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Completeness of the polynomial invariants of compact groups

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1. Introduction. Let (G, X) be an abstract geometric object [10], [13]. A real valued function $\sigma\colon X \to R$ is called an *invariant* (a *scalar invariant*) if $\sigma(x) = \sigma(gx)$ for every $g \in G$, $x \in X$, [2], [8], [12]. Denote by $\mathcal{N}(G, X)$ the set of invariants of the object (G, X).

We introduce two classes of subsets of $\mathcal{N}(G, X)$, [2], [8], [12].

Definition 1. A set $Z \subset \mathcal{N}(G, X)$ will be called a *complete set of invariants of the object* (G, X) iff from the equalities

$$\sigma(x) = \sigma(y) \quad \text{for every } \sigma \in Z,$$

it follows, that the points x, y belong to the same transitive fiber of (G, X) (i.e. there exists $g \in G$ such that $y = gx$).

Consider the space of transitive fibers X_G of the object (G, X). Let $\Pi\colon X \to X_G$ be the quotient transformation. Thus $\Pi(x) = [x]$ is the transitive fiber of point $x \in X$. Every invariant σ of (G, X) determines a real valued function $\varphi_\sigma\colon X_G \to R$ by the formula

$$\varphi_\sigma([x]) = \sigma(x).$$

Conversely, a function $\varphi\colon X_G \to R$ determines an invariant of (G, X) by the formula $\sigma = \varphi \circ \Pi$.

From the definition of a complete set of invariants it follows that the family Z distinguishes points of the space X_G, i.e. if $[x] \neq [y]$, then there exists $\sigma \in Z$ such that $\sigma(x) \neq \sigma(y)$.

Suppose $F \subset \mathcal{N}(G, X)$. Let R^F be the Cartesian product of F copies of the set of real numbers R (for F a finite or infinite set). Denote by Δ_F the diagonal transformation, $\Delta_F\colon X \to R^F$, of family of invariants F. It follows, that

$$(p_\sigma \circ \Delta_F)(x) = \sigma(x) \quad \text{for } \sigma \in F,$$

where $p_\sigma\colon R^F \to R$ is the projection on to the σ-coordinate of R^F.

Definition 2. A set $F \subset \mathcal{N}(G, X)$ is called a *functionally complete* *set of scalar invariants* if, for every invariant $a \in \mathcal{N}(G, X)$, there exists a function $\bar{a}$: $R^F \to R$ such that $a(x) = (\bar{a} \circ \Delta_F)(x)$.

In Section 3 we shall prove the following

LEMMA 1. *A subset $Z \subset \mathcal{N}(G, X)$ is complete iff Z is functionally complete.*

According to Lemma 1, complete sets of invariants are identical to functionally complete sets of invariants. However, if we restrict ourselves to continuous mappings and topological spaces, then the above property is false in general [4].

In this paper we consider the geometric objects determined by tensors, i.e. $(\mathrm{GL}(n, R), R^m)$, where $\mathrm{GL}(n, R)$ denotes the full linear group of the n-dimensional vector space, $m = n^{p+q}$, with the transformation law

$$t^{\lambda_1', \ldots, \lambda_p'}{}_{\mu_1', \ldots, \mu_q'} = A^{\lambda_1'}_{\lambda_1} \ldots A^{\lambda_p'}_{\lambda_p} A^{\mu_1}_{\mu_1'} \ldots A^{\mu_q}_{\mu_q'} t^{\lambda_1, \ldots, \lambda_p}{}_{\mu_1, \ldots, \mu_q}.$$

We shall prove, that polynomial invariants with respect to compact groups determine the complete set of invariants (concerning the vector invariants see [14], p. 441). These results will be applied to the study of the scalar differential invariants of a positive metric tensor. In Section 2 we shall give the main results of this paper (Theorems 1-3). Proofs of these theorems are presented in Sections 3-5.

2. Results. Let $a_1, \ldots, a_M$ be a finite set of tensors. The tensor a_i is an object the form $(\mathrm{GL}(n_i, R), R^{m_i})$. Let λ_i: $G \to \mathrm{GL}(n_i, R)$ be a representation of compact group G, $i = 1, \ldots, M$. Denote by (G, X) the cartesian product of the objects $(\lambda_i(G), R^{m_i})$, [12], p. 40.

Let $\sigma_1, \ldots, \sigma_b$ be a set of basic polynomial invariants ([2], p. 138, [12], p. 49-50 and 367) of the object (G, X) i.e. σ_i are polynomials of the components of tensors $a_1, \ldots, a_M$ and every polynomial invariant of (G, X) is a polynomial of the invariants $\sigma_1, \ldots, \sigma_b$.

