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Hellinger–Hahn type decompositions of the domain of a Borel function

by

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Abstract. In this paper we give decompositions of the domain of a Borel function f from a complete separable metric space X (of cardinality $\mathfrak{c}$) into another complete separable metric space Y where X and Y are equipped with their usual Borel σ-algebras and X is further equipped with a finite non-atomic measure μ. These decompositions depend on a deep theorem of Lusin which says that if f is "countable to one" then X can be partitioned into countable number of Borel sets on each of which f is one-one. We also give a decomposition of X when f is not "countable to one".

Introduction. In this paper we give decompositions of the domain of a Borel function f from a complete separable metric space X (of cardinality $\mathfrak{c}$) into another complete separable metric space Y, where X and Y are equipped with their usual Borel σ-algebras and X is further equipped with finite non-atomic measure μ. These decompositions, which are given in Theorem 2.2 and 2.3. depend on a deep theorem of Lusin (Theorem 2.1) and the method used in the proof of Hellinger–Hahn theorem for spectral measures on a separable Hilbert space.

When f is a bounded complex valued Borel function, our decompositions of X completely describe the measures and their multiplicities that occur in the Hellinger–Hahn canonical representation of the spectral measure of the normal operator T_f on $L_2(X, \mu)$ consisting of multiplication by f. In Section 4 we indicate how this is so and also give some applications.

1.

Definition 1.1. Let m be a cardinal number. A function f defined on a subset of X into Y is said to be m to 1 if the inverse image of every singleton is of cardinality m. It is said to be *countable to one* if the inverse image of every singleton is either of finite cardinality or of cardinality $\aleph_0$.

Proposition 1.1. *Let f be a Borel function on X with values in Y. Then X can be partitioned into two Borel sets C and D such that*

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1) $f|_C$, *the restriction of* f *to* C, *is countable to one,*

2) $f|_D$ *is countable to one on no subset of positive* μ*-measure in* D.

Proof. Let $\mathfrak{F} = \{A\colon \mu(A) > 0, f|_A \text{ is countable to one}\}$. If $\mathfrak{F}$ is empty we take $D = X$. If $\mathfrak{F}$ is not empty let $a = \sup_{A\in\mathfrak{F}} \mu(A)$ and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of Borel sets such that $\mu(A_n) \to a$. We take $C = \bigcup_{n=1}^{\infty} A_n$. Then clearly $f|_C$ is countable to one and $f|_{X-C}$ is countable to one on no subset of positive measure. Take $D = X - C$. ■

2. In this section we give two forms of Hellinger–Hahn type decompositions of the domain of a countable to one Borel function. First of all we need.

THEOREM 2.1 (LUSIN). *Let* f *be a Borel function on* X *with values in* Y *such that inverse image of every singleton is countable. Then* X *can be decomposed into pairwise disjoint Borel sets* $A_1, A_2, A_3, \ldots$ *such that* $f|_{A_k}$ *is one-one for each* k.

For a proof of this we refer to ([2], p. 234).

LEMMA 2.1. *Let* f *be a Borel function on* X *with values in* Y *such that inverse image of every singleton is countable. Then* X *can be decomposed into pairwise disjoint Borels sets* $N, A_1, A_2, A_3, \ldots$ *such that*

1) $\mu(N) = 0$,

2) $\mu(A_i) > 0$ *for each* i,

3) $f|_{A_i}$ *is one-one for each* i.

DEFINITION 2.1. A Borel function of f on X into Y is said to be *uniformly* m *to one* if X can be partitioned into m Borel sets such that

1) f is one-one on each member of the partition,

2) measures induced by the restriction f to these sets in the partition are mutually absolutely continuous. f is said to be *essentially* m *to one* if f is uniformly m to one after removal of a μ-null Borel set.

In the above definition we require $m \leqslant \aleph_0$. We note that if f is essentially m to one and also essentially n to one then $m = n$.

DEFINITION 2.2. Let f be a Borel function on a Borel subset of X with values in Y. We say that the domain of f has *Hellinger–Hahn decomposition of first kind* if it can be decomposed into pair-wise disjoint Borel sets $N, C_1, C_2, C_3, \ldots$ (this sequence may be finite) such that

1) $\mu(N) = 0$,

2) $\mu(C_i) > 0$ for each i,

3) $f|_{C_i}$ is one-one for each i,

4) measure induced by $f|_{C_{i+1}}$ is absolutely continuous with respect to the one induced by $f|_{C_i}$.

