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# Uniformly non-$l^{(1)}$ and $B$-convex Banach spaces*

by

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**Abstract.** Banach space is uniformly non-$l^{(1)}$ if and only if it is $B$-convex if and only if $l^{(1)}$ is not finitely representable in it. If all $B$-convex Banach spaces are reflexive, then $B$-convexity is equivalent to super-reflexivity. The non-reflexive space $J$ which is isometrically isomorphic to $J^{**}$ is not only not $B$-convex, but posseses a property which is sufficient but not necessary for non-$B$-convexity ($c_0$ is finitely representable in $J$).

It has long been known that a Banach space is reflexive if it is uniformly non-square. It is not known whether a Banach space is reflexive if it is uniformly non-$l^{(1)}$. It is shown that if this conjecture is correct, then a Banach space is super-reflexive if and only if it is uniformly non-$l^{(1)}$. The space $J$ that is nonreflexive and isometric to $J^{**}$ might have been a prime candidate for a counterexample to this conjecture, but it is shown that both $c_0$ and $l^{(1)}$ are finitely representable in $J$. It also is shown that, if $n \geqslant 2$ and every uniformly non-$l_n^{(1)}$ Banach space is reflexive, then every uniformly non-$l_n^{(1)}$ space is super-reflexive.

DEFINITION 1. For $n \geqslant 2$ and $\varepsilon > 0$, a normed linear space being *$(n, \varepsilon)$-convex* means that there does not exist a subset $\{x_1, \ldots, x_n\}$ of the unit ball such that, for all choices of signs,

$$\|x_1 \pm x_2 \pm \ldots \pm x_n\| > n(1-\varepsilon).$$

For $n \geqslant 2$, a *uniformly non-$l_n^{(1)}$* normed linear space is a normed linear space that is $(n, \varepsilon)$-convex for some $\varepsilon > 0$. A *$B$-convex* normed linear space is a normed linear space that is uniformly non-$l_n^{(1)}$ for some $n \geqslant 2$.

A $B$-convex Banach space is known to be reflexive if it has an *unconditional basis* (see [3], Theorem III.6, p. 142 or [9], Theorem 2.2), or (in the real case) if it can be endowed with a partial order under which it becomes a normed Riesz space (equivalently, a normed linear vector lattice; see [5]). Beck proved that a Banach space is $B$-convex if and only if a certain law of large numbers is valid for random variables with ranges in the space [1], which implies that $B$-convexity is isomorphically

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invariant. Also, a normed linear space $X$ is $B$-convex if and only if $X^*$ is $B$-convex ([3], Corollary II.4, p. 129).

A normed linear space $X$ being *finitely representable* in a normed linear space $Y$ means that, for each finite-dimensional subspace $X_n$ of $X$ and each number $\lambda > 1$, there is an isomorphism $L$ of $X_n$ into $Y$ for which

$$\lambda^{-1}\|x\| \leqslant \|L(x)\| \leqslant \lambda\|x\| \quad \text{if } x \in X_n.$$

DEFINITION 2. A *uniformly non-$l^{(1)}$ normed linear space* is a normed linear space $X$ which has the property that, for each positive number $\Delta$, there is a positive integer $n$ for which there does not exist a subset $\{x_1, \ldots, x_n\}$ of $X$ such that, for all numbers $\{a_1, \ldots, a_n\}$,

$$(1) \qquad \Delta \cdot \sum_1^n |a_1| \leqslant \Big\| \sum_1^n a_i x_i \Big\| \leqslant \sum_1^n |a_i|.$$

A space $X$ is *not uniformly non-$l^{(1)}$* if and only if there is a positive number $\Delta$ such that for each positive integer $n$ there is a subset $\{x_1, \ldots, x_n\}$ of $X$ such that

$$(2) \qquad \Delta \cdot \sum_1^n |a_i| \leqslant \Big\| \sum_1^n a_i x_i \Big\| \leqslant \sum_1^n |a_i|$$

for all numbers $\{a_1, \ldots, a_n\}$. This property is isomorphically invariant. In fact, if $L$ is an isomorphism of $X$ onto $Y$, $\alpha$ and $\beta$ are numbers for which

$$\alpha\|x\| \leqslant \|L(x)\| \leqslant \beta\|x\| \quad \text{if } x \in X,$$

