Let $(f_1, \ldots, f_n) = \hat{f} \in \bigoplus_{i=1}^{n} H_i$ $(g_1, \ldots, g_n) = \hat{g} \in \bigoplus_{i=1}^{n} H_i$, where $H_i = H$. By the Proposition 1.1 we can take approximating summs for $\int_\Omega G_2(w) X_{ik}(w)\,d\mu$ of the form: $\sum_{j=1}^{\infty} G_2(w_j) X_{ik}(w_j) \mu(\sigma_j)$.

We have now

$$\Big| \sum_{ik=1}^{n} \Big(\sum_{j=1}^{\infty} G_2(w_j) X_{ik}(w_j) \mu(\sigma_j) f_k, g_i \Big) \Big|$$
$$= \Big| \sum_{j=1}^{\infty} \sum_{ik=1}^{n} \big(G_2(w_j) X_{ik}(w_j) \mu(\sigma_j) f_k, g_i \big) \Big|$$
$$= \Big| \sum_{j=1}^{\infty} \sum_{ik=1}^{n} \big(X_{ik}(w_j) G_1^{1/2}(w_j) \mu(\sigma_j) f_k, G_2^{1/2}(w_j) g_i \big) \Big|$$
$$\leqslant \sum_{j=1}^{\infty} \sup_w \|(X_{ik}(w))\|_{L(\oplus H_i)} \Big(\sum_{k=1}^{n} \|G_1^{1/2}(w_j) \mu^{1/2}(\sigma_j) f_k\|^2 \Big)^{1/2} \cdot \Big(\sum_{i=1}^{n} \|G_2^{1/2}(w_j) \mu^{1/2}(\sigma_j) g_i\|^2 \Big)^{1/2}$$
$$\leqslant \sup_w \big\| \big(X_{ik}(w)\big) \big\|_{L(\oplus H_i)} \Big(\sum_{j=1}^{\infty} \sum_{k=1}^{n} \|G_1^{1/2}(w_j) \mu^{1/2}(\sigma_j) f_k\|^2 \Big)^{1/2} \cdot \Big(\sum_{j=1}^{\infty} \sum_{i=1}^{n} \|G_2^{1/2}(w_j) \mu^{1/2}(\sigma_j) g_i\|^2 \Big)^{1/2} \leqslant \sup_w \big\| \big(X_{ik}(w)\big) \big\|_{L(\oplus H_i)} \|\hat{f}\| \cdot \|\hat{g}\|.$$

Since f and g are arbitrary the proof is complete.

If $G_1 = G_2 = G$ in the above theorem, then $\mathscr{T}$ becomes a C^*-algebra with involution $X^*(w) = X(w)^*$. From Theorem 1.1. we derive

COROLLARY 1.1. *If* $G_1 = G_2 = G$ *then the linear map* $\mathscr{T} \ni X(\cdot) \to \int_\Omega X(w) G(w)\,d\mu$ *is completely contractive.*

Now let us define a class of $\hat{\mathscr{B}}$-measurable operator valued functions.

DEFINITION 3. The bounded operator valued function $X\colon \Omega \to L(H)$ is called $\hat{\mathscr{B}}$-measurable, if there is a sequence of simple functions $X_n(w)$ such that $\sup_w \|X(w) - X_n(w)\| \underset{n\to\infty}{\to} 0$.

Let $X\colon \Omega \to L(H)$ be $\hat{\mathscr{B}}$-measurable and let $F_i\colon \Omega \to L(H)$ $(i = 1, 2)$ be semi-spectral measures. Suppose that $X(w)$ intertwines F_2 and F_1 i.e.

$$F_2(\sigma) X(w) = X(w) F_1(\sigma) \quad \text{for all } w \in \Omega \text{ and } \sigma \in \mathscr{B}. \tag{2}$$

Denote by $\mathscr{C}$ the set of all $X(\cdot)$ such that (2) holds true. For $X(\cdot)\in\mathscr{C}$ one can define the integral $\int_\Omega X(w)dF_1$ $(=\int_\Omega dF_2X(w))$ (see [6], [7] — for references). We have the following

