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On commutative approximate identities and cyclic vectors of induced representations

by

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Abstract. It is shown that every locally compact group has a commutative approximate identity for $L_1(G)$ which consists of continuous positive functions which decrease very rapidly at infinity. This is applied to a construction of a cyclic vector for a representation of a locally compact first countable group induced by a cyclic representation.

The aim of this paper is twofold. To show that every locally compact group has a commutative approximate identity for $L_1(G)$ which consists of continuous positive functions which decrease very rapidly at infinity and apply this to a construction of a cyclic vector for a representation of a locally compact first countable group induced by a cyclic representation.

A construction of commutative approximate identity for a C^*-algebra was given by J. F. Aarnes and R. V. Kadison [1]. Their method uses C^*-algebras technique and does not apply to the group algebras. It would be interesting to know whether there exists an approximate identity for $L_1(G)$ consisting of commuting continuous functions with compact support.

The fact that for a first countable group representations induced by cyclic representations are cyclic was first proved by F. Greenleaf and M. Moskowitz [5] and [6] and a construction of a cyclic vector for such representations was claimed by the authors [7]. Unfortunately [7] makes use of a statement in [2], p. 49, which is false, as it has been recently discovered by R. Goodman. The construction presented here avoids this difficulty and (for induced representations) improves the construction given in [7].

Very briefly the idea is the following. For a Lie group G the fundamental solution $u(g, t) = p^t(g)$ of the heat equation is a one-parameter semi-group of non-negative functions p^t, that is $p^s * p^t = p^{s+t}$ for all positive real s, t. Moreover $p^t * f$ tends to f as t tends to zero, and for a fixed t the function p^t decreases faster than exponentially at infinity. In short, functions p^t, $t \in \boldsymbol{R}^+$, form an approximate identity for $L_1(G)$ consisting of commuting rapidly decreasing functions.

On the other hand, the positive-definite measures μ which define representations induced by cyclic representations are of at most exponential growth. Thus functions p^t can be used in the same way as functions e_n were used in [7] to construct a cyclic vector for the representation I^{μ}.

Another method of saving the construction in [7] for Lie groups has been recently developed by R. Goodman [4].

1. The space of very rapidly decreasing functions. Let G be a locally compact group. We say that a non-negative function φ on G is submultiplicative if

$$\varphi(gh) \leqslant \varphi(g)\varphi(h) \quad \text{for all } g, h \text{ in } G.$$

Let Φ denote the set of all submultiplicative continuous functions on G. It is clear that Φ is closed under multiplication and that the modular function δ_G is in $\Phi(G)$.

Let $\{G_i\}_{i\in I}$ be the family of all compactly generated open subgroups of G directed by inclusion. For an i in I we define $\boldsymbol{E}(G_i)$ to be the space of continuous functions f such that $\operatorname{supp} f$ is contained in G_i and for each φ in $\Phi(G)$

$$\sup\{|f(g)\varphi(g)|:\ g\in G\} = \|f\|_\varphi < \infty.$$

The set of pseudo-norms $\|\cdot\|_\varphi$ define a locally convex topology in $\boldsymbol{E}(G_i)$. The space $\boldsymbol{E}(G) = \bigcup_{i\in I} \boldsymbol{E}(G_i)$ is topologiezed as the inductive limit of the spaces $\boldsymbol{E}(G_i)$. It is easy to see that $\boldsymbol{E}(G)$ is complete and that the functions with compact support form a dense subset of $\boldsymbol{E}(G)$. $\boldsymbol{E}(G)$ is invariant under left and right translations and both left and right regular representation of G on $\boldsymbol{E}(G)$ are jointly continuous.

