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Spaces of continuous functions into a Banach space

by

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Abstract. Let X be a compact Hausdorff space and E be a real Banach space. Let $C(X, E)$ denote the Banach space of evalued continuous functions with the usual supremum norm. The Banach–Stone theorem asserts that if X, Y are compact Hausdorff spaces then X is homeomorphic with Y if and only if there is a linear isometry on $C(X, \boldsymbol{R})$ onto $C(Y, \boldsymbol{R})$. The corresponding theorem for Banach space-valued continuous functions is not true even when E is the two dimensional Banach space R^2 with supremum norm and X, Y are compact metric spaces. However, an analogue of the Banach–Stone theorem is obtained for Banach spaces E with a cylindrical unit cell of which the base is strictly convex and smooth.

§ 1. Introduction. Let X be a compact Hausdorff space and E be a real Banach space. Let $C(X, E)$ denote the Banach space of E-valued continuous functions equipped with the usual supremum norm. The Banach-Stone theorem, Day [2], asserts that if X, Y are compact Hausdorff spaces then X is homeomorphic with Y if and only if there is a linear isometry on $C(X, \boldsymbol{R})$ onto $C(Y, \boldsymbol{R})$ where $\boldsymbol{R}$ is the real line. Subsequently Jerison [5], investigated the problem of extending Banach–Stone theorem with $\boldsymbol{R}$ replaced by an arbitrary Banach space. In [5] it is proved that the theorem remains true if (*) any two T-sets in E are discrepant. It is further shown that the Banach–Stone theorem is not true if E is the infinite dimensional space $C(I, R)$ where I is the unit interval. The results in this paper were obtained in an attempt to extend Banach–Stone theorem to the case of finite dimensional E. Surprisingly enough, as shown in the last section of this paper, it is found that the theorem is not true even when E is the two dimensional Banach space R^2 with the supremum norm and X, Y are compact metric spaces.

The plan of the paper is as follows. We recall few useful known theorems and other preliminaries in Section 2. In Section 3 we obtain an analogue of Banach–Stone theorem for Banach spaces E with a cylindrical unit cell of which the base is strictly convex and smooth. The two dimensional Banach space mentioned in the preceeding paragraph is one such. Lastly we discuss a counter example thus justifying the theorem in Section 3.

§ 2. Preliminaries. Throughout the paper, E is a fixed real Banach space of dimension at least 2. If B is a Banach space we denote the dual space of B by B^*. The norms of the various Banach spaces that enter our discussion are all denoted by the same symbol $\|\cdot\|$ as there is no occasion for confusion. The unit cell of $B(B^*)$ is denoted by $U_B(U_B^*)$ and $S_B(S_B^*)$ is the boundary of $U_B(U_B^*)$. If X is a compact Hausdorff space the unit cells of $C(X,E)$ and $C(X,E)^*$ are denoted by U_X and U_X^* respectively. The set of extreme points of a set K is denoted by $\mathrm{Ext}K$.

We recall few geometric properties of a Banach space. If B is a Banach space and $x\in S_B$ then a functional $l\in B^*$ is said to support U_B at x if $\|l\| = 1 = l(x)$. The cell U_B is said to be smooth at x if there is only one hyperplane of support at x. A Banach space B is said to be smooth if U_B is smooth at all points $x\in S_B$. An M-set in a Banach space B is a maximal convex subset of S_B. A T-set is a half-cone of nonnegative multiples of vectors in a M-set. For a discussion of these sets we refer to [9]. Two T-sets T_1, T_2 in B are said to be discrepant if either $T_1\cap T_2 = \{0\}$ or there exists a T-set T_3 such that $T_3\cap T_1 = T_3\cap T_2 = \{0\}$. It is verified by applying Zorn's lemma if $x\in S_B$ then there is an M-set M_x containing x. Further if $x\in S_B$ and $\{x\}$ is an M-set then any two T-sets are discrepant. Thus in a strictly convex Banach space B any two T-sets are discrepant. We summarize few properties of M-sets in the accompanying remark.