THEOREM 1.2. *If $F_i(\Omega)=I$ for $i=1,2$, then the linear* map $\mathscr{C}\ni X\to\int_\Omega X(w)dF_1$ *for every $n=1,2,\ldots$ satisfies the inequality*:

$$\Big\|\Big(\int_\Omega X_{ik}(w)dF_1\Big)\Big\|_{L(H\oplus\ldots\oplus H)}\leqslant\sup\big\|\big(X_{ik}(w)\big)\big\|_{L(H\oplus\ldots\oplus H)}\qquad(i,k=1,\ldots,n).$$

The proof of Theorem 1.2 is just the same as that of Theorem 1.1. In particular, if $F_1=F_2=F$, then $\mathscr{C}$ becomes C^*-algebra and so we have

COROLLARY 1.2. *If $F_1=F_2=F$ then linear* map $\mathscr{C}\ni X(\cdot)\to\int_\Omega X(w)dF$ *is completely contractive provided* (2) *holds true i.e. the values of $X(\cdot)$ and $F(\cdot)$ commute.*

Note by the way, that from the above corollary one can deduce, that every contractive representation $T\colon A\to L(H)$ (such that $T(1)=I$) of the Dirichlet algebra $A\subset C(\Omega)$ (Ω-compact Hausdorff space) is completely contractive. Indeed, it is known (see [2]), that then $T(u)=\int_\Omega u dF$, where F is a unique, regular, normalized semi-spectral measure on Borel subsets of Ω. Taking in the Corollary 1.2 $\mathscr{C}=C(\Omega)$ we get the claim. We emphasize that our proof is direct and avoids the Naimark dilation theorem for semi-spectral measure F.

Let $X\colon\Omega\to L(H)$ be $\hat{\mathscr{B}}$-measurable and let F be a normalized semi-spectral measure. Assume that the values of $X(\cdot)$ and $F(\cdot)$ commute. We will prove the following theorem

THEOREM 1.3. *Let $X\colon\Omega\to L(H)$ and F be as above. Then for every $f\in H$ we have the inequality*

$$(\alpha)\qquad\Big\|\Big(\int_\Omega X(w)dF\Big)f\Big\|^2\leqslant\int_\Omega\|X(w)\|^2d(Ff,f).$$

Proof. If $f,g\in H$ are arbitrary and $\{\sigma_i\}$ is a partition of Ω, then we have for $w_i\in\sigma_i$

$$\Big|\sum_{i=1}^\infty\big(X(w_i)F(\sigma_i)f,g\big)\Big|\leqslant\sum_{i=1}^\infty\big|\big(X(w_i)F^{1/2}(\sigma_i)f,\,F^{1/2}(\sigma_i)g\big)\big|$$
$$\leqslant\Big(\sum_{i=1}^\infty\|X(w_i)F^{1/2}(\sigma_i)f\|^2\Big)^{1/2}\cdot\Big(\sum_{i=1}^\infty\|F^{1/2}(\sigma_i)g\|^2\Big)^{1/2}$$
$$=\Big(\sum_{i=1}^\infty\|X(w_i)F^{1/2}(\sigma_i)f\|^2\Big)^{1/2}\cdot\|g\|.$$

Thus

$$\Big\|\sum_{i=1}^\infty X(w_i)F(\sigma_i)f\Big\|^2\leqslant\sum_{i=1}^\infty\|X(w_i)F^{1/2}(\sigma_i)f\|^2\leqslant\sum_{i=1}^\infty\|X(w_i)\|^2\big(F(\sigma_i)f,f\big).$$

Since the partition $\{\sigma_i\}$ is arbitrary the theorem is proved. Now let $X\colon\Omega\to L(H)$ be $\mathscr{B}$-measurable and let $0\leqslant G(w)\leqslant I$ $(w\in\Omega)$ be also $\mathscr{B}$-measurable. Assume that the values of $X(\cdot)$ and $G(\cdot)$ commute μ-almost everywhere. Then we have an analogous to Theorem 1.3

THEOREM 1.4. *Let $X\colon\Omega\to L(H)$ and G be as above. For every $f\in H$ we have the inequality*:

$$(\beta)\qquad\Big\|\Big(\int_\Omega X(w)G(w)d\mu\Big)f\Big\|^2\leqslant\int_\Omega\|X(w)\|^2\big(G(w)f,f\big)d\mu.$$

The proof is just the same as that of Theorem 1.3.