PROPOSITION 1.1. *If G is compactly generated, there is a φ_0 in $\Phi(G)$ such that $\int\varphi_0(g)^{-1}dg < \infty$ and for every φ in $\Phi(G)$ there is a constant M and a positive integer k such that*

$$\varphi(g) \leqslant M\varphi_0(g)^k \quad \textit{for all } g \textit{ in } G. \tag{1.1}$$

Proof. Let U be a precompact neighbourhood of the unit element in G such that $\bigcup_{n=1}^{\infty} U^n = G$ and F a finite set such that $U^2 \subset FU$. Then, of course, $U^n \subset F^{n-1}U$. For a g in G we put

$$\tau(g) = \inf\{n:\ g\in U^n\}$$

and

$$\tau_0(g) = |U|^{-1}\int_U \tau(hg)\,dh.$$

Since τ_0 is left uniformly continuous on G, the function

$$\sigma(g) = \sup\{|\tau_0(gh)-\tau_0(h)|:\ h\in G\}$$

is a non-negative function on G continuous at the unity and

$$\sigma(gh) \leqslant \sigma(g)+\sigma(h),$$
$$\sigma(g) \geqslant \tau_0(g) \geqslant \tau(g)-2.$$

The continuity of σ at a point h in G follows from the inequalities

$$-\sigma(g^{-1}) \leqslant \sigma(gh)-\sigma(h) \leqslant \sigma(g).$$

The function φ_0 is defined as follows

$$\varphi_0(g) = C^{2\sigma(g)}, \quad \text{where } C = \operatorname{card} F. \tag{1.2}$$

We have

$$\int_G \varphi_0^{-1}(g)\,dg = \sum_{n=1}^{\infty}\int_{U^n\setminus U^{n-1}} C^{-2\sigma(g)}dg \leqslant \sum_{n=1}^{\infty}\int_{U^n\setminus U^{n-1}} C^{-2\tau(g)+4}dg$$
$$= \sum_{n=1}^{\infty} C^{-2n+4}|U^n\setminus U^{n-1}| \leqslant C^3|U|\sum_{n=1}^{\infty}C^{-n} < \infty.$$

Now let φ be an arbitrary function in $\Phi(G)$ and let $M_0 = \sup\{\varphi(g):\ g\in U\}$. For an element g in G we have $g = g_1\ldots g_n$ with $g_j\in U$ and $\tau(g) = n$. Hence

$$\varphi(g) \leqslant \varphi(g_1)\ldots\varphi(g_n) \leqslant M_0^n = M_0^{\tau(g)} < C^{2k(\tau(g)+2)} = M\varphi_0(g)^k, \tag{1.3}$$

where $k = \frac{1}{2}\log_C M_0$ and $M = M_0^{4k}$.

COROLLARY 1.2. *$\boldsymbol{E}(G)$ is a *-subalgebra of $L_1(G)$.*

COROLLARY 1.3. *If $f\in\boldsymbol{E}(G)$, then for every φ in $\Phi(G)$ the function $f\varphi$ belongs to $L_1(G)$.*

COROLLARY 1.4. *Suppose G is generated by a relatively compact neighbourhood U of the unit element of G. If a set $\boldsymbol{F}$ in $\boldsymbol{E}(G)$ is bounded, i.e. $\|f\|_\varphi \leqslant C_\varphi$ for every f in $\boldsymbol{F}$ and φ in $\Phi(G)$, then for every φ in $\Phi(G)$ $f(g)\varphi(g)$ tends to zero uniformly on $\boldsymbol{F}$ as g tends to infinity.*

Proof. Take φ in $\Phi(G)$ and $\varepsilon > 0$. Let φ_0 be the function defined in (1.2) by means of U. Since $\boldsymbol{F}$ is bounded, we have

$$|f(g)\varphi(g)\varphi_0(g)| \leqslant C_{\varphi\varphi_0} \quad \text{for all } f \text{ in } \boldsymbol{F}.$$

Hence, by (1.3),

$$|f(g)\varphi(g)| \leqslant C_{\varphi\varphi_0}M_0^{-\tau(g)} = C_{\varphi\varphi_0}M_0^{-n} < \varepsilon,$$

for all $g\notin U^n$ and n large enough.

2. Approximate identity for $\boldsymbol{E}(G)$. In this section we prove the existence of a commutative approximate identity in $\boldsymbol{E}(G)$. For Lie groups this is a consequence of several well-known properties of the fundamental solution of the heat equation. For other locally compact groups we obtain the result by Yamabe approximation theorem.