THEOREM 2.2. *Let* f *be a countable to one Borel function on* X *with values in* Y. *Then* X *has a Hellinger–Hahn decomposition* $N, C_1, C_2, C_3, \ldots$ *of first kind. If* $N', C_1', C_2', C_3', \ldots$ *be another such decomposition then the measures induced by* $f|_{C_i}$ *and* $f|_{C_i'}$ *are mutually absolutely continuous.*

Proof. Let $N, A_1, A_2, A_3 \ldots$ be a sequence of Borel sets satisfying conditions of Lemma 2.1. Let A_2 be partitioned into Borel sets A_{21} and A_{22} such that measures induced by $f|_{A_{21}}$ and $f|_{A_{22}}$ are respectively singular and absolutely continuous with respect to the one induced by $f|_{A_1}$. Now the facts that 1) $f|_{A_1}$ and $f|_{A_{21}}$ are one-one Borel functions 2) measures induced by them are mutually singular together imply that f is essentially one-one on $A_1 \cup A_{21}$. For $n \geqslant 3$, let A_n be partitioned into Borel sets A_{n1} and A_{n2} such that measures induced by $f|_{A_{n1}}$ and $f|_{A_{n2}}$ are singular and absolutely continuous with respect to the one induced by $f|_{A_1\cup A_{21}\cup\ldots\cup A_{n-1,1}}$. The function f is essentially one-one on $A_1 \cup A_{21} \cup \ldots \cup A_{n1}$ and the measures induced by $f|_{A_{p2}}$, $2 \leqslant p \leqslant n$ are absolutely continuous with respect to the one induced by $f|_{A_1\cup A_{21}\cup\ldots\cup A_{n1}}$. Let $D_1 = A_1 \cup A_{21} \cup \ldots \cup A_{n1} \ldots$ and let D_2 be obtained from $A_{22}, A_{32}, A_{42}, \ldots$ by the above procedure leaving residual subsets $A_{33}, A_{43}, A_{53}, \ldots$ of $A_{32}, A_{42}, A_{52} \ldots$ respectively. Proceeding thus we get a sequence $D_1, D_2, D_3, \ldots$ of pair-wise disjoint Borel sets such that for each k

1) f is essentially one-one on D_k,

2) $A_k \subset D_1 \cup D_2 \cup \ldots \cup D_k$,

3) measure induced by $f|_{D_{k+1}}$ is absolutely continuous with respect to the one induced by $f|_{D_k}$.

Let C_k be obtained from D_k by removing a μ-null Borel set N_k so that f is one-one on C_k. Let N_k's be absorbed in N and the resulting set still denoted by N. The sets $N, C_1, C_2, C_3, \ldots$ then form a Hellinger–Hahn decomposition of X of first kind.

Let $N', C_1', C_2', \ldots$ be another Hellinger–Hahn decomposition of X of first kind. Measures induced by $f|_{C_1}$ and $f|_{C_1'}$ are mutually absolutely continuous since they are equivalent (in the sense of mutual absolute continuity) to the measure induced by f itself. Now assume, to use induction, that measures induced by $f|_{C_i}$ and $f|_{C_i'}$ are mutually absolutely continuous for $1 \leqslant i \leqslant n-1$. We show that $f|_{C_n}$ and $f|_{C_n'}$ induce equivalent measures. Suppose they do not. We may suppose then that there is a Borel set $B \subset C_n'$ of positive μ-measure such that $f(B)$ has $f|_{C_n}$ induced measure zero. Then the set $f^{-1}(f(B))$ is equal to $\bigcup_{i=1}^{n-1} f|_{C_i}^{-1}(f(B))$ upto a μ-null set and the restriction of f to this set is essentially $(n-1)$ to one. But since $\mu(B) > 0$ and $B \subset C_n$ the restriction of f to $f^{-1}(f(B))$ is not essentially

$(n-1)$ to one. This is a contradiction. Hence $f|_{C_n}$ and $f|_{C'_n}$ induce mutually absolutely continuous measures. ■

DEFINITION 2.3. Let f be a Borel function on X with values in Y. We say that X has *Hellinger–Hahn decomposition of 2nd kind* if X can be decomposed into pair-wise disjoint Borel sets $\eta, \gamma_\infty, \gamma_1, \gamma_2, \gamma_3, \ldots$ (this sequence may be finite) such that

1) $\mu(\eta) = 0$,

2) $f(\gamma_i) \cap f(\gamma_j) = \emptyset$ if $i \neq j$,

3) $f|_{\gamma_k}$ is uniformly k to 1 for each $k < \infty$,

4) $f|_{\gamma_\infty}$ is uniformly $\aleph_0$ to one.

THEOREM 2.3. *Let f be a countable to one Borel function on X with values in Y. Then X has Hellinger–Hahn decomposition $\eta, \gamma_\infty, \gamma_1, \gamma_2, \ldots$ of 2nd kind. If $\eta', \gamma'_\infty, \gamma'_1, \gamma'_2, \ldots$ be another such decomposition, then for each k, $\mu(\gamma_k \triangle \gamma'_k) = 0$.*