$\{x_1, \ldots, x_n\}$ satisfies (2), and $y_i = \beta^{-1}L(x_i)$ for each $i$, then

$$\alpha\Delta\beta^{-1} \sum_1^n |a_i| \leqslant \Big\| \sum_1^n a_i y_i \Big\| \leqslant \sum_1^n |a_i|.$$

THEOREM 1. *For a normed linear space $X$, the following are equivalent:*

(i) *$X$ is $B$-convex.*

(ii) *$X$ is uniformly non-$l^{(1)}$.*

(iii) *$l^{(1)}$ is not finitely representable in $X$.*

Proof. The proof that $B$-convexity of $X$ implies (ii) is known (see [3], Lemma I. 4, p. 119 and [4]). To give an alternative proof, let us suppose $X$ is not uniformly non-$l^{(1)}$. It then follows that there is a space $Y$ which is isomorphic to $l^{(1)}$ and is finitely representable in $X$ (see the proof of Lemma $B$ in [10]). It follows from this that for each positive number $\delta$ there is an infinite-dimensional subspace $Y_\delta$ of $Y$ which has a basis $\{u_i\}$ such that

$$(1-\delta)\sum |a_i| \leqslant \Big\| \sum a_i u_i \Big\| \leqslant \sum |a_i|$$

if $\sum |a_i| < \infty$ ([9], Lemma 2.1). This implies $X$ is not $B$-convex.



Clearly $l^{(1)}$ is not finitely representable in $X$ if $X$ is uniformly non-$l^{(1)}$.

The equivalence of $B$-convexity and (iii) follows from ([3], Lemma I. 6, p. 123). However, the following proof that (iii) implies $B$-convexity is much easier. Let us suppose $X$ is not $B$-convex. That is, for each $n$, $X$ is not uniformly non-$l_n^{(1)}$. Therefore, for each $n$ and each positive number $\varepsilon$, there is a subset $\{x_1, \ldots, x_n\}$ of the unit ball such that

$$\|x_1 \pm x_2 \pm \ldots \pm x_n\| > n(1-\varepsilon)$$

for all choices of signs. This implies that, for all numbers $\{a_i\}$, if $k$ is chosen so that $|a_k| = \max\{|a_i|: 1 \leqslant i \leqslant n\}$, then

$$\begin{aligned}
\Big\| \sum_{i=1}^n a_i x_i \Big\| &= \Big\| \sum_{i=1}^n [\operatorname{sign}(a_i)|a_k|] x_i - \sum_{i=1}^n [\operatorname{sign}(a_i)|a_k| - a_i] x_i \Big\| \\
&\geqslant |a_k| \Big\| \sum_{i=1}^n \operatorname{sign}(a_i) x_i \Big\| - \sum_{i=1}^n |(|a_k| - |a_i|)| \, \|x_i\| \\
&\geqslant n|a_k|(1-\varepsilon) + \sum_{i=1}^n (|a_i| - |a_k|) \\
&= \sum_{i=1}^n |a_i| - n\varepsilon|a_k| \geqslant (1-n\varepsilon)\sum_{i=1}^n |a_i|,
\end{aligned}$$

so that

$$(1-n\varepsilon)\sum_{i=1}^n |a_i| \leqslant \Big\| \sum_{i=1}^n a_i x_i \Big\| \leqslant \sum_{i=1}^n |a_i|$$

and $l^{(1)}$ is finitely representable in $X$. This completes the proof of Theorem 1.

As we have noted, if normed linear spaces $X$ and $Y$ are isomorphic, then $X$ is uniformly non-$l^{(1)}$ if and only if $Y$ is also. Thus we obtain as a Corollary of Theorem 1 that $B$-convexity is isomorphically invariant ([3], Corollary II. 6, p. 130; [1], p. 33).

DEFINITION 3. A *super-reflexive Banach space* is a Banach space $Y$ which has the property that no non-reflexive Banach space is finitely representable in $Y$.

There are many characterizations of super-reflexive spaces [11]. A Banach space is super-reflexive if and only if it is isomorphic to a space whose unit ball is uniformly convex [2]. We shall now give some relations between super-reflexivity and the uniformly non-$l^{(1)}$ property.