Let G be a Lie group. We denote by $BC^\infty(G)$ the space of bounded infinitely differentiable functions whose all the derivatives are bounded. Let $\boldsymbol{L}(G)$ be the algebra of linear operators on $BC^\infty(G)$ which commute with right translations on G. The Lie algebra LG of G viewed as right-invariant differential operators of the first order on G is contained in $\boldsymbol{L}(G)$ and so is the envelopping algebra of LG.

Let G' be another Lie group and let

$$\pi\colon G \to G'$$

be a homomorphism of G onto G'. For a function f in $BC^\infty(G')$ we correspond a function $\pi^* f$ in $BC^\infty(G)$ defined by $\pi^* f(g) = f(\pi(g))$. Clearly, since π is onto, π^* is one-to-one and the subspace $\pi^* BC^\infty(G')$ of $BC^\infty(G)$ is stable under $\boldsymbol{L}(G)$.

For a function f on G which is constant on the cosets modulo $\ker\pi$ let $\mathring{f}$ be the unique function on G' such that $\pi^*\mathring{f} = f$. We have the algebra homomorphism

$$\pi_*\colon \boldsymbol{L}(G) \ni T \to \pi_* T \in \boldsymbol{L}(G')$$

defined by

$$(\pi_* T)f = (T\pi^* f)^\circ.$$

Clearly

$$\pi_* LG = LG'$$

and π_* coincides with $d\pi$, if LG and LG' are viewed as tangent spaces at the unit elements of G and G', respectively.

Let $p \in L_1(G)$ and let $T_p f = p * f$, where $f \in BC^\infty(G)$. Then $T_p \in \boldsymbol{L}(G)$ and

$$\pi_* T_p = T^\circ_{p'} \qquad \text{where } p'(g) = \int_{\ker\pi} p(hg)\,dh. \tag{2.1}$$

We say that a metric locally compact group G is a *Yamabe group*, if there exists a sequence $U_1 \supset U_2 \supset \ldots$ of relatively compact neighbourhoods of the unit element in G with $\bigcap_{j=1}^\infty U_j = 1$ and a descending sequence $N_1, N_2, \ldots$ of normal subgroups of G such that $N_j \subset U_j$ and $G_j = G/N_j$ is a Lie group. (A discrete group is regarded as a 0-dimensional Lie group.)

Let G be a Yamabe group and let π^j and π_j be the natural homomorphisms

$$\pi^j\colon G \to G_j, \qquad \pi_j\colon G_{j+1} \to G_j.$$

Of course, $\pi^{j-1} = \pi_{j-1}\circ\pi^j$.

By an easy inductive procedure, for each j we select a basis

$$X_{j1}, \ldots, X_{jn(j)}$$

in the Lie algebra LG_j of G_j in such a way that

$$\pi_{j-1*}X_{jk} = \begin{cases} X_{j-1k} & \text{for } k \leqslant \dim G_{j-1}, \\ 0 & \text{for } k > \dim G_{j-1}. \end{cases}$$

Then, if

$$\Delta_j = \sum_{i=1}^{n(j)} X_{ji}^2,$$

we have $\Delta_j \in \boldsymbol{L}(G_j)$ and, moreover,

$$\pi_{j-1*}\Delta_j = \Delta_{j-1}. \tag{2.2}$$

The following properties of the operator Δ_j were established by L. Garding [3] and E. Nelson [9], cf. also [10].

(2.3)
- (i) Δ_j is a self-adjoint negative operator on $L_2(G_j)$,
- (ii) for real positive t the operator $\exp(t\Delta_j)$ is of the form $\exp(t\Delta_j)f = p_j^t * f$, where $p_j^t(g) \geqslant 0$ and $\int_{G_j} p_j^t(g)\,dg = 1$, consequently
- (iii) $\exp(t\Delta_j) \in \boldsymbol{L}(G_j)$,
- (iv) for each φ in $\Phi(G_j)$, for every positive real s there is a constant C such that $\int_{G_j} p_j^t(g)\varphi(g)\,dg \leqslant C$ for all $t < s$, whence also $\int_{G_j} |p_j^t(g)\varphi(g)|^2\,dg < \infty$ for all positive t,
- (v) for every f in $L_p(G_j)$, $1 \leqslant p < \infty$, $\lim_{t\to 0}\|p_j^t * f - f\|_p = 0$ and also, if f is uniformly continuous, $\lim_{t\to 0}\|p_j^t * f - f\|_\infty = 0$.