Proof. Let $N, C_1, C_2, C_3, \ldots$ be a first kind Hellinger-Hahn decomposition of X as in Theorem 2.2. Let

$$\begin{aligned} C_1 &= \gamma_{11} \cup \gamma_{12} \cup \ldots \cup \gamma_{1\infty}, \\ C_2 &= \qquad \gamma_{22} \cup \ldots \cup \gamma_{2\infty}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_n &= \qquad\qquad \gamma_{nn} \cup \gamma_{n,n+1} \cup \ldots \cup \gamma_{n\infty}, \end{aligned}$$

where γ_{kj}, $j = k, k+1, k+2, \ldots$ are pair-wise disjoint Borel subsets of C_k such that measure induced $f|_{\gamma_{kj}}$ is absolutely continuous with respect to the one induced by $f|_{C_j}$ but singular with respect to one induced by $f|_{C_{j+1}}$. Further $\gamma_{k\infty}$ is disjoint from γ_{kj} and measure induced by $f|_{\gamma_{k\infty}}$ is absolutely continuous with respect to the one induced by $f|_{C_j}$ for each j. Now take $\eta = N$ and

$$\begin{aligned} \gamma_1 &= \gamma_{11} \\ \gamma_2 &= \gamma_{12} \cup \gamma_{22} \\ &\ldots\ldots\ldots\ldots \\ \gamma_n &= \gamma_{1n} \cup \gamma_{2n} \cup \ldots \cup \gamma_{nn} \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

We note that $\mu(\eta) = 0$ and $\gamma_1, \gamma_2, \gamma_3 \ldots \gamma_\infty$ are pair-wise disjoint Borel sets such that $f|_{\gamma_k}$ is uniformly k to one, $\gamma_k = \gamma_{1k} \cup \gamma_{2k} \cup \ldots \cup \gamma_{kk}$ being a decomposition of γ_k such that $f|_{\gamma_{ik}}$ is one-one and measures induced by $f|_{\gamma_{ik}}$ is equivalent to the part of the measure induced by $f|_{C_k}$ which is singular to the one induced by $f|_{C_{k+1}}$. After removing a μ-null Borel set if necessary we can choose $\gamma_1, \gamma_2, \ldots, \gamma_\infty$ such that their images are disjoint.

Now suppose that $\eta', \gamma'_1, \gamma'_2, \ldots, \gamma'_\infty$ is another Hellinger–Hahn decomposition of X of 2nd kind. Suppose $\mu(\gamma_1 \triangle \gamma'_1) \neq 0$. Then either γ_1 or γ'_1 has a subset of positive measure which does not intersect the other. Suppose there is a set $E \subset \gamma_1$ of positive measure which does not intersect γ'_1. Now the multiple valued function f^{-1} is one-one on $f(E)$ since $E \subset \gamma_1$ and it is not one-one on $f(E)$ since $E \cap \gamma'_1 = \emptyset$. This is a contradiction. Hence $\mu(\gamma_1 \triangle \gamma'_1) = 0$. An inductive argument shows that for each n, $\mu(\gamma_n \triangle \gamma'_n) = 0$. ■

Let Z be another complete separable metric space of cardinality c and let ν be a non-atomic finite measure on Z.

DEFINITION 2.4. Two Borel functions $f\colon X \to Y$ and $\varphi\colon Z \to Y$ are said to be *equivalent* if there exists a Borel isomorphism $\tau\colon X \to Z$, such that

1) the measures $\mu\tau$ and ν are equivalent,

2) $f = \varphi(\tau)$ a.e. μ.

DEFINITION 2.5. Assume further that f and φ are countable to one and let $N, C_1, C_2, C_3, \ldots$ and $M, D_1, D_2, D_3, \ldots$ be the respective Hellinger–Hahn decomposition of first kind. We say that these decompositions are *equivalent* if for each k the measures induced by $f|_{C_k}$ and $\varphi|_{D_k}$ are equivalent.

An elementary argument yields:

THEOREM 2.5. *Let f and φ be countable to one Borel functions on X and Z respectively with values in Y. Then f and φ are equivalent if and only if the corresponding Hellinger–Hahn decompositions of first kind are equivalent.*

3. In this section we consider Borel functions which are essentially uncountable to one, which we define to mean functions which are one-one on no subset of positive measure. In view of Theorem 2.1 it is clear that a Borel function is essentially uncountable to one if and only if it is countable to one on no subset of positive measure. We prove

THEOREM 3.1. *Let f be a Borel function on X with values in Y such that it is essentially uncountable to one. Then there exists a Borel set $A \subset X$ such that $f|_A$ and $f|_{X-A}$ induce equivalent measures.*

For a proof of this theorem we need following two lemmas:

LEMMA 3.1. *Let f be a function on a set E with range $f(E) = F$. Let $\pi_n = \{A_{n,1}, \ldots, A_{n,k_n}\}$ be a sequence of partitions of E such that*

1) *atoms of $\bigcup_{n=1}^{\infty} \pi_n$ are singletons,*

2) *for each n, $\{f(A_{n,1}), \ldots, f(A_{n,k_n})\}$ are pair-wise disjoint.*

Then f is one-one on E.