Remark 1.1. Note that inequality (α) implies the results obtained by Mlak and Ryll-Nardzewski in [7]. The estimation (α) is better than the inequality

$$\Big\|\int_\Omega X(w)dF\Big\|\leqslant\sup_w\|X(w)\|\cdot\|F(\Omega)\|.$$

Remark 1.2. Let $T\in L(H)$, $\|T\|\leqslant1$ and let F denote semi-spectral measure of T (see [5]). Consider a matrix (p_{ij}) $1\leqslant i,j\leqslant n$, of polynomials $p_{ij}(z)=p_{ij}(z,A_{0ij},\ldots,A_{kij})$ $(|z|=1)$ with operator coeficients A_{sij} commuting with T and T^*. For each fixed z, the matrix $\big(p_{ij}(z)\big)$ is considered as an operator on $\bigoplus_{i=1}^n H_i=M$, $H_i=H$. Since T is a contraction, we have the equality:

$$p_{ij}(T,A_{0ij},\ldots,A_{kij})=\sum_{s=0}^kA_{sij}T^s=\int_{|z|=1}\sum_{s=0}^kz^sA_{sij}dF$$
$$=\int_{|z|=1}p_{ij}(z,A_{0ij},\ldots,A_{kij})dF.$$

Corollary 2.1 proves that

$$\big\|\big(p_{ij}(T,A_{0ij},\ldots,A_{kij})\big)\big\|_{L(M)}\leqslant\sup_{|z|=1}\big\|\big(p_{ij}(z,A_{0ij},\ldots,A_{kij})\big)\big\|_{L(M)}.$$

This inequality generalizes the inequality obtained by Parrott [9] in a different way.

Assume now that $T\in L(H)$ has a unitary ϱ-dilation $(\varrho>0)$ i.e.

$$T^n=\operatorname{pr}\varrho\cdot U^n\qquad(n=1,2,\ldots)$$

where U is unitary (see [5] Ch. I for details). Let E be the spectral measure of U and $F = \mathrm{pr}E$. Let $B_{kij} \in L(H)$ $k = 0, \ldots, n$, $i, j = i, \ldots, m$ commute with T and T^*. Then $B_{kij}F(\sigma) = F(\sigma)B_{kij}$ for $\sigma \in \mathscr{B}$. Next we have

$$\sum_{k=0}^{n} B_{kij}T^k = \varrho \cdot \sum_{k=0}^{n} B_{kij} \int_{|z|=1} z^k dF + (1-\varrho) B_{0ij} \int_{|z|=1} z^0 dF$$
$$= \int_{|z|=1} \Big(\varrho \sum_{k=0}^{n} B_{kij} z^k + (1-\varrho) B_{0ij}\Big) dF .$$

Applying Corollary 2.1 once more we conclude that

$$\left\| \begin{bmatrix} \sum_{k=0}^{n} B_{k11} T^k, \ldots, \sum_{k=0}^{n} B_{k1m} T^k \\ \sum_{k=0}^{n} B_{km1} T^k, \ldots, \sum_{k=0}^{n} B_{kmm} T^k \end{bmatrix} \right\|_{L(M)}$$
$$\leqslant \sup_{|z|=1} \left\| \begin{bmatrix} \varrho \sum_{k=0}^{n} B_{k11} z^k, \ldots, \varrho \sum_{k=0}^{n} B_{k1m} z^k \\ \varrho \sum_{k=0}^{n} B_{km1} z^k, \ldots, \varrho \sum_{k=0}^{n} B_{kmm} z^k \end{bmatrix} + (1-\varrho) \begin{bmatrix} B_{011}, \ldots, B_{01m} \\ B_{0m1}, \ldots, B_{0mm} \end{bmatrix} \right\|_{L(M)}$$

where $M = \underbrace{H \oplus \ldots \oplus H}_{n}$.