THEOREM 2. *If every uniformly non-$l^{(1)}$ Banach space is reflexive, then a Banach space is super-reflexive if and only if it is uniformly non-$l^{(1)}$.*

Proof. It follows from Theorem 1 that if a Banach space $Y$ is not uniformly non-$l^{(1)}$, then $l^{(1)}$ is finitely representable in $Y$ and $Y$ is not super-reflexive. Now suppose $Y$ is not super-reflexive, so that there is a non-reflexive Banach space $X$ that is finitely representable in $Y$. Also suppose every uniformly non-$l^{(1)}$ Banach space is reflexive, so that $X$ is not uniformly non-$l^{(1)}$. That $Y$ is not uniformly non-$l^{(1)}$ now follows from the facts that $X$ is not uniformly non-$l^{(1)}$ and $X$ is finitely representable in $Y$.

THEOREM 3. *If $n \geqslant 2$, $\varepsilon > 0$, and each $(n, \varepsilon)$-convex Banach space is reflexive, then each $(n, \varepsilon)$-convex Banach space is super-reflexive.*

Proof. Suppose each $(n, \varepsilon)$-convex space is reflexive. Let $Y$ be $(n, \varepsilon)$-convex and let $X$ be finitely representable in $Y$. Then $X$ is $(n, \varepsilon)$-convex. Therefore $X$ is reflexive and $Y$ is super-reflexive.

COROLLARY. *If $n \geqslant 2$ and each uniformly non-$l_n^{(1)}$ Banach space is reflexive, then each uniformly non-$l_n^{(1)}$ Banach space is super-reflexive.*

It is known that all uniformly non-square Banach spaces are reflexive ([9], Theorem 1.1). It has long been conjectured that a Banach space $X$ is reflexive if there is an $n \geqslant 2$ for which $X$ is uniformly non-$l_n^{(1)}$. A natural candidate for a counterexample is the space $J$ described below.

We shall show $J$ is not a counterexample. That is, there is no $n$ for which $J$ is uniformly non-$l_n^{(1)}$. It then follows that a counterexample must be a non-reflexive space $B$ that satisfies the conditions:

(i) $B$ has no subspace isomorphic to $c_0$, $l^{(1)}$, $J$, or any other space that is not uniformly non-$l^{(1)}$.

(ii) No non-reflexive subspace of $B$ admits an equivalent norm and a partial ordering under which it is a normed Riesz space (normed linear vector lattice) [5]. This includes the following two cases.

(iii) No non-reflexive subspace of $B$ has an unconditional basis ([3], Theorem III. 6; [9] Theorem 2.2).

(iv) $B$ is not an Orlicz space [11].

DEFINITION 4. *If $x = \{x(0), x(1), \ldots\}$ is a sequence of real numbers, define the numbers $\|x\|$ and $|||x|||$ by*

$$\|x\| = \sup\Big\{\sum_{i=1}^{n-1} [x(p_i) - x(p_{i+1})]^2\Big\}^{\frac{1}{2}},$$

and

$$|||x||| = \sup\Big\{[x(p_n) - x(p_1)]^2 + \sum_{i=1}^{n-1} [x(p_i) - x(p_{i+1})]^2\Big\}^{\frac{1}{2}},$$

where in each case the supremum is for all $n \geqslant 2$ and all increasing sequences $\{p_i\}$ of $n$ non-negative integers. Let $J$ be the collection of all real-valued sequences $x$ such that $\lim\limits_{n\to\infty} x(n) = 0$ and $|||x||| < \infty$.

It was shown in [7] and [8] that $(J, |||\cdot|||)$ is a non-reflexive Banach space with only conditional bases, that the canonical image of $J$ in its second dual has codimension 1, and that there is an isometric isomorphism of $J$ onto its second dual. Moreover, it is easily verified that $|||\cdot|||$ and $\|\cdot\|$ are equivalent, with

$$\|x\| \leqslant |||x||| \leqslant 2^{\frac{1}{2}}\|x\| \quad \text{if } x \in J.$$



It has long been known by both authors that $l^{(1)}$ is finitely representable in $(J, \|\cdot\|)$ — a lengthy argument is given in [6], with a brief outline in [3]. We will show that both $c_0$ and $l^{(1)}$ are finitely representable in both $(J, \|\cdot\|)$ and $(J, |||\cdot|||)$. This is an improvement over previous results, since $c_0$ is not finitely representable in $l^{(1)}$. Also, the present proof is much easier.