Proof. Let $A_1, A_2, A_3, \ldots$ be an enumeration of $\bigcup_{n=1}^{\infty} \pi_n$. Let $A_k^0 = A_k$ and $A_k^1 = E - A_k$. Then because of 1) for any given $x \in E$ there exists a sequence $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ of zeros and ones such that

$$x = A_1^{\varepsilon_1} \cap A_2^{\varepsilon_2} \cap \ldots \cap A_n^{\varepsilon_n} \cap \ldots$$

Then

$$f(x) = f(A_1^{\varepsilon_1} \cap A_2^{\varepsilon_2} \cap \ldots) \subseteq \bigcap_{i=1}^{\infty} f(A_i^{\varepsilon_i}) = \bigcap_{i=1}^{\infty} \big(f(A_i)\big)^{\varepsilon_i}$$

where the last equality is true because of 2). Thus $f(x)$ belongs to an atom generated by $f(A_1), f(A_2), \ldots$ It follows that f is one-one. ∎

LEMMA 3.2. *A Borel function on a Borel subset of X is essentially one-one if and only if the measures induced by restrictions of f to disjoint Borel subsets are mutually singular.*

Proof. That the condition is necessary is obvious. To prove that it is sufficient let $\pi_n = \{A_{n1}, \ldots, A_{n,k_n}\}$ be a sequence of partitions into Borel sets of the Domain D of f such that $\bigcup_{n=1}^{\infty} \pi_n$ generates the σ-algebra of D. Since the restriction of f to disjoint Borel sets induce mutually singular measures, for each n there exists a null Borel set N_n such that images under f of $A_{n,1} - N_n, \ldots, A_{n,k_n} - N_n$ are pair-wise disjoint. Applying Lemma 3.1 to $D - \bigcup_{n=1}^{\infty} N_n$ with $\pi_n = \{A_{n,1} - N, \ldots, A_{n,k_n} - N\}$ we see that f is one-one in $D - N$. ∎

A consequence of above lemma is that if f is one-one on no subset of positive measure, then given any set A of positive measure, there exist disjoint Borel sets C and D in A of positive measure such that $f|_C$ and $f|_D$ induce mutually absolutely continuous measures. This remark is used in the

Proof of Theorem 3.1. Let $\mathfrak{F}$ denote the collection of pairs (A, B) of disjoint Borel subsets of X of positive measure such that $f|_A$ and $f|_B$ induce mutually absolutely continuous measures. Partially order $\mathfrak{F}$ by writing $(A, B) > (C, D)$ if $\mu(A - C) = \mu(B - D) = 0$. We show that every chain in $\mathfrak{F}$ has an upper bound. Let $(A_a, B_a)\, a \in I$ be a chain in $\mathfrak{F}$. Let $a = \sup_{a \in I} \mu(A_a)$ and $b = \sup_{a \in I} \mu(B_a)$. Let $(a_n)_{n=1}^{\infty}$ be a sequence of indices such that $\mu(A_{a_n}) \to a$ and $(B_{a_n}) \to b$. Then the pair (A, B) where $A = \bigcup_{n=1}^{\infty} A_{a_n} - \bigcup_{n=1}^{\infty} B_{a_n}$ and $B = \bigcup_{n=1}^{\infty} B_{a_n} - \bigcup_{n=1}^{\infty} A_{a_n}$ is an upper bound of the chain (A_a, B_a), $a \in I$. Hence by Zorn's lemma there exists a maximal element (E, F) in $\mathfrak{F}$. By our remark $X - E \cup F$ must have μ-measure zero. Now take $A = E - (X - E \cup F)$. Then $f|_A$ and $f|_{X-A}$ induce measures on Y which are mutually absolutely continuous.

Remark. We can choose A such that $\mu(A) = \mu(X)\cdot\frac{1}{2}$.

EXAMPLE 3.1. Let $X = R^2$ and let μ be any measure absolutely continuous with respect to the plane Lebesgue measure on X. Let $f(x, y) = x$. Then Fubini theorem shows that f is essentially uncountable to one. Theorem 3.1 shows that for any given n we can decompose X into n disjoint Borel sets $B_1, B_2, \ldots, B_n$ such that measures induced on R by $f|_{B_k}$, $k = 1, \ldots, n$ are mutually absolutely continuous.

4. Let f be a bounded complex valued Borel function on X and let T_f denote the normal operator on $L_2(X, \mu)$ consisting of multiplication by f. The objective of this section is to describe the spectral measure associated with T_f. For this purpose we first of all recall some of the relevant results about spectral measures ([1], [3], [4]).

Let $\mathfrak{H}$ be a non-trivial separable Hilbert space. Let $\boldsymbol{C}$ denote the complex plane and $\mathfrak{B}$ its Borel σ-algebra. By a spectral measure E we shall mean a countably additive function on $\mathfrak{B}$, the values of E being orthogonal projections in $\mathfrak{H}$ and $E(\boldsymbol{C})$ being equal to the identity map of $\mathfrak{H}$. For any $x \in \mathfrak{H}$ we have a non-negative measure μ_x defined by $\mu_x(\sigma) = \big(E(\sigma)x, x\big)$, $\sigma \in \mathfrak{B}$. If $\mathfrak{H}_x$ denotes the subspace spanned by $\{E(\sigma)x\colon \sigma \in \mathfrak{B}\}$ then $\mathfrak{H}_x$ is invariant under E and the mapping $S_x\colon E(\sigma)x = 1_\sigma$ extends to an invertible isometry from $\mathfrak{H}_x$ onto $L_2(\boldsymbol{C}, \mu_x)$ in such a manner that $S_x E S_x^{-1} = F_x$, where F_x is the spectral measure on $L_2(\boldsymbol{C}, \mu_x)$ defined by $F_x(\sigma) =$ multiplication by 1_σ(1_σ = characteristic function of σ).