2. Let B be a C^*-algebra with the unit element e and φ_i: $B \to L(H_i)$ $i = 1, 2$ a completely positive linear map. Stinespring proved (see [1]) that every such φ_i has the form $\varphi_i(u) = V_i^* \psi_i(u) V_i$ where ψ_i is a representation of B on some Hilbert space K_i and V_i is a bounded linear operator from H_i to K_i. Setting $L_i = [\psi_i(B) V_i H_i]$, then the restriction $\bar{\psi}_i$ of ψ_i to L_i also satisfies the equality $\varphi_i(u) = V_i^* \bar{\psi}_i(u) V_i$, so there is no loss of generality, if we require that $[\psi_i(B) V_i H_i] = K_i$.

The pair (ψ_i, V_i) is called *minimal* if $[\psi_i(B) V_i H_i] = K_i$.

We will prove later, that the map (considered before) $\mathscr{T} \ni X(\cdot) \to \int_\Omega X(w) G(w)\, d\mu$ is completely positive. Also the map $\varphi \ni X(\cdot) \to \int_\Omega X(w)\, dF$ is completely positive, since it is completely contractive and $F(\Omega) = I$.

Let X: $H_1 \to H_2$ be a bounded operator which intertwines F_i $(i = 1, 2)$ — the normalized semi-spectral measures. By the theorem due to Lebow [4], see also Mlak [8], X extends uniquely to an operator $\tilde{X} \in L(K_1, K_2)$ such that

a) $\tilde{X} \cdot E_1(\sigma) = E_2(\sigma) \cdot \tilde{X}$,

b) $\|\tilde{X}\| = \|X\|$

E_i stands here for the minimal spectral dilation of F_i.

A more general theorem holds true for $X \in L(H_1, H_2)$ which intertwine completely positive linear maps φ_1 and φ_2 of a general C^*-algebra. It's proof reduces to the use of suitably modified arguments given by Arveson in the proof of Theorem 1. 3. 1 of [1].

THEOREM 2.1. *Let B be a C^*-algebra with the unit e. Assume that φ_1 and φ_2 are completely positive linear maps of B. If X: $H_1 \to H_2$ is a bounded operator such that*

$$X\varphi_1(u) = \varphi_2(u) X \quad \textit{for } u \in B$$

then there is a unique bounded operator $\tilde{X}$: $K_1 \to K_2$ which satisfies the following equalities

$$\tilde{X}\psi_1(u) = \psi_2(u)\tilde{X}, \tag{1}$$

$$\tilde{X}V_1 = V_2 X, \tag{2}$$

$$V_2^*\tilde{X} = XV_1^* \quad \textit{and} \quad \tilde{X}V_1V_1^* = V_2V_2^*\tilde{X}. \tag{3}$$

Proof. Let $f_1, \ldots, f_n \in H$ and $u_1, \ldots, u_n \in B$. We claim that

$$\Big\| \sum_{j=1}^{n} \psi_2(u_j) V_2 X f_j \Big\|^2 \leqslant \|X\|^2 \Big\| \sum_{j=1}^{n} \psi_1(u_j) V_1 f_j \Big\|^2 .$$

If $n = 1$, then for $f \in H$ and $u \in B$ we have

$$\begin{aligned} \|\psi_2(u) V_2 X f\|^2 &= (V_2^* \psi_2(u^* u) V_2 X f, Xf) \\ &= (\varphi_2(u^* u) Xf, Xf) = (X^* X \varphi_1(u^* u) f, f) \\ &= ((X^* X)(V_1^* \psi_1(u^*) V_1) f, f) \end{aligned} \tag{*}$$

$V_1^* \psi_1(u^*) \psi_1(u) V_1$ is positive operator which commutes with X^*X. Indeed $X^* X \varphi_1(u^* u) = X^* \varphi_2(u^* u) X = \varphi_1(u^* u) X^* X$ and so does the positive square root of $\varphi_1(u^* u)$. Denoting this square root by S we obtain from (*)

$$\|\psi_2(u) V_2 X f\|^2 = (X^* X S^2 f, f) = \|XSf\|^2 \leqslant \|X\|^2 \|Sf\|^2 = \|X\|^2 \|\psi_1(u) V f\|^2 .$$

The case $n > 1$ is reduced to the preceeding $(n = 1)$ in the following way.