Remark 1. If M is a M-set in a Banach space B then it follows from the separation theorem that there exists an $f\in B^*$, $\|f\| = 1$ such that $\inf_{x\in M} f(x) \geqslant \sup_{x\in U_B} f(x)$. Hence if H is the hyperplane $f^{-1}(1)$, then $M\subset H$. Further since $H\cap S_B = H\cap U_B$ is a convex set and M is a maximal convex set it follows that $H\cap S_B = M$. We refer to H as a hyperplane supporting U_B along M. Further we note that if there is a smooth point x in M then the hyperplane supporting U_B along M is unique.

We conclude this section with a definition of S-cylinders and few known results useful in the subsequent discussion.

DEFINITION 1. The unit cell of a Banach space $(B, \|\ \|)$ is said to be *S-cylindrical* if there exists a closed subspace L of B of codimension 1 such that $(L, \|\ \|)$ is smooth and strictly convex and there is a point p, $\|p\| = 1$ such that $U_B = [-p,p]+U_B\cap L$. The Banach space B is said to be S-cylindrical if the unit cell of B is S-cylindrical.

The two dimensional Banach space R^2 with supremum norm is S-cylindrical. More generally if B is a smooth Banach space with a strictly convex norm and L is a closed subspace of codimension 1 such that a translate of L supports U_B at x then let $V = [-x,x]+U_B\cap L$. It is verified that if p is the gauge of V then (B,p) is a S-cylindrical Banach space linearly homeomorphic with B. In passing we note that if the unit cell of a Banach space B is S-cylindrical then the T-sets in B need not be discrepant. In this connection we refer to example 4.5 in [5].

For convenience of reference we state three useful theorems.

THEOREM 1. (Jerison). *Let X, Y be compact Hausdorff spaces and E be a real Banach space such that any two T-sets in E are discrepant. Then X is homeomorphic with Y if and only if there is an isometry on $C(X,E)$ onto $C(Y,E)$.*

For a proof of this theorem we refer to 5.2 in [5]. Before proceeding to the statement of the next theorem let us note that there is a natural map e on $E^*\times X$ into $C(X,E)^*$ defined by $e(l,p)(f) = l(f(p))$. It is verified that $\|e(l,p)\| = \|l\|$ and for a fixed $p\in X$, $e(\cdot,p)$ is a linear isometry on E^* into $C(X,E)^*$.

THEOREM 2. (Singer). *Let X be a compact Hausdorff space and E be a Banach space. Then*

$$\mathrm{Ext}(U_X^*) = e(\mathrm{Ext}\,U_E^*\times X)$$

where e is the map defined earlier.

For a proof of this theorem we refer to Singer [8].

THEOREM 3. (Sundaresan). *If X is a compact Hausdorff space and $f\in C(X,E)$, $\|f\| = 1$ then the unit cell of $C(X,E)$ is smooth at f if and only if there is a point $q\in X$ such that $1 = \|f(q)\| > \|f(q')\|$ for all $q'\neq q$ and U_E is smooth at $f(q)$.*

A proof of this theorem is provided in [8].

In this context we recall a theorem of Mazur [6] stating that if U_E is smooth at x and f is the supporting functional at x then $\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists for all $y\in E$ and is equal to $f(y)$. Further if the preceeding limit exists for all $y\in E$ then U_E is smooth at x.

§ 3. Banach–Stone theorem for spaces $C(X,E)$. We proceed to the main theorem of the paper after stating few lemmas.