LEMMA 2.1. *For all $t \in \boldsymbol{R}^+$ we have $p_j^t \in \boldsymbol{E}(G_j)$.*

The proof follows from the following evaluation and (2.3) iv.

$$\begin{aligned} \int p_j^{2t}(g)\varphi(g)\,\delta_{G_j}^{-1}(g) &= \int p_j^t(h)p_j^t(h^{-1}g)\varphi(g)\,\delta_{G_j}^{-1}(g)\,dh \\ &\leqslant \int p_j^t(h)\varphi(h)p_j^t(h^{-1}g)\varphi(h^{-1}g)\,\delta_{G_j}(g^{-1})\,dh \\ &\leqslant \Big(\int |p_j^t(h)\varphi(h)|^2\,dh\Big)^{1/2}\Big(\int |p_j^t(h^{-1}g)\varphi(h^{-1}g)\,\delta_{G_j}(g^{-1})|^2\,dh\Big)^{1/2} \\ &= \Big(\int |p_j^t(h)\varphi(h)|^2\,dh\Big)^{1/2}\Big(\int |p_j^t(h)\varphi(h)\,\delta_{G_j}(h)|^2\,dh\Big)^{1/2}. \end{aligned}$$

LEMMA 2.2. *If $f \in \boldsymbol{E}(G_j)$, then for a φ in $\Phi(G_j)$ we have*

$$\lim_{t\to 0}\|p_j^t * f - f\|_\varphi = 0.$$

Proof. We take an f in $\boldsymbol{E}(G_j)$ and a φ in $\Phi(G_j)$ and we note that for an $s>0$ there is a constant C such that

$$\|p_j^t*f\|_\varphi \leqslant C \text{ for all } 0<t<s. \tag{2.4}$$

In fact,

$$\begin{aligned}|p_j^t*f(g)\varphi(g)| &\leqslant \int p_j^t(h)|f(h^{-1}g)|\varphi(h)\varphi(h^{-1}g)\,dh\\ &\leqslant \int p_j^t(h)\varphi(h)\,dh\cdot\sup_h|f(h^{-1}g)\varphi(h^{-1}g)|\\ &= \int p_j^t(h)\varphi(h)\,dh\cdot\|f\|_\varphi,\end{aligned}$$

whence (2.4) follows by (2.3) iv.

Now for an $\varepsilon>0$, in virtue of (2.4), we apply Corollary 1.4 to select a compact set A in G_j such that

$$|p_j^t*f(g)\varphi(g)|\leqslant\varepsilon/2 \quad\text{and}\quad |f(g)\varphi(g)|\leqslant\varepsilon/2$$

for all g outside A and $t<s$. Then

$$|p_j^t*f(g)-f(g)|\varphi(g)\leqslant\sup_{g\in A}\varphi(g)\|p_j^t*f-f\|_\infty$$

for g in A and all $t\in\boldsymbol{R}^+$, and

$$|p_j^t*f(g)-f(g)|\varphi(g)\leqslant\varepsilon \quad\text{for } g\notin A \text{ and } t<s,$$

which, in virtue of (2.3) v completes the proof of the lemma.