For any non-negative finite measure λ on $\mathfrak{B}$ and any cardinal number $n \leqslant \aleph_0$ we shall denote by $nL_2(\boldsymbol{C}, \lambda)$ the Hilbert space which is the direct sum of n copies of $L_2(\boldsymbol{C}, \lambda)$. If $\bar{f} = (f_1, f_2, f_3, \ldots)$ (this sequence is finite if $n < \aleph_0$) be an element of $nL_2(\boldsymbol{C}, \lambda)$, we write

$$F_n(\sigma)\bar{f} = (1_\sigma f_1, 1_\sigma f_2, \ldots), \quad \sigma \in \mathfrak{B}.$$

Then F_n is a spectral measure on $\mathfrak{B}$.

We now state Hellinger–Hahn theorem for spectral measures in two different forms.

HELLINGER–HAHN THEOREM (first-form). *Let E be a spectral measure. Then $\exists$ finite measures $\lambda_1, \lambda_2, \lambda_3, \ldots$ on $\mathfrak{B}$ and an invertible isometry S from $\mathfrak{H}$ onto the direct sum $\sum_{n=1}^{\infty} L_2(\boldsymbol{C}, \lambda_n)$ such that*

(i) *for each n, λ_{n+1} is absolutely continuous with respect to λ_n*

(ii) *SES^{-1} = multiplication by characteristic function, i.e. if $\bar{f} = (f_1, f_2, f_3, \ldots)$ belongs to $\sum_{n=1}^{\infty} L_2(\boldsymbol{C}, \lambda_n)$, then*

$$SE(\sigma)S^{-1}\bar{f} = (1_\sigma f_1, 1_\sigma f_2, \ldots), \ \forall \sigma \in \mathfrak{B}.$$

If $\lambda_1', \lambda_2', \lambda_3', \ldots$ be another sequence of finite measures for which $\exists$ an invertible isometry S' from $\mathfrak{H}$ onto $\sum_{n=1}^{\infty} L_2(C, \lambda_n')$ such that (i) *and* (ii) *are satisfied with respect to $\lambda_1', \lambda_2', \lambda_3', \ldots$ and S', then for each n, λ_n and λ_n' are mutually absolutely continuous.*

For a proof of above theorem we refer to [(4], Chapter VII).

HELLINGER–HAHN THEOREM (Second form). *Let E be a spectral measure. Then $\exists$ mutually singular finite measures $\lambda_\infty, \lambda_1, \lambda_2, \ldots$ on $\mathfrak{B}$ and an invertible isometry S from $\mathfrak{H}$ onto the direct sum of n $L_2(C, \lambda_n)$, $n = \infty, 1, 2, 3, \ldots$ such that $SES^{-1} = F$ = multiplication by characteristic function, i.e., for each n, the restriction of F to $nL_2(C, \lambda_n)$ is F_n. Further if $\lambda_\infty', \lambda_1', \lambda_2', \ldots$ be another sequence of mutually singular measures for which $\exists$ an isometry S' from $\mathfrak{H}$ onto the direct sum of n $L_2(C, \lambda_n')$, $n = \infty, 1, 2, 3, \ldots$ such that $S'ES'^{-1} = F'$ = multiplication by characteristic function, then for each n, λ_n and λ_n' are mutually absolutely continuous.*

A proof of above theorem can be obtained by specializing the results of ([1], Chapter III) to the case of separable Hilbert space.

For any finite measure λ on $\mathfrak{B}$ we shall write $\bar{\lambda}$ to denote the class of σ-finite measures on $\mathfrak{B}$ which are mutually absolutely continuous with respect to λ and call $\bar{\lambda}$ the measure class of λ. It follows from second form of Hellinger–Hahn theorem that any spectral measure E uniquely determines mutually singular measure classes $\bar{\lambda}_\infty, \bar{\lambda}_1, \bar{\lambda}_2, \ldots$ so that $\lambda_\infty, \lambda_1, \lambda_2, \ldots$ satisfy the conditions of that theorem.

DEFINITION 4.1. We say that E has *uniform multiplicity n with associated measure class $\bar{\lambda}_n$* if in the Hellinger–Hahn theorem of second form $\bar{\lambda}_k = \bar{0}$ for $k \neq n$.

Remark. It can be shown using Radon–Nikodym derivatives that E has uniform multiplicity n if and only if in the Hellinger–Hahn theorem of first form $\lambda_1, \lambda_2, \ldots, \lambda_n$ belong to same measure class and λ_{n+1} is the zero measure.