LEMMA 1. *If X is a compact Hausdorff space and f is an extreme point of the unit cell of $C(X,E)$ then $\|f(t)\| = 1$ for all $t\in X$.*

Proof. Let $f\in \mathrm{Ext}\,U_X$. If for some $t_0\in X$ $\|f(t_0)\|\neq 1$ then there exists a δ, $0\leqslant\delta<1$, a compact neighborhood N of t_0 such that $\|f(t)\|\leqslant\delta<1$ for all $t\in N$. Since X is a compact Hausdorff there exists a continuous function φ: $X\to[0,1-\delta]$ such that $\varphi(t_0) = 1-\delta$ and $\varphi(t) = 0$ if $t\notin N$. Let a be a vector in E such that $\|a\| = 1$. Let g be the function on $X\to E$ defined by $g(t) = \varphi(t)a$. Then $g\in C(X,E)$, $\|f\pm g\|\leqslant 1$ and $g\neq 0$. Thus $f\notin \mathrm{Ext}\,U_X$. This completes the proof of the lemma.

LEMMA 2. *If M is an M-set in a Banach space E then* $\mathrm{Ext}M \subset \mathrm{Ext}\,U_E$.

LEMMA 3. *Let E be a Banach space with a S-cylindrical unit cell U_E. Let $U_E = [-p,p]+U_E\cap L$ where p and L are as in Definition 1. Then*

(1) $\operatorname{Ext} U_E = \operatorname{Ext} M_1 \cup \operatorname{Ext} M_2$ *where* $M_1 = p + U_E \cap L$ *and* $M_2 = -p + U_E \cap L$.

(2) *M is an M-set in E if and only if* $M = M_i$, $i = 1, 2$, *or* $M = [p+\xi, p-\xi]$ *for some* $\xi \in S_E \cap L$.

(3) *If P_1, P_2 are distinct M-sets in E then* $\operatorname{card}(P_1 \cap P_2) \leqslant 1$. (4) *Interior M_i relative to S_E is* $M_i \sim \operatorname{Ext} M_i$.

The proofs of Lemmas 2 and 3 are rather straightforward and the details are omitted.

Remark 2. We make a note of some implications of Lemma 3.

(a) $\operatorname{Ext} U_E$ is a closed subset of S_E. Further from (2) of the lemma it follows that if M is an M-set in E, $\operatorname{card} M \geqslant 2$. Thus $M \cap D \neq \emptyset$ for every M-set (D is the set of points of smoothness of S_E), M in E. Hence the hyperplane supporting U_E along a fixed M-set is unique. Further if $x \in S_E \sim (M_1 \cup M_2)$ then from (2) it is verified that there exist a unique pair of extreme points e_x^1, e_x^2 with $e_x^i \in M_i$, $i = 1, 2$ and $e_x^1 - e_x^2 = 2p$.

(b) The property (4) implies that if E is an S-cylindrical Banach space and M_i $i = 1, 2$ are as in the preceeding lemma then if $g \in \operatorname{Ext} U_X$ then $g(p) \notin (M_1 \sim \operatorname{Ext} M_1) \cup (M_2 \sim \operatorname{Ext} M_2)$. If possible let $p \in X$ such that $g(p) \in (M_1 \sim \operatorname{Ext} M_1)$. From Lemma 1 it follows that $\|g(t)\| = 1$ for all $t \in X$. Since g is a continuous function, from (4) it follows that there is a compact neighborhood C of p such that $g(C) \subset M_1 \sim \operatorname{Ext} M_1$. Since C is compact, with f as in the proof of (4), it is found that there exists a $\delta > 0$ such that $\|h\| < \delta$, $f(h) = 0$ imply $g(z) + h \in M_1$ for all $z \in C$. Let h be such a fixed non-zero vector. Since X is a compact Hausdorff space there exists a continuous function $\varphi\colon X \to [0,1]$ such that $\varphi(p) = 1$ and $\varphi(X \sim \operatorname{Int} C) \subset \{0\}$. Let g_1, g_2 be the functions on X to E defined by $g_1(t) = f(t) + \varphi(t)h$ and $g_2(t) = f(t) - \varphi(t)h$. It is verified that $g_1, g_2 \in C(X, E)$, $\|g_i\| = 1$, $i = 1, 2$ and $g = (g_1+g_2)/2$. Since $g_1 \neq g_2$, $g \notin \operatorname{Ext} U_X$. Thus the proof is completed.