Now let m_j be the normalized Haar measure on N_j viewed as a Borel measure on G. Since N_j is a normal subgroup, then for all continuous functions f on G we have

$$m_j*f=f*m_j,\qquad j=1,2,\ldots \tag{2.5}$$

Moreover, for $k\leqslant j$, we have

$$m_j*m_k=m_k.$$

For a positive t and a $j=1,2,\ldots$ we write

$$\overline{p}_j^t(g)=p_j^t(\pi^j(g)),\qquad g\in G.$$

In virtue of (2.2) we have

$$\pi_{j-1*}\exp(t\Delta_j)=\exp(t\Delta_{j-1}),$$

whence, by (2.1),

$$p_{j-1}^t(\pi_j(g))=\int\limits_{\ker\pi_{j-1}}p_j^t(hg)\,dh,$$

or

$$\overline{p}_j^t*m_k=\overline{p}_k^t \quad\text{for all } k\leqslant j. \tag{2.6}$$

We also have

$$\overline{p}_j^t\in\boldsymbol{E}(G)\quad\text{for all } t \text{ and } j=1,2,\ldots \tag{2.7}$$

In fact, let $\varphi\in\Phi(G)$. We put

$$\varphi_j(g)=\sup\{\varphi(gh):\ h\in N_j\}. \tag{2.8}$$

Then, clearly, φ_j is a submultiplicative function on G_j and, in view of Lemma 2.1,

$$\sup\{\overline{p}_j^t(g)\varphi(g):\ g\in G\}\leqslant\sup\{p_j^t(g)\varphi_j(g):\ g\in G_j\}<\infty.$$

Let

$$p_j=\overline{p}_j^{1/j},\qquad j=1,2,\ldots \tag{2.9}$$

Now we are ready to prove a theorem on the existence of a commutative approximate identity in $\boldsymbol{E}(G)$. As above we deal only with the metric groups, since this is the case in which we are going to apply the theorem. It is clear how to generalize this to an arbitrary locally compact group on the expense of more complicated notation.

THEOREM 2.1. *Let G be a locally compact metric group. For every open compactly generated group H there exists a sequence of functions $\{p_j\}$ $j=1,2,\ldots$ in $\boldsymbol{E}(G)$ with $\operatorname{supp}p_j\subset H$ which has the following properties*:

(i) $p_j(g)\geqslant 0$,

(ii) $p_j^*=p_j$,

(iii) $p_i*p_j=p_j*p_i$ *for all* $i,j=1,2,\ldots$,

(iv) *for every φ in $\Phi(G)$ there is a constant C_φ such that $\|p_j*f\|_\varphi\leqslant C_\varphi\|f\|_\varphi$ for all f in $\boldsymbol{E}(G)$ and $j=1,2,\ldots$,*

(v) *for every f in $\boldsymbol{E}(G)$ the sequence p_j*f is convergent to f in the topology of $\boldsymbol{E}(G)$.*

Proof. First we note that if H is an open subgroup of G and the theorem is proved for H in place of G, then it holds for G. We verify (v) only, the rest being easy. Let $f\in\boldsymbol{E}(G)$. Since $\operatorname{supp}f$ is contained in a compactly generated subgroup, we see that

$$f=\sum_{n=1}^{\infty}f_n\quad\text{with }\operatorname{supp}f_n\subset Hg_n$$

for a sequence of elements $g_1,g_2,\ldots$ of G. By Corollary 1.4, for every φ in $\Phi(G)$ we then have $\|f_n\|_\varphi\to 0$ as $n\to\infty$. Consequently it is sufficient to show that for every n $\lim_{j\to\infty}\|p_j*f_n-f_n\|_\varphi=0$. For a fixed n we translate the function f_n from the right to get a function f_n' such that $\operatorname{supp}f_n'\subset H$. Hence, by assumption, $\lim_{j\to\infty}\|p_j*f_n'-f_n'\|_\varphi=0$ and, since convolution from the left commutes with translation from the right, and the latter is continuous in $\boldsymbol{E}(G)$, we get the result.

Since every locally compact group contains an open Yamabe subgroup, cf. [8], chapter IV, we may assume that G is Yamabe.

Now let p_j, $j = 1, 2, \ldots$ be defined as in (2.7). We prove that they satisfy (i)–(v) of the theorem.