The principal properties of the basic polynomial invariants are given in the following theorems.

THEOREM 1. *Every continuous invariant σ of the object (G, X) determined by tensors $a_1, \ldots, a_M$ and a compact group G has the form*

$$\sigma(x) = \bar{\sigma}(\sigma_1(x), \ldots, \sigma_b(x)),$$

where $\bar{\sigma}$ is a real valued function of b variables, continuous on R^b.

Concerning the completeness of the basic polynomial invariants σ_i we have the following

THEOREM 2. *The set of basic polynomial invariants σ_i of the object (G, X) determined by tensors $a_1, \ldots, a_M$ and a compact group G is a complete* *and functionally complete set of invariants. The mapping* $\Lambda: X_G \to R^b$ *of the space* X_G *of transitive fibers of* (G, X), *given by the formula*

$$\Lambda([x]) = \big(\sigma_1(x), \ldots, \sigma_b(x)\big),$$

is a closed embedding of X_G *into* R^b.

Now we shall consider scalar differential invariants of a positive metric tensor.

Denote by

$$g^{\lambda\mu}, R_{\varrho\mu\nu\lambda}, \nabla_{\nu_1} R_{\varrho\mu\nu\lambda}, \ldots, \nabla_{\nu_{p-2},\ldots,\nu_1} R_{\varrho\mu\nu\lambda}, \tag{1}$$

the metric contravariant tensor, the Riemann curvature tensor and its covariant derivatives determined by the Christoffel symbols belonging to the tensor $g_{\lambda\mu}$, respectively. In Section 4 we shall prove the following

THEOREM 3. *Polynomial invariants with respect to the full linear group* $\mathrm{GL}(n, R)$, [2], *of the tensors* (1) *constitutes a complete set of scalar differential invariants of order* p *of a positive metric tensor.*

First we shall deal with the scalar differential invariants of second order of a positive metric tensor.

Denote by $\langle \nu_1, \ldots, \nu_l \rangle \Pi$ the Young symmetrization ([12], p. 166) applied to a set of l indices. We have the following

THEOREM 4. *The invariants*

$$\sigma_p^{\Pi} = g^{\langle\lambda_1\mu_1} g^{\nu_1\varrho_1} \ldots g^{\lambda_p\mu_p} g^{\nu_p\varrho_p\rangle \Pi} R_{\langle\lambda_1\mu_1\nu_1\varrho_1} \ldots R_{\lambda_p\mu_p\nu_p\varrho_p\rangle \Pi},$$

where Π *runs over the Young symmetrizations with property*

$$g^{\langle\lambda_1\mu_1} g^{\nu_1\varrho_1} \ldots g^{\lambda_p\mu_p} g^{\nu_p\varrho_p\rangle \Pi} \neq 0, \quad p \leqslant n,$$

form a complete set of invariants of $g^{\lambda\mu}$, $R_{\varrho\mu\nu\lambda}$.

Thus the problem of finding all differential invariants of second order of a positive metric tensor is reduced by Theorem 4 to finding a Young symmetrizations with the above property.

THEOREM 5. *A base of polynomial invariants of tensors* $g^{\lambda\mu}$, $R_{\varrho\mu\nu\lambda}$ *contains at last*

$$u = \frac{n^2(n^2-1)}{12} - \frac{n(n-1)}{2}$$

invariants. The derivative of the mapping $\Delta: x \to \big(\sigma_1(x), \ldots, \sigma_b(x)\big)$ *has rank* $d\Delta(x) \leqslant u$, *where* σ_i *are basic polynomial invariants of* $g^{\lambda\mu}, R_{\varrho\mu\nu\lambda}$.

3. The proofs of Theorems 1-3 will be preceded by lemmas.

Proof of Lemma 1. Suppose that Z is a complete set of invariants of (G, X). Then the diagonal transformation $\Delta_Z: X \to R^Z$ of the family Z is a (1-1) function on the space X_G. Let σ be an invariant of (G, X).

Then we define $\bar{\sigma}\colon R^Z \to R$ as follows:

$$\bar{\sigma}(p) = \begin{cases} \sigma(x) & \text{for } p = \Delta_Z(x), \\ 1 & \text{for } p \notin \Delta_Z(X), \end{cases}$$

consequently we have $\sigma(x) = (\bar{\sigma}\circ \Delta_Z)(x)$. Thus Z is functionally complete.