Let $H_l^{\mid} = \bigoplus_{i=1}^{n} H_i^l$ $(H_i^l = H_l)$ for $l = 1, 2$ $K_l^{\mid} = \bigoplus_{i=1}^{n} K_i^l$ $(K_i^l = K_l)$ for $l = 1, 2$. Let $X^{\mid} \in L(H_1^{\mid}, H_2^{\mid})$ be the operator given by the matrix

$$\begin{bmatrix} X & & 0 \\ & X & \\ & & \ddots \\ 0 & & X \end{bmatrix},$$

$V_i^{|} \in L(H_i^{|}, K_i^{|})$ given by the matrix

$$\begin{bmatrix} V_i & & & 0 \\ & V_i & & \\ & & \ddots & \\ 0 & & & V_i \end{bmatrix}$$

and $A_i^{|} \in L(K_i^{|})$ given by the matrix

$$\begin{bmatrix} \psi_i(u_1), \ldots, \psi_i(u_n) \\ 0 \qquad\qquad 0 \\ \vdots \qquad\qquad \vdots \\ 0 \qquad\qquad 0 \end{bmatrix}.$$

Then $V_i^{|*} A_i^{|*} A_i^{|} V_i^{|} = B_i^{|} \in L(H_i^{|})$ has the matrix (b_{jk}^i), where $b_{jk}^i = V_i^* \psi_i(u_j^*) \psi_i(u_k) V_i$. The equality $X b_{jk}^1 = b_{jk}^2 X$, for $j, k = 1, \ldots, n$ implies that

$$X^{|} B_1^{|} = B_2^{|} X^{|}.$$

Now for $(f_1, \ldots, f_n) = \hat{f} \in H_1^{|}$ we have

$$\Big\| \sum_{j=1}^{n} \varphi_2(u_j) V_2 X f_j \Big\|^2 = \sum_{ij=1}^{n} \big(V_2^* \varphi_2(u_i^*) \varphi_1(u_j) V_2 X f_j, f_i\big)$$
$$= (V_2^{|*} A_2^{|*} A_2^{|} V_2^{|} X^{|} \hat{f}, X^{|} \hat{f}) \leqslant \|X^{|}\|^2 \|A_1^{|} V_1^{|} \hat{f}\|^2$$
$$= \|X\|^2 \Big\| \sum_{j=1}^{n} \psi_1(u_j) V_1 f_j \Big\|^2$$

and that proves the claim. We infer that the operator determined by the equality $\hat{X}\big(\sum_j \psi_1(u_j) V_1 f_j\big) = \sum_j \psi_2(u_j) V_2 X f_j$ is well defined and extends uniquely to an operator $\tilde{X} \in L(K_1, K_2)$ on $[\psi_1(B) V_1 H_1] = K_1$. It is obvious that $\|\tilde{X}\| \leqslant \|X\|$. It is easy to check that $\tilde{X}\psi_1(u) = \psi_2(u)\tilde{X}$. To see this write

$$\tilde{X}\psi_1(u) \sum_j \psi_1(u_j) V_1 f_j = \tilde{X} \sum_j \psi_1(u u_j) V_1 f_j = \sum_j \psi_2(u u_j) V_2 X f_j$$
$$= \psi_2(u) \sum_j \psi_2(u_j) V_2 X f_j = \psi_2(u) \tilde{X} \sum_j \psi_1(u_j) V_1 f_j.$$

We obtained equality on a dense subset of K_1, so it holds true on all of K_1. The equality $\tilde{X} V_1 = V_2 X$ is obvious because $e \in B$. The uniqueness of the operator $\tilde{X}$ satisfying (1) and (2) is a consequence of the equality $[\psi_1(B) V_1 H_1] = K_1$.

The proof of (3) is also immediate. Indeed

$$V_2^* \tilde{X} \sum_j \psi_1(u_j) V_1 f_j = V_2^* \sum_j \psi_2(u_j) \tilde{X} V_1 f_j = V_2^* \sum_j \psi_2(u_j) V_2 X f_j$$
$$= \sum_j \varphi_2(u_j) X f_j = \sum_j \varphi_1(u_j) f_j = X V_1^* \sum_j \psi_1(u_j) V_1 f_j.$$

The conclusion follows since $[\psi_1(B) V_1 H_1] = K_1$. The proof is complete.