(i) and (ii) follow immediately form the definition. (iii) is an easy consequence of (2.5) and (2.6). Thus only (iv) and (v) require any proving. To this end we prove first

LEMMA 2.3. *For every φ in $\Phi(G)$ there is a constant C_φ such that*

$$\int p_j(g)\varphi(g)\,dg \leqslant C_\varphi. \tag{2.10}$$

Proof. Let $\varphi \in \Phi(G)$ and $\varphi(s) > 0$ for all $g \in G$ and let φ_1 be defined as in (2.8) (for $j = 1$). Then, since all functions $\overline{p}_j^t$, φ and the measures m_j are non-negative, by (2.6), we have

$$\begin{aligned}\int p_1^t(g)\varphi_1(g)\,dg &\geqslant \int \overline{p}_1^t(g)\varphi(g)\,dg = \int \overline{p}_j^t * m_1(g)\varphi(g)\,dg\\ &\geqslant \int\int \overline{p}_j^t(h^{-1}g)\,dm_1(h)\varphi(h^{-1}g)\varphi(h)^{-1}dg\\ &= \int\int \overline{p}_j^t(h^{-1}g)\varphi(h^{-1}g)\,dg\,\varphi(h)^{-1}dm_1(h)\\ &= \int\int \overline{p}_j^t(g)\varphi(g)\,dg\,\varphi(h)^{-1}dm_1(h).\end{aligned}$$

Hence

$$\int \overline{p}_j^t(g)\varphi(g)\,dg \leqslant K\int p_1^t(g)\varphi_1(g)\,dg \leqslant KC \quad \text{for all } t \in \boldsymbol{R}^+$$

and $j = 1, 2, \ldots$, where $K = \left(\int \varphi(g)^{-1}dm_1(g)\right)^{-1}$ and C is the bound for $\int p_1^t(g)\varphi_1(g)\,dg$ stated in (2.3) iv. Thus, by (2.9), the lemma follows.

From Lemma 2.3 property (iv) follows almost immediately. Let $\varphi \in \Phi(G)$ and let C_φ be the constant established in Lemma 2.3. Then, for an f in $\boldsymbol{E}(G)$, we have

$$\begin{aligned}|p_j*f(g)\varphi(g)| &\leqslant \int p_j(h)|f(h^{-1}g)|\varphi(g)\,dh\\ &\leqslant \int p_j(h)\varphi(h)\,dh\cdot\sup_g |f(h^{-1}g)\varphi(h^{-1}g)| \leqslant C_\varphi\|f\|_\varphi.\end{aligned}$$

To prove (v) we take a φ in $\Phi(G)$, an f in $\boldsymbol{E}(G)$, an $\varepsilon > 0$ and we select a j_0 such that

$$\|m_j*f - f\|_\varphi < \varepsilon/3C \quad \text{for all } j \geqslant j_0, \tag{2.11}$$

where $C = \max\{C_\varphi, 1\}$. Then by Lemma 2.2 we select a j_1 such that

$$\|\overline{p}_{j_0}^{1/i} * m_{j_0} * f - m_{j_0} * f\|_\varphi < \varepsilon/3 \quad \text{for all } i > j_1.$$

Hence, in virtue of (2.6), we have

$$\|\overline{p}_i^{1/i} * m_{j_0} * f - m_{j_0} * f\|_\varphi < \varepsilon/3 \quad \text{for all } i > j_1. \tag{2.12}$$

Now, by Lemma 2.3, from (2.11) we infer

$$\|p_i * m_{j_0} * f - p_i * f\|_\varphi < \varepsilon/3 \quad \text{for all } i > j_1,$$

which together with (2.12) and (2.11) again proves (v) and at the same time completes the proof of Theorem 2.1.

3. Cyclic vectors of induced representations. Let G be a locally compact group. Let $\boldsymbol{K}(G)$ denote the space of continuous functions with compact support on G. As we have noticed before, $\boldsymbol{K}(G)$ is dense in $\boldsymbol{E}(G)$. The classical Blattner construction shows that a unitary representation of G induced by a cyclic representation is defined by a positive-definite measure $\mu \in \boldsymbol{K}(G)'$, cf. e.g. [2]. We shall prove later that μ is bounded on $\boldsymbol{E}(G)$, that is μ belongs to $\boldsymbol{E}(G)'$. Let us first recall Blattner construction.