Conversely, let F be a functionally complete set of invariants of (G, X). Suppose that there exist points $x_0, y_0 \in X$ such that $[x_0] \neq [y_0]$ and

$$\sigma(x_0) = \sigma(y_0) \quad \text{for every } \sigma \in F. \tag{3.1}$$

Choose an invariant a of (G, X) such that $a(x_0) \neq a(y_0)$.

By the definition of a complete set of invariants there exists a function $\bar{a}\colon R^F \to R$ with property

$$a(x) = (\bar{a}\circ \Delta_F)(x).$$

According to (3.1) we have $\Delta_F(x_0) = \Delta_F(y_0)$. Consequently we obtain $a(x_0) = (\bar{a}\circ\Delta_F)(x_0) = (\bar{a}\circ\Delta_F)(y_0) = a(y_0)$. A contradiction.

LEMMA 2. *Every continuous invariant σ of the tensors $a_1, \ldots, a_M$ is the limit of a sequence of polynomial invariants ω_n,*

$$\sigma(x) = \lim \omega_n(x),$$

where the convergence is almost uniform on X.

Before we pass to the proof of Lemma 2, we recall the fundamental properties of the invariant integral m defined by the Haar integral over a compact group G, [3], p. 367-368 and [12], p. 230-236. The invariant integral is a functional on the space of continuous real-valued functions defined on the space X, which fulfils the following conditions:

1° $m(1) = (1)$, $m(\sigma) = \sigma$, if σ is an invariant of (G, X),

2° $m(f)$ is an invariant of (G, X) for each real-valued function continuous on X,

3° $m(w)$ is a polynomial invariant of (G, X) if w is a polynomial of the components $\{a_i\}$.

Our further considerations depend upon the following Theorem, which follows from Theorem (VIII, 14.A), [12], p. 364.

(*) THEOREM. *For every object (G, X) defined by tensors $a_1, \ldots, a_M$ and compact group G there exists a finite set of polynomial invariants $\sigma_1, \ldots, \sigma_l$. Invariants σ_i fulfil a finite system of polynomial equations. Every polynomial invariant of (G, X) is a polynomial of basic invariants σ_i.*

Proof of Lemma 2. Let σ be a continuous invariant of (G, X). By the Weierstrass approximation theorem, σ is the limit

$$\sigma(x) = \lim w_n(x), \quad x \in X,$$

of an almost uniformly covergent sequences of polynomials w_n. According to 1° and 3° we have $\sigma = m(\sigma)$ and $m(w_n) = \omega_n$, where ω_n are polynomial invariants of (G, X). Since the integration is performed over a compact set G, we obtain

$$(3.2) \qquad \sigma(x) = \lim \omega_n(x), \quad x \in X,$$

where ω_n is a sequence of polynomial invariants, almost uniformly covergent to σ. This proves Lemma 2.

4. Proof of Theorem 1. Denote by $\Delta\colon X \to R^b$ the mapping

$$x \to (\sigma_1(x), \ldots, \sigma_b(x)).$$

The coordinates of the mapping Δ are polynomials. Consequently Δ is a closed mapping. The image $\Delta(A)$ of a subset $A \subset X$ is compact iff A is a compact subset of X. Let σ be a continuous invariant of (G, X). From Lemma 2 it follows that σ is the limit of an almost uniformly covergent sequence of polynomial invariants,

$$\sigma(x) = \lim \omega_n(x).$$

According to (*) Theorem we have

$$(4.1) \qquad \omega_n(x) = \theta_n(\sigma_1(x), \ldots, \sigma_b(x)),$$

where θ_n are polynomials of b variables $(\xi_1, \ldots, \xi_b) \in R^b$. Thus we have

$$(4.2) \qquad \sigma(x) = \lim \theta_n(\sigma_1(x), \ldots, \sigma_b(x)).$$

We shall prove that the sequence θ_n is almost uniformly covergent on the image $\Delta(X) \subset R^b$. Indeed, let $K \subset \Delta(X)$ be a compact subset. Then $\Delta^{-1}(K)$ is a compact subset of X. Consequently, for $\varepsilon > 0$ there exists n_0 such that $|\omega_k(x) - \omega_l(x)| < \varepsilon$ for every $x \in \Delta^{-1}(K)$ and every $k, l > n_0$. Hence the inequality $|\theta_k(\xi) - \theta_l(\xi)| < \varepsilon$ is valid for every $\xi \in K$ and every $k, l > n_0$. This proves our assumption. Thus $\theta = \lim \theta_n$ is a continuous function on the closed subspace $\Delta(X) \subset R^b$. By the Tietze theorem, there exists an extension $\bar{\sigma}\colon R^b \to R$ of the mapping θ. From equality (4.2) we obtain $\sigma(x) = \bar{\sigma}(\sigma_1(x), \ldots, \sigma_b(x))$ for $x \in X$. This com) pletes the proof of Theorem 1.