Returning to Hellinger–Hahn theorem in its second form, if S be the isometry of that theorem, then the subspaces $\mathfrak{H}_n = S^{-1}(L_2(C, \lambda_n))$ are such that $\mathfrak{H}$ is their direct sum, $E\mathfrak{H}_n = \mathfrak{H}_n$, the restriction of E to $\mathfrak{H}_n$ has uniform multiplicity n with associated measure class $\bar{\lambda}_n$. If $x \in \mathfrak{H}$ then μ_x is always absolutely continuous with respect to $\lambda_\infty + \lambda_1 + \lambda_2 + \ldots$ (this is a σ-finite measure). Further $x \in \mathfrak{H}_n$ if and only if μ_x is absolutely continuous with respect to λ_n, and indeed $\mathfrak{H}_n = \{x: \mu_x$ is absolutely continuous with respect to $\lambda_n\}$. From this it follows that $\mathfrak{H}_n$ does not depend on choice of S or $\lambda_\infty, \lambda_1, \lambda_2, \ldots$

LEMMA 4.1. *Let λ be a finite positive measure on $\mathfrak{B}$ and let $\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_m \in n L_2(C, \lambda)$ be such that*

(i) $\forall \sigma \in \mathfrak{B}$, $1_\sigma \bar{f}_i \perp \bar{f}_j$ *if* $i \neq j$,

(ii) *the measures ν_i, $\nu_i(\sigma) = \|1_\sigma \bar{f}_i\|^2$, belongs to the measure class $\bar{\lambda}$. Then $m \leqslant n$.*

Proof. The conditions of the lemma imply that for a.e. x with respect to λ the vectors $\bar{f}_1(x), \bar{f}_2(x), \ldots, \bar{f}_m(x)$ are non-zero and they are orthogonal. Hence it is clear that $m \leqslant n$.

LEMMA 4.2. *Let E be a spectral measure of uniform multiplicity n with associated measure class $\bar{\lambda} \neq \bar{0}$. Let $x_1, x_2, \ldots, x_m \in \mathfrak{H}$ be such that*

(i) $\forall \sigma$, $E(\sigma)x_i \perp x_j$ if $i \neq j$

(ii) *the measures $\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_m}$ are all of measure class $\bar{\lambda}$. Then $m \leqslant n$.*

This lemma follows from Lemma 4.1 by making use of the isometry S.

LEMMA 4.3. *Let E be a spectral measure with associated measure classes $\bar{\lambda}_\infty, \bar{\lambda}_1, \bar{\lambda}_2, \ldots$ according to the second form of Hellinger–Hahn theorem. Suppose that for every positive integer n, $\exists$ $x_1, x_2, \ldots, x_n \in \mathfrak{H}$ such that*

(i) $\forall \sigma \in \mathfrak{B}$, $E(\sigma)x_i \perp x_j$ *if* $i \neq j$,

(ii) *for each i, the measure class μ_{x_i} belongs to the measure class of $\lambda_\infty + \lambda_1 + \lambda_2 + \lambda_3 \ldots$ Then $\lambda_k = 0$ for $k < \infty$.*

Proof. Suppose λ_k is non-zero for some $k < \infty$. Let $x_1, x_2, \ldots, x_{k+1}$ be elements in $\mathfrak{H}$ satisfying conditions (i) and (ii) of the lemma. Let $y_i = P_{\mathfrak{H}_k} x_i$, where $P_{\mathfrak{H}_k}$ denotes the orthogonal projection on $\mathfrak{H}_k$. Let E_k denote the restriction of E to $\mathfrak{H}_k$. Then we have

(a) E_k has uniform multiplicity k with associated measure class $\bar{\lambda}_k$

(b) $\forall \sigma \in \mathfrak{B}$, $E_k(\sigma)y_i \perp y_j$ if $i \neq j$,

(c) for each i, the measures ν_i defined by $\nu_i(\sigma) = (E_k(\sigma)y_i, y_i)$ belong to the measure class $\bar{\lambda}_k$. By Lemma 4.2 this is impossible. Hence $\lambda_k = 0$. ■

Now let A be a bounded normal operator on $\mathfrak{H}$. Then by the spectral theorem ([1], p. 71] $\exists$ a spectral measure E supported on the spectrum of A such that

$$A = \int_C z E(dz).$$

If p is any polynomial in z and $\bar{z}$, then

$$p(A, A^*) = \int_C p(z, \bar{z}) E(dz)$$

Further for any $x \in \mathfrak{H}$, the subspace spanned by $\{A^m x, A^{*m} x\}$, $m = 0, 1, 2, 3, \ldots$ is the same as the subspace spanned by $\{E(\sigma)x: \sigma \in \mathfrak{B}\}$. Thus $E(\sigma)x \perp y \forall \sigma$ if and only if $\forall m$, $A^m x \perp y$, $A^{*m} x \perp y$. Now take $\mathfrak{H} = L_2(X, \mu)$ and $A = T_f$. For any $\varphi \in L_2(X, \mu)$ write $\mu^\varphi(B) = \int_B |\varphi|^2(u) \mu(du)$ and $\lambda_\varphi = \mu^\varphi f^{-1}$, i.e., λ_φ is the measure on $\mathfrak{B}$ defined by $\lambda_\varphi(\sigma) = \mu^\varphi(f^{-1}(\sigma))$, $\sigma \in \mathfrak{B}$. Let λ be the measure μf^{-1}. It is easy to see that λ_φ is aways absolutely continuous with respect to λ.