Now we will show that the mapping $X \to \tilde{X}$ preserves certain properties of X. Note that if $X\varphi_1(u) = \varphi_2(u) X$ for $u \in B$, then $\varphi_1(u) X^* = X^* \varphi_2(u)$ and so by the symmetry of the role of φ_1 and φ_2 it makes a sense an extension $\widetilde{X^*}$ of the X^*. First we observe that $\widetilde{X^*} = \tilde{X}^*$.

To prove this write

$$\Big(\sum_j \psi_1(u_j) V_1 f_j, \tilde{X}^* \sum_i \psi_2(u_i) V_2 g_i\Big) = \Big(\tilde{X} \sum_j \psi_1(u_j) V_1 f_j, \sum_i \psi_2(u_i) V_2 g_i\Big)$$
$$= \Big(\sum_j \psi_2(u_j) V_2 X f_j, \sum_i \psi_2(u_i) V_2 g_i\Big)$$
$$= \sum_{ij} \big(\varphi_2(u_i^* u_j) X f_j, g_i\big)$$
$$= \sum_{ij} \big(\varphi_1(u_i^* u_j) f_j, X^* g_i\big)$$
$$= \Big(\sum_j \psi_1(u_j) V_1 f_j, \sum_i \psi_1(u_i) V_1 X^* g_i\Big)$$
$$= \Big(\sum_j \psi_1(u_j) V_1 f_j, \widetilde{X^*} \sum_i \psi_2(u_i) V_2 g_i\Big)$$

and since $[\psi_i(B) V_i H_i] = K_i$, the proof is complete.

Moreover, we have the following corollary:

COROLLARY 2.1. *Let B be a C^*-algebra and let φ_1, φ_2, X and $\tilde{X}$ be as in the Theorem 2.1.*

The following implications holds true:

a) *If $X^* = X^{-1}$ then $\tilde{X}^* = \tilde{X}^{-1}$.*

b) *If $X^* X = I_{H_1}$ then $\tilde{X}^* \tilde{X} = I_{K_1}$.*

c) *If $\overline{R(X)} = H_2$ then $R(\tilde{X}) = K_2$.*

d) *If X is strictly invertible so is $\tilde{X}$ and $\tilde{X}^{-1} = \widetilde{X^{-1}}$.*

3. As we mentioned before when using Theorems 1.2 and 2.1 one can give an easy proof of the following fact due to Lebow [4] (see also Mlak [8]).

(T) If F_i $(i = 1, 2)$ denotes a normalized semi-spectral measure on Ω, then every operator $X \in L(H_1, H_2)$ such that $F_2(\sigma)X = XF_1(\sigma)$ for all $\sigma \in \mathscr{B}$ extends uniquely to an operator $\tilde{X} \in L(K_1, K_2)$ such that

i) $\tilde{X}E_1(\sigma) = E_2(\sigma)\tilde{X}$,

ii) $\|\tilde{X}\| = \|X\|$

where E_i — is the minimal spectral dilation of F_i.

To see this note that the condition $XF_1(\sigma) = F_2(\sigma)X$ for all $\sigma \in \mathscr{B}$ implies the equality $X \cdot \int_\Omega Y(w)\,dF_1 = \int_\Omega Y(w)\,dF_2 \cdot X$ for every scalar valued $\mathscr{B}$-measurable, bounded function $Y\colon \Omega \to C$. By Theorem 1.2 the mappings $Y \to \int_\Omega Y(w)\,dF_i$ $(i = 1, 2)$ are completely contractive and so they are completely positive (see [1]). Theorem 2.1 asserts that there exists an operator $\tilde{X} \in L(K_1, K_2)$ such that

i) $\tilde{X} \cdot \int_\Omega Y(w)\,dE_1 = \int_\Omega Y(w)\,dE_2 \cdot \tilde{X}$,

ii) $\|\tilde{X}\| = \|\tilde{X}\|$,

iii) $\tilde{X}f = Xf$ for all $f \in H$.