Proof of Theorem 2. Let Π be the quotient mapping, $\Pi\colon X \to X_G$, where X_G is the space of transitive fibers of (G, X).

Every transitive fiber of (G, X) is a compact subset of X as the image of the compact group G under the mapping $g \to gx$.

The space X_G is a normal topological space. Indeed, X is the countable union of closed balls B_k with center at the origin and radius k. Write $\bar{B}_k = \bigcup \{gx\colon g \in G, x \in B_k\}$; the sets $\bar{B}_k$ form a family of compact subsets of X. Consequently, the sets $A_k = \{[x]: x \in \bar{B}_k\}$ are compact subsets of X_G and $X_G = \bigcup A_k$. Therefore ([0], p. 205) the space X_G is normal.

Suppose that there exist points $x, y \in X$ such that $[x] \neq [y]$ and

$$(4.3) \qquad \sigma_i(x) = \sigma_i(y) \quad \text{for } i = 1, \ldots, b.$$

Hence

$$(4.4) \qquad \sigma(x) = \sigma(y)$$

for every continuous invariant of (G, X).

Denote by $\Lambda: X_G \to R^b$ the mapping induced by Δ on the quotient space X_G,

$$\Lambda([x]) = \Delta(x) = \big(\sigma_1(x), \ldots, \sigma_b(x)\big).$$

According to equality (4.4), every real-valued continuous function has equal values at $[x]$ and $[y]$, contrary to the Tietze Theorem. Indeed, for the mapping $\varphi: \{[x], [y]\} \to R$ with $\varphi([x]) \neq \varphi([y])$ there exist no extension to the space X_G.

Thus the mapping Λ is a (1-1) function. The mapping Δ and consequently also Λ is a closed mapping. Hence Λ is a closed embedding X_G into R^b. This proves Theorem 2.

Remark. Theorems 2 and 3 are valid for every closed G-invariant subset of X (a set $H \subset X$ is G-invariant if $gx \in H$ for each $g \in G$, $x \in H$).

5. Now we are going to prove the following

LEMMA 3. *Let (G, X) be a transitive geometric object* ([10], [13]). *Then to every invariant of the product $(G, X \times Y)$ of the objects (G, X), (G, Y) there corresponds an invariant of the object (S_{x_0}, Y), where S_{x_0} is the stability group* ([10], [13]) *of a fixed point $x_0 \in X$, and vice versa.*

Proof of Lemma 3. Let $\sigma: X \times Y \to R$ be an invariant of $(G, X \times Y)$ (i.e. $\sigma(gx, gy) = \sigma(x, y)$ for $(x, y) \in X \times Y$, $g \in G$). Then the restriction $\sigma|_{x_0 \times Y}$ is an invariant of (S_{x_0}, Y).

Conversely, suppose that $a: Y \to R$ is an invariant of (S_{x_0}, Y). Then we put

$$(5.1) \qquad \sigma(x, y) = a(g^{-1}y) \quad \text{for } x = gx_0.$$

Let $x = hx_0$. Then $x_0 = h^{-1} gx_0$ and consequently $h^{-1} g \in S_{x_0}$. Hence we get $a(h^{-1}y) = a(h^{-1}gg^{-1}y) = a(g^{-1}y)$. Thus $\sigma(x, y)$ is well defined. The function $\sigma(x, y)$ is an invariant of $(G, X \times Y)$. Indeed, suppose $\sigma(gx, gy) = \sigma(ghx_0, gy)$, where $hx_0 = x$. From (5.1) we obtain $\sigma(gx, gy) = \sigma(ghx_0, gy) = a\big((gh)^{-1}gy\big) = a(h^{-1}y) = \sigma(x, y)$. This completes the proof of Lemma 3.

Proof of Theorem 3. According to Lemma 3, the problem of finding invariants of the system

$$(5.2)\qquad g^{\lambda\mu},\ R_{\varrho\mu\nu\lambda},\ \nabla_{\nu_1}R_{\varrho\mu\nu\lambda},\ \ldots,\ \nabla_{\nu_{p-2},\ldots,\nu_1}R_{\varrho\mu\nu\lambda},$$

is equivalent to that of finding invariants of the tensors

$$(5.3)\qquad R_{\varrho\mu\nu\lambda},\ \nabla_{\nu_1}R_{\varrho\mu\nu\lambda},\ \ldots,\ \nabla_{\nu_{p-2},\ldots,\nu_1}R_{\varrho\mu\nu\lambda}$$

with respect to the orthogonal group $O(n, R)$, i.e. the stability group of the canonical form of the positive metric $g_{\lambda\mu}$. We say: orthogonal invariants.