Lemma 4. *Let E be a Banach space with a S-cylindrical unit cell. Then if $f \in C(X, E)$ is an extreme point of U_X then $f(p) \in \operatorname{Ext} U_E$ for all $p \in X$.*

Proof. Let $f \in \operatorname{Ext} U_X$. As noted in Lemma 1, range $f \subset S_E$. From the preceeding remark it is further seen that range $f \subset [S_E \sim (M_1 \cup M_2)] \cup \cup (\operatorname{Ext} M_1 \cup \operatorname{Ext} M_2)$. Thus if for some $p \in X$, $f(p) \notin \operatorname{Ext} U_E$ then since $M_1 \cup M_2$ is a closed subset of S_E, there exists a compact neighborhood C of p such that $f(C) \subset S_E \sim (M_1 \cup M_2)$. From the preceeding remark it follows that for each point $t \in C$, there is a unique pair of points e_t^i, $i = 1, 2$ such that $e_t^i \in \operatorname{Ext} M_i$, $f(t) \in [e_t^1, e_t^2]$ and $e_t^1 - e_t^2 = 2p$. Since $f(t) \notin M_i$, $i = 1, 2$, there exists a function $\lambda\colon C \to]0, 1[$ such that $f(t) = \lambda(t) e_t^1 + (1-\lambda(t)) e_t^2$. Further if $l \in E^*$, $\|l\| = 1$ is such that $l^{-1}(0) = L$, $l(p) = 1$ then $l(f(t)) = 2\lambda(t) - 1$. Thus λ is a continuous function. Since C is a compact set there exists an $a > 0$ such that $a < \lambda(t) < 1 - a$ for $t \in C$. Let $G = \operatorname{Int} C$ and $\varphi\colon X \to [0, 1]$ be a continuous function such that $\varphi(p) = 1$ and $\varphi(X \sim G) \subset \{0\}$. Let g_1, g_2 be the functions on $X \to E$ defined by $g_1(t) = f(t) + \varphi(t) a [e_t^1 - e_t^2]$ and $g_2(t) = f(t) - \varphi(t) a [e_t^1 - e_t^2]$. Since $e_t^1 - e_t^2 = 2p$, g_1, g_2 are in $C(X, E)$, and it is further verified that $\|g_i\| = 1$, $i = 1, 2$, $f = (g_1+g_2)/2$ and $g_1 \neq g_2$. Thus $f \notin \operatorname{Ext} U_X$ contradicting the hypothesis. Thus the proof of the lemma is complete.

Lemma 5. *A set $P \subset C(X, E)$ is an M-set if and only if there is an M-set $M \subset E$ and a point $p \in X$ such that*

$$P = \{f \mid f \in C(X, E),\ \|f\| = 1 \text{ and } f(p) \in M\}.$$

Thus each M-set P in $C(X, E)$ could be represented as $P = (M, p)$ where M and p are chosen as above, and two M-sets P_1, P_2 where $P_1 = (M_1, p_1)$, $P_2 = (M_2, p_2)$ are equal if and only if $M_1 = M_2$ and $p_1 = p_2$.

This lemma is an immediate consequence of Theorem 4.1 and Lemma 4.3 in [9]. Hence the details of a proof are omitted.

Before proceeding to the main theorem we note that if X, Y are compact Hausdorff spaces and $T\colon X \to Y$ is a homeomorphism then the operator $T\colon C(Y, E) \to C(X, E)$ defined by $T(f)(p) = f(Tp)$ is a linear isometry onto $C(X, E)$. Thus in the subsequent discussion we consider only the converse question.