Note that by Corollary 2.1 this extension $\tilde{X}$ shares the series of properties of X, mentioned in Corollary 2.1. One can also prove a similar theorem concerning X intertwining densities G_1 and G_2 considered in Theorem 1.1.

The fact that each X, which commutes with semi-spectral measure F $(F(\Omega) = I)$ extends uniquely to $\tilde{X}$, which commutes with its minimal spectral dilation E, implies completely contractivity of the map $\mathscr{C} \ni X(\cdot) \to \int_\Omega X(w)\,dF$. Indeed, to every $X(\cdot) \in \mathscr{C}$ there corresponds an operator valued function $\tilde{X}(\cdot)$, which is also $\tilde{\mathscr{B}}$-measurable, because the mapping $X \to \tilde{X}$ is isometric. We denote the $*$-isomorphizm $X \to \tilde{X}$ by τ. Let us denote by $\tilde{\mathscr{C}}$ the C^*-algebra $\tau(\mathscr{C}) \subset L(K)$. Let the sequence $X_n = \sum_{i=1}^{\infty} X(w_i^n)F(\sigma_i^n)$ tend to $\int_\Omega X(w)\,dF$, as $n \to \infty$.

Write $\sum_{i=1}^{\infty} X(w_i^n)F(\sigma_i^n) = P\sum_{i=1}^{\infty} X(w_i^n)E(\sigma_i^n)$, $P\colon K \to H$-projection. Passing to the limit in the above equality we have

$$\int_\Omega X(w)\,dF = P\int_\Omega \tilde{X}(w)\,dE$$

or equivalently $T(X) = P\tilde{T}(\tau(X))$, where $\tilde{T}(\tau(X)) = \int_\Omega \tilde{X}(w)\,dE$. We will show that $\tilde{T}$ is multiplicative on $\tilde{\mathscr{C}}$. It suffices to prove the multiplicativity of $\tilde{T}$ for simple functions. Without loosing generality, we can take simple functions $\psi_1, \psi_2 \in \tilde{\mathscr{C}}$ of the following form: $\psi_1(w) = \sum_{i=1}^{\infty} \tilde{X}_1(w_i^1) \times \psi_{\sigma_i}(w)$, $\psi_2(w) = \sum_{i=1}^{\infty} \tilde{X}_2(w_i^2)\psi_{\sigma_i}(w)$.

Therefore

$$\tilde{T}(\psi_1) \cdot T(\psi_2) = \sum_{i=1}^{\infty} X_1(w_i^1)E(\sigma_i) \cdot \sum_{i=1}^{\infty} \tilde{X}_2(w_i^2)E(\sigma_i)$$

$$= \sum_{i=1}^{\infty} \tilde{X}_1(w_i^1) \cdot \tilde{X}_2(w_i^2)E(\sigma_i) = \tilde{T}(\psi_1 \cdot \psi_2).$$

We conclude from the above that T is a projection of the representation $\tilde{T} \circ \tau$ of $\mathscr{C}$. It is known that every linear map of a general C^*-algebra into $L(H)$, which is a projection of a representation of this C^*-algebra is completely contractive. Consequently T is completely contractive.

It follows from Corollary 2.1 that if $X(\cdot) \in \mathscr{C}$ and for every $w \in \Omega$ $X(w)$ is isometric, unitary, normal etc. then for every $w \in \Omega$, $\tilde{X}(w)$ is isometric, unitary, normal etc. Conclusion: if $X(w)$ are isometric, normal, unitary etc. then $\int_\Omega \tilde{X}\,dE$ is isometric, normal, unitary etc. Let us reconsider the situation as in Corollary 1.1. Above all observe that the map

$$\varphi\colon X(\cdot) \to \int_\Omega X(w) \cdot G(w)\,d\mu, \quad \text{for } X(\cdot) \in \mathscr{T}$$

is a completely positive linear map of $\mathscr{T}$ into $L(H)$. Indeed, if $(X_{ij}(\cdot))$ is a positive $n \times n$ matrix over $\mathscr{T}$ and $f_1, \ldots, f_n \in H$, then one can choose $Z_{ij}(\cdot)$ such that $(X_{ij}(\cdot)) = (Z_{ij}(\cdot))^*(Z_{ij}(\cdot))$.