Let E denote the spectral measure of T_f. Then $\forall m \geqslant 0$

$$(T_f^m\varphi, \varphi) = \int_X f^m(u)|\varphi(u)|^2\mu(du) = \int_C \bar{z}^m\lambda_\varphi(dz) = \int_C \bar{z}^m(E(dz)\varphi, \varphi),$$

$$(T_f^{*m}\varphi, \varphi) = \int_X \bar{f}^m(u)|\varphi(u)|^2\mu(du) = \int_C z^m\lambda_\varphi(dz) = \int_C \bar{z}^m(E(dz)\varphi, \varphi),$$

$$(T_f^m 1, 1) = \int_C z^m\lambda(dz) = \int_C z^m(E(dz)1, 1),$$

$$(T_f^{*m} 1, 1) = \int_C \bar{z}^m\lambda(dz) = \int_C \bar{z}^m(E(dz)1, 1).$$

The second equality in the first two equations follows from transformation of variable formula. It is clear from these formulas that $\lambda = \mu f^{-1}$ belongs to the measure class of $\lambda_\infty + \lambda_1 + \lambda_2 + \lambda_3 \ldots$ where $\bar{\lambda}_\infty, \bar{\lambda}_1, \bar{\lambda}_2, \ldots$ are the measure classes associated with E according to Hellinger–Hahn theorem in its second form.

THEOREM 4.1. *If f is essentially uncountable to one then T_f has uniform multiplicity $\aleph_0$ with associated measure class $\bar{\lambda}$ where $\lambda = \mu f^{-1}$.*

Proof. Let n be any positive integer. Let $A_1, A_2, \ldots, A_n$ be a decomposition of X in the fashion of Theorem 3.1, i.e., measures induced by restriction of f to A_i are all mutually absolutely continuous and indeed belong to the measure class $\bar{\lambda}$. Let $x_1 = 1_{A_1}, \ldots, x_n = 1_{A_n}$. Then it is clear that for every integer $m \geqslant 0$ $T_f^m x_i$, $T_f^{*m}x_i$ are both orthogonal to x_j if $j \neq i$, hence for all $\sigma \in \mathfrak{B}$, $E(\sigma)x_i \perp x_j$ whenever $i \neq j$. Next for each i, the measure μ_{x_i}: $\mu_{x_i}(\sigma) = (E(\sigma)x_i, x_i)$ is the measure induced by the restriction of f to A_i, i.e., $\mu_{x_i} = \mu f|_{A_i}^{-1}$ and belongs to the measure class $\bar{\lambda}$. Hence by Lemma 4.3 T_f has uniform multiplicity $\aleph_0$ with associated measure λ. ■

Now let us assume that f is countable to one. Let $\gamma_\infty, \gamma_1, \gamma_2, \ldots$ be a Hellinger–Hahn decomposition of X of second kind and let λ_n denote the measure induced by restriction of f to γ_n. Then $\bar{\lambda}_\infty, \bar{\lambda}_1, \bar{\lambda}_2, \ldots$ is the sequence of mutually singular measure classes associated with the spectral measure E of T_f according to second form of Hellinger–Hahn theorem. To see this it is enough to note that

(i) $L_2(X, \mu) = L_2(X, \mu|_{\gamma_\infty}) + L_2(X, \mu|_{\gamma_1}) \oplus \ldots$

(ii) $L_2(X, \mu|_{\gamma_n}) = \sum_{i=1}^{n} L_2(X, \mu|_{\gamma_{ni}})$, where $\gamma_n = \bigcup_{i=1}^{n} \gamma_{ni}, \gamma_{ni} \cap \gamma_{nj} = \emptyset$ if $i \neq j$. $f|\gamma_{ni}$ is one-one for every i and measures induced by $f|_{\gamma_{ni}}$ are mutually absolutely continuous, $1 \leqslant i \leqslant n$ [1].

(iii) If λ_{ni} be the measure induced by the restriction of f to γ_{ni}, then, since f is one-one on γ_{ni}, the mapping S_{ni}: $S_{ni}g = g \circ f|_{\gamma_{ni}}^{-1}$ is an invertible isometry from $L_2(X, \mu|_{\gamma_{ni}})$ and $L_2(C, \lambda_{ni})$ such that for any $h \in L_2(C, \lambda_{ni})$

$$(S_{ni}T_fS_{ni}^{-1}h)(z) = zh(z)$$

from which it follows that $\forall \sigma \in \mathfrak{B}$

$$S_{ni}E(\sigma)S_{ni}^{-1}h = 1_\sigma h.$$

(iv) if $\mathfrak{M}_n = \sum_{i=1}^{n} L_2(C, \lambda_{ni})$

$$S_n = \sum_{i=1}^{n} S_{ni}$$

then $S = S_\infty + S_1 + S_2 + \ldots$ is an invertible isometry from $L_2(X, \mu)$ onto $\mathfrak{M}_\infty \oplus \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \ldots$ such that SES^{-1} acts on each $L_2(\bar{C}, \lambda_{ni})$ in the manner given in (iii) above.