Remark 3. If $x \in E$ let K_x be the function in $C(X, E)$ with range $= \{x\}$. In § 2 we defined the function $e\colon E^* \times X \to C(X, E)^*$. We note that $e|\ (E^* \sim \{0\}) \times X \to C(X, E)^*$ is a 1–1 map. For let $l, m \in E^* \sim \{0\}$ and $p, q \in X$. Suppose $e(l, p) = e(m, q)$. For each $x \in E$, $e(l, p)(K_x) = e(m, q)(K_x)$. Thus $l(x) = m(x)$ for all $x \in E$. Hence $l = m$. Now if $p \neq q$ let $x \in E$ such that $l(x) \neq 0$. Since X is a compact space there is a function $f \in C(X, E)$ such that $f(p) = x$ and $f(q) = 0$. For such a function f, $e(l, p) f \neq e(m, q) f$. Hence a contradiction. Thus $p = q$ justifying the claim.

We denote the map corresponding to the map e, defined on $E^* \times Y$ into $C(Y, E)^*$ also by the same symbol e.

Theorem. *Let E be a Banach space with a S-cylindrical unit cell and X, Y be compact first countable Hausdorff spaces. Let T be a linear isometry on $C(X, E)$ onto $C(Y, E)$ such that corresponding to each point $t \in X$ there are at least two points $x_1, x_2 \in \operatorname{Ext} M_1$ for which $TK_{x_1}(t) \neq TK_{x_2}(t)$. Then X is homeomorphic with Y.*

Proof. Let E be a S-cylindrical Banach space and T be a linear isometry satisfying the hypothesis in the theorem. Let p, L, M_1, M_2 be as in the Lemma 4. Let $l_1 (l_2 = -l_1)$ be the functionals in E^* supporting U_E along $M_1 (M_2)$. From the uniqueness of l_1 it is verified that $l_1 \in \operatorname{Ext} U_E^*$.

We note that since T is an isometry, T^*, the adjoint of T, is an isometry on $C(Y,E)^*$ onto $C(X,E)^*$. For each point $t \in X$ consider the functional $e(l_1, t)$. From Singer's theorem (see § 2), $e(l_1, t) \in \operatorname{Ext} U_X^*$. Since linear isometries preserve extreme points there exists a point $e(l_t^1, \tau(t)) \in \operatorname{Ext} U_Y^*$ such that $T^* e(l_t^1, \tau(t)) = e(l_1, t)$. From Remark 3 it follows that τ is a function on $X \to Y$. We proceed to verify that τ is a homeomorphism on X onto Y.

1. $\tau(X) = Y$: As an initial step we verify that if $t' \in Y$ then there exist two points $y, z \in M_1$ such that $TK_y(t')$, $TK_z(t')$ are in the same M-set in E. For consider the mapping $\beta\colon E \to E$ defined by $\beta(x) = TK_x(t')$. It is verified that β is a continuous mapping. Further from Lemma 4 after noting that for each $x \in \operatorname{Ext} M_1$, K_x (hence TK_x) is an extreme point of $U_X(U_Y)$ it follows that $\beta(\operatorname{Ext} M_1) \subset \operatorname{Ext} U_E$. Thus $\beta(\operatorname{Ext} M_1) \subset M_1$ or $\beta(\operatorname{Ext} M_1) \subset M_2$. In either case the assertion is verified since $\operatorname{card} \operatorname{Ext} M_1 \geqslant 2$. Next, if $\dim E = 2$, then since U_E is a parallelogram, using Lemma 4 and enumerating possible positions of $TK_x(t')$ for $x \in \operatorname{Ext} M_1$, the assertion is verified.