A Radon measure μ on G is bounded on $\boldsymbol{E}(G)$ if $\int |f|d|\mu| < \infty$, we then write $\mu \in \boldsymbol{E}(G)'$.

Let $\mu \in \boldsymbol{E}(G)'$. If μ is positive-definite, i.e. $\langle f^* * f, \mu\rangle \geqslant 0$ for all f in $\boldsymbol{E}(G)$, it defines a unitary representation of G as follows. Let $I = \{x \in \boldsymbol{E}(G)\colon \langle x^* * x, \mu\rangle = 0\}$. Then $H_\mu^0 = \boldsymbol{E}(G)/I$ is a pre-Hilbert space with a strictly positive-definite inner product $(\overline{x}, \overline{y}) = \langle y^* * x, \mu\rangle$, where $x \to \overline{x}$ is the natural mapping of $\boldsymbol{E}(G)$ onto $\boldsymbol{E}(G)/I$. Moreover, if $L_g x(h) = x(g^{-1}h)$, $g, h \in G$, then I is stable under L_g, $g \in G$, and so L_g acts on H_μ^0 and is unitary with respect to $(\cdot,\cdot)$. As such it extends to the completion H_μ and is denoted by L_g^μ. The mapping $g \to L_g^\mu$ is the required representation. For a function f in $\boldsymbol{E}(G)$ we write $L_f^\mu = \int_G f(g)L_g^\mu\,dg$. We then have

$$L_f^\mu \overline{x} = (f*x)^-. \tag{3.1}$$

Let H be an open compactly generated subgroup of G and let p_j, $j = 1, 2, \ldots$ be the approximate identity in $\boldsymbol{E}(G)$ which satisfies (i)–(v) of Theorem 2.1. and supp $p_j \subset H$. Let a_j be a sequence of positive numbers such that the series

$$\sum_{j=1}^\infty a_j p_j * p_j \tag{3.2}$$

is convergent to a function ξ in $\boldsymbol{E}(G)$. To see that such a_j $j = 1, 2, \ldots$ exist we take the function φ_0 in $\Phi(H)$ as defined in Proposition 1.2 in $\Phi(H)$ and select a_j such that $\sum_{j=1}^\infty a_j\|p_j*p_j\|_{\varphi_0^j} < \infty$.

THEOREM 3.1. *If μ is a positive-definite measure bounded on $\boldsymbol{E}(G)$, then the vector $\overline{\xi}$, where ξ is defined by (3.1) is cyclic for the representation $G \ni g \to L_g^\mu$.*

Proof. Let P be an orthogonal projection in H^μ which commutes with every L_g^μ, $g \in G$, and $P\overline{\xi} = 0$. Then P commutes with every L_f^μ,

$f \in \boldsymbol{E}(G)$. For an arbitrary i we have

$$0 = \big(P_{\bar{\xi}}, (p_i * p_i)^{\sim}\big) = \sum_{j=1}^{\infty} a_j \big(P(p_j * p_j)^{\sim}, (p_i * p_i)^{\sim}\big). \tag{3.3}$$

But for all $i, j = 1, 2, \ldots$, by (3.1), we have

$$\begin{aligned}\big(P(p_j*p_j)^{\sim}, (p_i*p_i)^{\sim}\big) &= \big(L^{\mu}_{p_i} P\overline{p}_i, (p_j*p_j)^{\sim}\big)\\ &= \big(P\overline{p}_i, L^{\mu}_{p_i}(p_j*p_j)^{\sim}\big) = \big(P\overline{p}_i, (p_i*p_j*p_j)^{\sim}\big)\\ &= \big(P\overline{p}_i, (p_j*p_j*p_i)^{\sim}\big) = \big(P\overline{p}_i, L^{\mu}_{p_j}(p_j*p_i)^{\sim}\big)\\ &= \big(L^{\mu}_{p_j} P\overline{p}_i, (p_j*p_i)^{\sim}\big) = \big(P(p_j*p_i)^{\sim}, (p_j*p_i)^{\sim}\big).\end{aligned}$$