Note that we are letting sequences be functions on the non-negative integers. This is for notational convenience. In particular, $z_{2k}$ is the sequence $\{0, (2k)^{-\frac{1}{2}}, 0, (2k)^{-\frac{1}{2}}, \ldots, 0, (2k)^{-\frac{1}{2}}, 0, 0, \ldots\}$ that has exactly $k$ non-zero entries, each equal to $(2k)^{-\frac{1}{2}}$ and located at one of the positions $1, 3, \ldots$ $\ldots, 2k-1$. If $x$ is a sequence and $n$ an integer, denote by $T_n x$ the sequence $y$ for which $y$ is linear on the interval $[kn, (k+1)n]$ and

$$y(kn) = x(k) \quad \text{if } k \geqslant 0.$$

LEMMA 1. *Let $m$ be a positive integer and $\gamma$, $\delta$, $\varepsilon_1$, $\varepsilon_2$ and $N$ be positive numbers for which $\gamma < \delta$ and*

$$\gamma^{\frac{1}{2}} N^{-\frac{1}{2}} + \gamma^{\frac{1}{2}} < \delta^{\frac{1}{2}}, \; 2\varepsilon_1 + 4\gamma N^{-\frac{1}{2}} < \varepsilon_2. \tag{3}$$

*If $n > N$, if $x \in J$ has the properties*

(i) $x(i) = 0$ *if* $i \geqslant 2m$,

(ii) $\|x\|^2 < \sum\limits_{i=0}^{2m-1} |x(i) - x(i+1)|^2 + \varepsilon_1$,

(iii) $|x(i) - x(i+1)| < \left(\dfrac{\gamma}{2m}\right)^{\frac{1}{2}}$ *if* $i \geqslant 0$,

*and if $y = T_n x$, $z = \gamma^{\frac{1}{2}} z_{2mn}$, and $w = y + z$, then*

(i′) $w(i) = 0$ *if* $i \geqslant 2mn$,

(ii′) $\|w\|^2 < \sum\limits_{i=0}^{2mn-1} |w(i) - w(i+1)|^2 + \varepsilon_2$,

(iii′) $|w(i) - w(i+1)| < \left(\dfrac{\delta}{2mn}\right)^{\frac{1}{2}}$ *if* $i \geqslant 0$.

Proof. Suppose $n > N$. Clearly (i′) is satisfied. Also, it follows from (iii), the saw-tooth character of $z$, and (3) that

$$|w(i) - w(i+1)| \leqslant \frac{1}{n}\left(\frac{\gamma}{2m}\right)^{\frac{1}{2}} + \frac{\gamma^{\frac{1}{2}}}{(2mn)^{\frac{1}{2}}} = \frac{\gamma^{\frac{1}{2}} n^{-\frac{1}{2}} + \gamma^{\frac{1}{2}}}{(2mn)^{\frac{1}{2}}} < \left(\frac{\delta}{2mn}\right)^{\frac{1}{2}}$$

for every $i$, so (iii′) is satisfied. If $\{p_i\colon 1 \leqslant i \leqslant k\}$ is an arbitrary increasing sequence of positive integers for which $p_k = 2mn$, then (iii) implies

$$|y(p_i)-y(p_{i+1})| < \frac{p_{i+1}-p_i}{n}\left(\frac{\gamma}{2m}\right)^{\frac12},$$

and it follows from $z = \gamma^{\frac12} z_{2mn}$ that

$$(4)\qquad \sum_{i=1}^{k-1}|y(p_i)-y(p_{i+1})|\,|z(p_i)-z(p_{i+1})| \leqslant \sum_{i=1}^{k-1}|y(p_i)-y(p_{i+1})|\left(\frac{\gamma}{2mn}\right)^{\frac12}$$
$$< \frac{2mn}{n}\left(\frac{\gamma}{2m}\right)^{\frac12}\left(\frac{\gamma}{2mn}\right)^{\frac12} < \gamma N^{-\frac12}.$$

Now suppose $\{p_i\}$ also has the property that

$$\|w\|^2 = \sum_{i=1}^{k-1}|w(p_i)-w(p_{i+1})|^2.$$

Then it follows from (4) that

$$(5)\qquad \|w\|^2 < \sum_{i=1}^{k-1}|y(p_i)-y(p_{i+1})|^2 + \sum_{i=1}^{k-1}|z(p_i)-z(p_{i+1})|^2 + 2\gamma N^{-\frac12}.$$