Let y, z be two points in $\operatorname{Ext} M_1$ with the property in the preceeding paragraph. Let M_0 be an M-set in E such that $M_0 \supset \{TK_y(t'), TK_z(t')\}$. Consider the M-set (M_0, t') in the space $C(Y, E)$. For the definition of (M_0, t') see Lemma 5. Since T is a linear isometry $T^{-1}(M_0, t')$ is an M-set in $C(X,E)$. Thus from Lemma 5 it follows that there exists exactly one M-set L in E and a unique point $t \in X$ such that $T^{-1}(M_0, t') = (L, t)$. We claim that $L = M_1$. For, since $T\big((K_y + K_z)/2\big)(t') = TK_{(y+z)/2}(t') \in M_0$ it follows that $K_{(y+z)/2}(t) \in L$. Since $(y+z)/2$ is a smooth point of S_E it follows that $L = M_1$. Thus $T(M_1, t) = (M_0, t')$. Let m_0 be the functional supporting U_E along M_0. It will be shown that $T^* e(m_0, t') = e(l_1, t)$. Since X is a first countable compact space with $v = (y+z)/2$, there exists a function $f_v^t \in C(X, E)$ such that $f_v^t(t) = v$ and $\|f_v^t(q)\| < 1$ for all $q \in X$, $q \neq t$. Since v is a smooth point of S_E, it follows from Theorem 3 that f_v^t is a smooth point of U_X. Since T is an isometry it is readily verified from Mazur's theorem (see comments following Theorem 3 in § 2) that Tf_v^t is a smooth point of U_Y. Since $f_v^t(t) \in M_1$ and $T(M_1, t) = (M_0, t')$ it is verified that $Tf_v^t(t') \in M_0$. Thus the linear functionals $e(l_1, t)$ and $e(m_0, t')$ support U_X and U_Y along M_1 and M_0 respectively. Thus

$$\lim_{r \to 0} \frac{\|f_v^t + rf\| - \|f_v^t\|}{r} = e(l_1, t)(f)$$

and

$$\lim_{r \to 0} \frac{\|Tf_v^t + rTf\| - \|Tf_v^t\|}{r} = e(m_0, t')(Tf).$$

Since T is a linear isometry, $e(l_1, t)(f) = e(m_0, t')(Tf)$. Thus $T^* e(m_0, t') = e(l_1, t)$. Hence $\tau(t) = t'$ and $\tau(X) = Y$.

2. The map τ is injective: Let $t_1, t_2 \in X$ be such that $\tau(t_1) = \tau(t_2) = t'$. From the definition of the map τ it follows that there exist two functionals m_1', m_2' in $\operatorname{Ext} U_E^*$ such that $(*)$ $T^* e(m_1', t') = e(l_1, t_1)$ and $T^* e(m_2', t') = e(l_1, t_2)$. Let $T(M_1, t_1) = (N_1, t_1')$ and $T(M_1, t_2) = (N_2, t_2')$. Let y be a smooth point in M_1. Consider a function $f_y^{t_1} \in C(X, E)$ such that $f_y^{t_1}(t_1) = y$ and $\|f_y^{t_1}(s)\| < 1$ for all $s \in X \sim \{t_1\}$. As already noted in the proof of (1) such a function $f_y^{t_1}$ exists and further since $f_y^{t_1}(t_1) \in M_1$, $f_y^{t_1} \in (M_1, t_1)$. Hence from Theorem 3 it is concluded that t_1' is the only point in Y such that $\|Tf_y^{t_1}(t_1')\| = 1$. Thus $t_1' = t'$ since from equations $(*)$ it follows that $l_1(f_y^{t_1}(t_1')) = m_1'(Tf_y^{t_1}(t')) = 1$ and the last equation implies that $\|Tf_y^{t_1}(t')\| = 1$. Further from Theorem 3 it follows that $Tf_y^{t_1}(t_y')$ is a smooth point of U_E and since $Tf_y^{t_1}(t_1') \in N_1$, m_1' is the only functional supporting U_E along N_1. Similarly it is verified that m_2' is the only functional supporting U_E along N_2. From the additional hypothesis on T there are two points $x_1, x_2 \in \operatorname{Ext} M_1$ such that $TK_{x_1}(t') \neq TK_{x_2}(t')$. Since $K_{x_i} \in (M_1, t_1) \cap (M_1, t_2)$, from our choice of N_1, N_2, it follows that $TK_{x_i}(t_j') \in N_1 \cap N_2$. Thus $N_1 \cap N_2 \supset \{TK_{x_1}(t'), TK_{x_2}(t')\}$ since $t_i' = t'$. Hence $\operatorname{card}(N_1 \cap N_2) \geqslant 2$. Hence from the property (3) in Lemma 3 it follows that $N_1 = N_2$. Since m_1', m_2' are the unique supporting functionals along N_1, N_2 respectively we conclude that $m_1' = m_2'$. Thus $T^* e(m_1', t') = e(l_1, t_2)$ as seen from equations $(*)$. Since $e|\,(E^* \sim \{0\}) \times X$ is $1-1$ it follows that $t_1 = t_2$. Thus τ is injective.