Since $a_j > 0$, (3.3) implies $P(p_j*p_i)^{\sim} = 0$ for all $i, j = 1, 2, \ldots$ Hence, since p_j is an approximate identity in $\boldsymbol{E}(G)$, $P\overline{p}_j = 0$ for all $j = 1, 2, \ldots$ Finally, for arbitrary f, h in $\boldsymbol{E}(G)$

$$(P\bar{f}, \bar{h}) = \lim_{j\to\infty} \big(P(f*p_j)^{\sim}, \bar{h}\big) = \lim_{j\to\infty} \big(P\overline{p}_j, (f*h)^{\sim}\big) = 0.$$

Which shows that P is zero and so $\bar{\xi}$ is cyclic.

THEOREM 3.2. *If μ is a non-negative positive-definite Radon measure on G, then μ is bounded on $\boldsymbol{E}(G)$.*

Proof. It suffices to show that if H is an open compactly generated subgroup of G then $\boldsymbol{E}(H) \subset L_1(\mu)$. Since μ is positive definite, for every non-negative continuous function f with compact support in H, $f*\mu*f^{\sim}$ is a non-negative positive-definite continuous function and as such it is bounded. If φ_0 is a function in $\Phi(H)$ as defined in Proposition 1.1, φ_0^{-1} is integrable.

$$\infty > \int_H f*\mu*f^{\sim}(g)\varphi_0(g)^{-1}dg = \int_H f^* * \varphi_0^{-1} * f(g)\, d\mu(g).$$

But, since φ_0 is submultiplicative, there exists a positive constant M such that

$$M\cdot\varphi_0^{-1}(s) \leqslant f^* * \varphi_0^{-1} * f(g) \quad \text{for all } g \in H.$$

In fact,

$$\begin{aligned} f^* * \varphi_0^{-1} * f(g) &= \iint f^*(s)\varphi_0^{-1}(s^{-1}gh) f(h^{-1})\, ds\, dh\\ &\geqslant \int f^*(s)\varphi_0^{-1}(s^{-1})\, ds \int f(h^{-1})\varphi_0^{-1}(h)\, dh \cdot \varphi_0^{-1}(g).\end{aligned}$$

Consequently,

$$\begin{aligned}\int \varphi_0^{-1}(s)\, d\mu &< M^{-1} \int f^* * \varphi_0^{-1} * f(s)\, d\mu(s)\\ &= \int f*\mu*f(s)\varphi_0^{-1}(s)\, ds < \infty.\end{aligned}$$

But, for every $f \in \boldsymbol{E}(H)$ $|f|\varphi_0 < C$, hence $|f| < C\varphi_0^{-1}$, which shows that $f \in L_1(\mu)$, what we had to prove.

THEOREM 3.3. *A unitary representation of a first countable, locally compact group G induced by a cyclic representation of a subgroup H is cyclic.*

Proof. The induced representation is of the form $g \to L^{\mu}_g$, where μ is the Radon measure defined as follows

$$\int f(g)\, d\mu(g) = \int_H f(h)\, \delta_G^{1/2}(h)\, \delta_H^{-1/2}(h)\, \varrho(h)\, dh,$$

where $\varrho(h)$ is a normalized positive-definite continuous function on H (cf. [2]).

We show that $\mu \in \boldsymbol{E}(G)'$. In fact,

$$\Big|\int f(g)\, d\mu(g)\Big| \leqslant \int_H |f(h)|\, \delta_G^{1/2}(h)\, \delta_H^{-1/2}(h)\, dh \tag{3.4}$$

whence, since $\delta_G^{1/2}(h)\, \delta_H^{-1/2}(h)\, dh$ is a positive, positive-definite measure, by Theorem 3.2, the right hand side of (3.4) is finite for all $f \in \boldsymbol{E}(G)$, so $\mu \in \boldsymbol{E}(G)'$ and the theorem follows from Theorem 3.2.

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