(v) The conclusion follows from the remark following definition 4.1.

We continue with the assumption that f is essentially countable to one and show how the first kind Hellinger–Hahn decomposition of X yields the measure classes associated with the spectral measure of T_f according to the Hellinger–Hahn theorem in its first form. Let $C_1, C_2, C_3, \ldots$ be as in Theorem 2.2 and let, for each k, λ_k be the measure induced by restriction of f to C_k. Let $\mathfrak{K}$ denote the direct sum of Hilbert spaces $L_2(C, \lambda_k)$. For any Borel set $\sigma \in \mathfrak{B}$, let $F(\sigma)$ denote the projection operator $F(\sigma)(g_1, g_2, g_3, \ldots) = (1_\sigma g_1, 1_\sigma g_2, 1_\sigma g_3, \ldots)$ where $(g_1, g_2, g_3, \ldots) \in \mathfrak{K}$. Then F is unitarily equivalent to the spectral measure E of T_f. To see this it is enough to note that:

(i) if S_k denotes the invertible isometry between $L_2(X, \mu|_{C_k})$ and $L_2(C, \lambda_k)$ given by

$$S_kh = h \circ f|_{C_k}^{-1}, \quad h \in L_2(X, \mu|_{C_k}),$$

then

$$(S_kT_fS_k^{-1}\varphi)(z) = z\varphi(z), \quad \varphi \in L_2(C, \lambda_k)$$

from which it follows that for any $\varphi \in L_2(C, \lambda_k)$

$$S_kE(\sigma)S_k^{-1}\varphi = 1_\sigma\varphi, \quad \forall \sigma \in \mathfrak{B}.$$

(ii) $L_2(X, \mu) = \sum_{k=1}^{\infty} L_2(X, \mu|_{C_k})$.

Since λ_{k+1} is absolutely continuous with respect to λ_k, it is clear that $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \ldots$ give the measure classes associated with E according to Hellinger–Hahn theorem in its first form.

The following theorem is an easy consequence of the foregoing

THEOREM 4.2. *Let f be a bounded complex valued Borel function on X. Then the spectral measure of T_f is of uniform multiplicity $n < \infty$ if and only if f is essentially n to one. In particular it is of multiplicity one if and only if f is essentially one-one.*

For the next theorem whose proof is left to the reader, keep in view Definitions 2.4 and 2.5.

[1] Note that γ_{ni} here is γ_{in} in Theorem 2.3.

THEOREM 4.3. *Let f and φ be essentially countable to one bounded complex valued Borel function on X and Z respectively. Then T_f and T_φ are unitarily equivalent if and only if the corresponding first kind Hellinger–Hahn decompositions of X and Z are equivalent, i.e., if and only if f and φ are equivalent.*

Remark. It can happen that f is essentially uncountable to one, φ is countable to one and T_f and T_φ are unitarily equivalent. Indeed any bounded normal operator on a separable Hilbert space is unitarily equivalent to $T_{(\psi)}$ where (ψ) is the function on $I \times C$ (I = Set of positive integer) given by $(\psi)(n, x) = x$, and where a measure on $I \times C$ is determined by the operator in question. Note that (ψ) is always countable to one.

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Linear topologies which are suprema of dual-less topologies*

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Abstract. The first result of this paper is that every topological linear space of algebraic dimension at least the continuum is linearly homeomorphic to a subspace of a dual-less space (i.e., a topological linear space with zero dual) in such a way that the dimension and codimension of the image are equal. Using this result, it is then proved that the norm topology of many of the classical separable Banach spaces can be written as the supremum of a finite number of dual-less topologies. Some extensions of this are given for the non-separable case and for other topological linear spaces.

0. INTRODUCTION

It is well known that the topology of convergence in measure is one of the weakest topologies on a function space; for example, on the space of all Lebesgue measurable functions on $[0, 1]$ the only linear functional which is continuous for convergence in measure is the zero functional. In view of this it may be somewhat surprising that the norm topology on the classical Banach spaces can be expressed as simultaneous convergence in three topologies, each of which is an inverse image of a topology of convergence in measure. This is proved below as a consequence of more general results concerning the following problems:

a) which linear topologies on a vector space are restrictions of "very weak" topologies on a larger space?

b) which linear topologies on a vector space can be expressed as suprema of families of "very weak" topologies on it?

By a "very weak" topology we mean a linear topology that is at least dual-less in the sense that it does not have any non-trivial continuous linear functional. Theorems A, B, C below provide some answers to these problems.

Questions of this sort were investigated by Klee in [5], to which we refer the reader for background. In this paper, Klee proved that the supre-

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