3. The map τ is continuous: Since X, Y are first countable Hausdorff spaces it is enough to verify that if $\{t_n\}_{n \geqslant 1}$ is a sequence converging to $t \in X$ then $\tau(t_n) \to \tau(t)$ in Y. Let us denote for convenience $\tau(t_n) = t_n'$ and $\tau(t) = t'$. Let $T^* e(l_n', t_n') = e(l_1, t_n)$ where l_1 is as in the first paragraph of the proof. We note that $l_n' \in \operatorname{Ext} U_E^*$ for $n \geqslant 1$. If $t_n' \to t$ since Y is first countable compact space there exists a convergent subsequence $\{t_{n_i}'\}$ in $\{t_n'\}$. Let $t_{n_i}' \to c$. Let x_0 be a smooth point in M_1. Consider the function $f_{x_0}^t \in C(X, E)$ such that $f_{x_0}^t(t) = x_0$ and $\|f_{x_0}^t(q)\| < 1$ if $q \neq t$. Since $\|f_{x_0}^t(t_{n_i}) - f_{x_0}^t(t)\| \to 0$ it follows that $l_1(f_{x_0}^t(t_{n_i})) \to l_1(f_{x_0}^t(t))$. Thus $l_{n_i}'(Tf_{x_0}^t(t_{n_i}')) \to 1$ as readily seen from our choice of $\{e(l_n', t_n')\}$. Since $\|Tf_{x_0}^t(t_{n_i}') - Tf_{x_0}^t(c)\| \to 0$ and $\|Tf_{x_0}^t(c)\| \leqslant 1$ it is verified that $\|Tf_{x_0}^t(c)\| = 1$. Let $T^* e(l_1', t') = e(l_1, t)$. Thus $l_1(f_{x_0}^t(t)) = 1 = l_1'(Tf_{x_0}^t(t')) = \|l_1'\| = \|Tf_{x_0}^t\|$. Hence $\|Tf_{x_0}^t(t')\| = 1$. From Theorem 3 it follows that $f_{x_0}^t$ is a smooth point of U_X. Hence $Tf_{x_0}^t$ is a smooth point of U_Y. Further $\|f_{x_0}^t\| = \|Tf_{x_0}^t\| = 1$. Since $\|Tf_{x_0}^t(t')\| = 1$ once again appealing to Theorem 3 we conclude that $t' = c$. Thus every convergent subsequence of $\{\tau(t_n)\}$ converges to $\tau(t)$. Since Y is a first countable compact Hausdorff space $\tau(t_n) \to \tau(t)$. Since X, Y are compact spaces it follows that τ is a homeomorphism.

A counter example. We discuss an example to justify the additional hypothesis on the isometry T in the preceeding theorem. In what follows $\boldsymbol{R}^n$ is the n-dimensional real Banach space with supremum norm. For a definition of topological sum of two topological spaces, we refer to Dugundji [3].

LEMMA 6. *If X, Y are compact Hausdorff spaces then $C(X, \boldsymbol{R}^n)$ is isometric with $C(Y, \boldsymbol{R}^n)$ if and only if the sum of n copies of X is homeomorphic with the sum of n copies of Y.*

Proof. It is verified that $C(X, \boldsymbol{R}^n)$ is isometric with $C(Y, \boldsymbol{R}^n)$ if and only if $C(X\times n, \boldsymbol{R})$ is isometric with $C(Y\times n, \boldsymbol{R})$. Hence from Banach-Stone theorem $X\times n$ is homeomorphic with $Y\times n$ i.e. the topological sum of n copies of X is homeomorphic with the topological sum of n copies of Y.