For a particular $i$, suppose $r_1, \dots, r_\lambda$ are successive multiples of $n$ and

$$r_1 \leqslant p_i < r_2 < \dots < r_{\lambda-1} < p_{i+1} \leqslant r_\lambda.$$

If there is a $j$ for which $2 \leqslant j \leqslant \lambda-1$ and $y(r_j)$ is not between $y(p_i)$ and $y(p_{i+1})$, then the insertion of $r_j$ into the sequence $\{p_i\}$ does not decrease the right member of (5). If there is no such $r_j$, there is no loss of generality if we assume

$$y(r_1) \leqslant y(p_i) < y(r_j) < y(p_{i+1}) \leqslant y(r_\lambda) \qquad \text{for } 2 \leqslant j \leqslant \lambda-1.$$

Then the expression

$$(6)\qquad \Big[|y(p_i)-y(r_2)|^2 + \sum_{j=2}^{\lambda-2}|y(r_j)-y(r_{j+1})|^2 + |y(r_{\lambda-1})-y(p_{i+1})|^2 -$$
$$-|y(p_i)-y(p_{i+1})|^2\Big] - \Big[\sum_{j=1}^{\lambda-1}|y(r_j)-y(r_{j+1})|^2 - |y(r_1)-y(r_\lambda)|^2\Big]$$

is equal to twice

$$[y(r_{\lambda-1})-y(r_1)][y(r_\lambda)-y(r_2)] - [y(r_{\lambda-1})-y(p_i)][y(p_{i+1})-y(r_2)] \geqslant 0.$$

Also, let all pairs $(p_i, p_{i+1})$ for which there is at least one multiple of $n$ between $p_i$ and $p_{i+1}$ be placed alternately in two sets. Then if $(p_i, p_{i+1})$ and $(p_j, p_{j+1})$ belong to the same set and $p_{i+1} < p_j$, there are integers $r_1, r_\lambda, s_1, s_\mu$ which are multiples of $n$ and satisfy

$$r_1 \leqslant p_i < p_{i+1} \leqslant r_\lambda \leqslant s_1 \leqslant p_j < p_{j+1} \leqslant s_\mu.$$



Thus it follows from (ii) that the sum over all pairs $(p_i, p_{i+1})$ of the second bracketed expression in (6) is greater than $-2\varepsilon_1$. Therefore if $\{q_i\colon 1 \leqslant i \leqslant \varkappa\}$ is $\{p_i\}$ with all multiples of $n$ adjoined, then

$$(7)\qquad \|w\|^2 < \sum_{i=1}^{\varkappa}|y(q_i)-y(q_{i+1})|^2 + \sum_{i=1}^{\varkappa}|z(q_i)-z(q_{i+1})|^2 + 2\varepsilon_1 + 2\gamma N^{-1/2}.$$

For each $i$, $q_{i+1}-q_i \leqslant n$ and $y$ is linear between $q_i$ and $q_{i+1}$. Therefore,

$$|y(q_i)-y(q_{i+1})|^2 + |z(q_i)-z(q_{i+1})|^2 - \sum_{j=q_i}^{q_{i+1}-1}\big[|y(j)-y(j+1)|^2 + |z(j)-z(j+1)|^2\big]$$
$$\leqslant \left[\left(\frac{q_{i+1}-q_i}{n}\right)^2 - \frac{q_{i+1}-q_i}{n^2}\right]\left(\frac{\gamma}{2m}\right) + [1-(q_{i+1}-q_i)]\left(\frac{\gamma}{2mn}\right)$$
$$\leqslant [(q_{i+1}-q_i)^2 - (n+1)(q_{i+1}-q_i) + n]\frac{\gamma}{2mn^2} \leqslant 0,$$

so that (7) and another application of (4) give

$$\|w\|^2 < \sum_{i=1}^{2mn-1}|w(i)-w(i+1)|^2 + 2\varepsilon_1 + 4\gamma N^{-1/2}.$$

This and (3) imply (ii′).

The next lemma will be used several times, but its proof is relatively easy and will not be given.