By Proposition 1.1 we can take approximating summs for $\int_\Omega X_{ij}(w) \times G(w)\,d\mu$ of the from: $\sum_{i=1}^{\infty} X_{ij}(w_l) \cdot G(w_l)\mu(\sigma_l)$.

Now observe that

$$\sum_{ij=1}^{n}\Big(\sum_{k=1}^{n}\sum_{l=1}^{\infty} Z_{ki}^*(w_l)Z_{kj}(w_l)G(w_l)\mu(\sigma_l)f_j, f_i\Big)$$

$$= \sum_{l=1}^{\infty}\sum_{k=1}^{n}\sum_{ij=1}^{n}\big(Z_{ki}^*(w_l)Z_{kj}(w_l)G^{1/2}(w_l)\mu(\sigma_l)f_j, G^{1/2}(w_l)f_i\big)$$

$$= \sum_{l=1}^{\infty}\sum_{k=1}^{n}\Big\|\sum_{i=1}^{n} Z_{ki}(w_l)G^{1/2}(w_l)\mu^{1/2}(\sigma_l)f_i\Big\|^2 \geqslant 0$$

and so our assertion is true. As we know by Stinespring theorem (see [1] p. 145) $\varphi(X) = V^*\Pi(X)V$ where Π is a representation of $\mathscr{T}$ on K, $V \in L(H, K)$ and $K = [\Pi(\mathscr{T})VH]$.

We will find out some models of the space K assuming that H is separable. Let us define the space M of all measurable functions $f\colon\ \Omega\to H$ (i.e. the scalar function $(f(w), h)$ is measurable for every $h\in H$) for which

$$\int\limits_{\Omega}\big(G(w)f(w),\, f(w)\big)\,d\mu<+\infty.$$

M is a preunitary space with respect to the semi-inner product

$$(f,g)=\int\limits_{\Omega}\big(G(w)f(w),f(w)\big)\,d\mu.$$

Let $N=\{f\in M\colon\ (f,f)=0\}$. Then the quotient space M/N after completion becomes a Hilbert space. Denoting this Hilbert space by $\mathscr{L}^2(\mu G, H)$, we claim that $K=\mathscr{L}^2(\mu G, H)$.

To prove it we take $X_i=\sum_{k=1}^{\infty}D_k^i\psi\,\sigma_k\ (i=1,\ldots,m)$, the set of simple functions and $f_1,\ldots,f_m\in H$. Then we have

$$\begin{aligned}\Big\|\sum_{i=1}^{m}\Pi(X_i)Vf_i\Big\|^2&=\Big\|\sum_i\sum_k\Pi(D_k^{(i)})Vf_i\Big\|^2\\&=\Big\|\sum_k\sum_i\Pi(D_k^{(i)})Vf_i\Big\|^2=\sum_k\Big\|\sum_i\Pi(D_k^{(j)})Vf_i\Big\|^2\\&=\sum_k\sum_{ij}\Big(\int\limits_{\sigma_k}G(w)D_k^{(i)}\,d\mu f_i,\ D_k^{(j)}f_j\Big)\\&=\sum_k\sum_{ij}\int\limits_{\sigma_k}\big(G(w)D_k^{(i)}f_i,\ D_k^{(j)}f_j\big)\,d\mu\\&=\sum_{ij}\int\limits_{\Omega}\big(G(w)X_if_i,\ X_jf_j\big)\,d\mu=\Big\|\sum_i X_if_i\Big\|^2_{\mathscr{L}^2(G\mu,H)}.\end{aligned}$$

Let us define the map T by the equality:

$$T\Big(\sum_i\Pi(X_i)Vf_i\Big)=\sum_i X_if_i.$$

The above equality implies that T is isometric map from K to $\mathscr{L}^2(\mu G, H)$. Since simple functions $\sum_i X_if_i$ are dense in $\mathscr{L}^2(\mu G, H)$ so T extends to a unitary operator from K on $\mathscr{L}^2(\mu G, H)$.

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