It is known that for each integer $n \geqslant 2$, there are non-homeomorphic compact metric spaces X, Y such that $X\times n$ is homeomorphic with $Y\times n$, Hanf [4]. A concrete description of such spaces X, Y, $n = 2$ is provided in Sundaresan [10]. More generally there exist compact metric spaces X, Y such that $X\times k \neq Y\times k$ for $k = 1, 2, \ldots, n-1$, and $X\times n = Y\times n$, Kroonenberg [6].

It follows from Lemma 6 and preceeding remarks that there are non-homeomorphic compact metric spaces X, Y such that $C(X, \boldsymbol{R}^n)$ is isometric with $C(Y, \boldsymbol{R}^n)$ for $n \neq 2$. This justifies the additional hypothesis on the isometry in the preceeding theorem.

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Decompositions of set functions

by

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Abstract. Let $\mathscr{R}$ be a ring of sets. With each set $E\in\mathscr{R}$ a collection of classes $\mathscr{D}\subset\mathscr{R}$, consisting of disjoint sets, is associated in such a way that the set $\mathfrak{S}$ of all resulting pairs $(E, \mathscr{D})$ satisfies certain very natural conditions. The $\mathfrak{S}$ is then called an additivity on $\mathscr{R}$ (Section 2). Notions of $\mathfrak{S}$-additive and $\mathfrak{S}$-singular group valued set functions are next introduced and investigated to some degree; when specifying $\mathfrak{S}$ one obtains, e.g., notions of σ-additive and purely finitely additive or η-continuous and η-singular functions. For a very important class of the so called exhaustive (= strongly bounded) set functions a decomposition theorem (3.11) is proved, whose special cases are the Hewitt–Yosida and Lebesgue decompositions for group valued functions. Analogons of general and special decompositions are established also for some nonadditive functions (sumbeasures) and for Fréchet–Nikodym topologies on $\mathscr{R}$ (Section 4). By the way a theorem is given (2.14′) which contains the Vitali–Hahn–Saks, Nikodym and Brooks–Jewett theorems.

Introduction. Let $\mathscr{R}$ be a ring of sets and let μ, η be additive real-valued set functions on $\mathscr{R}$ with μ bounded and $\eta\geqslant 0$. We say that μ is η-continuous and write $\mu\ll\eta$ if, given $\varepsilon>0$, there is $\delta>0$ such that $|\mu(E)|\leqslant\varepsilon$ whenever $\eta(E)\leqslant\delta$, $E\in\mathscr{R}$. At first sight it is not seen at all that the properties "μ is countably additive" and "μ is η-continuous" have much in common. However, it can be proved ([10]; [7], II) that $\mu\ll\eta$ iff $\mu(E_n)\to 0$ provided $E_n\searrow$ and $\eta(E_n)\to 0$, $(E_n)\subset\mathscr{R}$. The latter condition can be equivalently formulated as follows: if (E_n) is a disjoint sequence of sets in $\mathscr{R}$, $E\in\mathscr{R}$, $\bigcup_{n=1}^{\infty} E_n\subset E$ and $\eta\big(E\setminus\bigcup_{k=1}^{n} E_k\big)\to 0$, then $\mu(E) = \sum_{n=1}^{\infty}\mu(E_n)$; the resemblance with the definition of countably additivity is striking. This observation was first made and employed by W. Orlicz in his study of absolute continuity of vector valued set functions [10]; it motivates the general notion of $\mathfrak{S}$-additivity introduced in Section 2. Also, it had suggested a quite natural conjecture that it should been possible to obtain the well known Hewitt–Yosida and Lebesgue decompositions of additive set function in a unified fashion. This is realized in the present paper for exhaustive additive set functions with values in an arbitrary abelian complete topological group G. The method we