LEMMA 2. *For each $n$, $T_n$ is an isometric isomorphism of $(J, \|\cdot\|)$ into itself. For each $k$, $\|z_{2k}\| = 1$.*

LEMMA 3. *For each $k \geqslant 1$ and $\varepsilon > 0$, there exist members $\xi_1, \dots, \xi_k$ of $J$ for which $\|\xi_i\| = 1$ for every $i$ and*

$$\|\xi_1 \pm \xi_2 \pm \dots \pm \xi_k\| < 1 + \varepsilon$$

*for all choices of signs.*

Proof. For arbitrary $k \geqslant 1$, suppose positive numbers $\{\gamma_i\}$ and $\{\varepsilon_i\}$ satisfy

$$\gamma_1 = 1 < \gamma_2 < \dots < \gamma_k, \qquad \varepsilon_1 < \tfrac12\varepsilon_2 < \tfrac14\varepsilon_3 < \dots < 2^{-(k-1)}\varepsilon_k.$$

Choose $N$ so that, if $1 \leqslant i < k$, then

$$\gamma_i^{1/2}N^{-1/2} + \gamma_i^{1/2} < (\gamma_{i+1})^{1/2}, \quad 2\varepsilon_i + 4\gamma_i N^{-1/2} < \varepsilon_{i+1}.$$

For $n > N$, let

$$x_i = T_{n^{k-i}}\gamma_i^{1/2} z_{2n^{i-1}} \qquad \text{if } 1 \leqslant i \leqslant k.$$

It follows from Lemma 2 that $\|z_{2m}\| = 1$ for all $m$. Thus

$$\|x_i\| = \gamma_i^{1/2} \qquad \text{for } 1 \leqslant i \leqslant k.$$

Then it follows from Lemmas 1 and 2 that

$$\|x_1 \pm x_2 \pm \ldots \pm x_k\| < 2n^{k-1}\left(\frac{\gamma_k}{2n^{k-1}}\right) + \varepsilon_k = \gamma_k + \varepsilon_k.$$

Since $\|x_i\| = \gamma_i^{1/2}$, for any $\varepsilon > 0$ we can choose $\varepsilon_k$ small enough and $\gamma_k$ near enough to 1 so that, if $\xi_i = x_i/\|x_i\|$, then

$$\|\xi_1 \pm \xi_2 \pm \ldots \pm \xi_k\| < 1 + \varepsilon$$

for all choices of signs.

THEOREM 4. *The spaces $c_0$ and $l^{(1)}$ are finitely representable in* $(J, \|\cdot\|)$ *and also in* $(J, |||\cdot|||)$.

Proof. Pick $\varepsilon > 0$ and a positive integer $k$. Let $\xi_1, \ldots, \xi_k$ be the members of $J$ given by Lemma 3, so that $\|\xi_i\| = 1$ for all $i$ and

$$\|\xi_1 \pm \xi_2 \pm \ldots \pm \xi_k\| < 1 + \varepsilon$$

for all choices of signs. If $\sup\{|a_i|\} \leqslant 1$, then $\sum_1^k a_i \xi_i$ belongs to the convex span of elements of type $\sum_1^k \pm \xi_i$. Therefore

$$\Big\|\sum_1^k a_i \xi_i\Big\| \leqslant (1+\varepsilon)\sup\{|a_i|\}.$$

Since $\|\xi_i\| = 1$ for all $i$, it follows that if $\sup\{|a_i|\} = |a_j|$, then

$$2|a_j| \leqslant \Big\|\sum_1^k a_i \xi_i\Big\| + \Big\|\sum_1^k a_i \xi_i - 2a_j \xi_j\Big\| < \Big\|\sum_1^k a_i \xi_i\Big\| + (1+\varepsilon)\sup\{|a_i|\},$$

so that $\|\sum_1^k a_i \xi_i\| > (1-\varepsilon)\sup\{|a_i|\}$. Thus $c_0$ is finitely representable in $(J, \|\cdot\|)$. Since $(J, \|\cdot\|)$ is isomorphic to $(J, |||\cdot|||)$, an argument similar to the proof of Lemma B in [10] shows that there is a space isomorphic to $c_0$ that is finitely representable in $(J, |||\cdot|||)$. This space can be as nearly isometric to $c_0$ as desired ([9], Lemma 2.2). Therefore $c_0$ is finitely representable in $(J, |||\cdot|||)$. Since $l^{(1)}$ is finitely representable in $c_0$, we conclude that $l^{(1)}$ is finitely representable in both $(J, \|\cdot\|)$ and $(J, |||\cdot|||)$.

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