From Theorem 2 it follows that a finite set of basic polynomial orthogonal invariants determines a complete set of invariants of the system (5.3).

Further considerations will be divided into two steps.

Step I. Every polynomial orthogonal invariant of the system (5.3) is a restriction of a polynomial invariant, with respect to $\mathrm{GL}(n, R)$, of the system (5.2). This fact follows from the theory of orthogonal invariants [12], p. 79-81 and p. 108-112. According to Theorem (II.9.A), p. 80, [12], every orthogonal invariant is a polynomial of the scalar product of covariant vectors $(u, v) = \sum_{\lambda} u_\lambda v_\lambda$. By means of the metric tensor we get the invariant expression for the product (u, v) as follows

$$(u, v) = g^{\lambda\mu}u_\lambda v_\mu.$$

After substitution the above formula to (5.1) we get an polynomial invariant of tensors (5.2) with respect to $\mathrm{GL}(n, R)$. This proves our statement.

Step II. Let $\partial^p g$ be a differential prolongation of rank p of the positive metric tensor $g_{\lambda\mu}$. By Theorem 2 the basic polynomial invariants of (5.2) determine a complete set of invariants.

Let $g_{\lambda\mu}$ and $\bar{g}_{\lambda\mu}$ denote positive metric tensors. Denote by $\partial^p g$ and $\partial^p \bar{g}$ their differential prolongations.

Let $\partial^p \bar{g}$ be obtained from $\partial^p g$ by a coordinate transformation. Consequently, they have the same normal coordinates of rank p, [7]. According to Theorem 5, cf. [7], they have the same system of tensors (5.2).

Let the tensors $\bar{g}^{\lambda\mu}$, $\bar{R}_{\varrho\mu\nu\lambda}$, $\bar{\nabla}_{\nu_1}\bar{R}_{\varrho\mu\nu\lambda}, \ldots, \bar{\nabla}_{\nu_{p-2},\ldots,\nu_1}\bar{R}_{\varrho\mu\nu\lambda}$ be obtained by a coordinate transformation of $g^{\lambda\mu}$, $R_{\varrho\mu\nu\lambda}$, $\nabla_{\nu_1}R_{\varrho\mu\nu\lambda}, \ldots, \nabla_{\nu_{p-2},\ldots,\nu_1}R_{\varrho\mu\nu\lambda}$. According to Theorem 1, cf. [7], the normal coordinates of $\partial^p\bar{g}$ are obtained from the normal coordinates of $\partial^p g$ by a transformation of group L_n^p of the form $(B_k^i, 0, \ldots, 0)$. Thus $\partial^p\bar{g}$ and $\partial^p g$ are contained in the same transitive fiber. Consequently, $\partial^p\bar{g}$ and $\partial^p g$ are contained in the same transitive fiber iff the tensors $\bar{g}^{\lambda\mu}$, $\bar{R}_{\varrho\mu\nu\lambda}$, $\bar{\nabla}_{\nu_1}\bar{R}_{\varrho\mu\nu\lambda}, \ldots, \bar{\nabla}_{\nu_{p-2},\ldots,\nu_1}\bar{R}_{\varrho\mu\nu\lambda}$ and $g^{\lambda\mu}$, $R_{\varrho\mu\nu\lambda}$, $\nabla_{\nu_1}R_{\varrho\mu\nu\lambda}, \ldots, \nabla_{\nu_{p-2},\ldots,\nu_1}R_{\varrho\mu\nu\lambda}$ have equal basic polynomial invariants. This together with Lemma 1 completes the proof of Theorem 3.

Proof of Theorem 4. According to Theorem (α), [2], p. 180, every basic polynomial invariant σ is a homogeneous polynomial of the components $g^{\lambda\mu}$, $R_{\varrho\mu\nu\lambda}$, formula (17.16), cf. [2], p. 181. Then

$$\sigma = \sum_{p=1}^{r!} c_p B_{l_1,\ldots,l_r}^{(l_1,\ldots,l_r)p}, \tag{5.4}$$

where c_p are constant numbers, $\{l_1, \ldots, l_r\}p$ is a permutation of indices $i_1, \ldots, i_r$. The tensor $B_{l_1,\ldots,l_r}^{d_1,\ldots,d_r}$ is obtained by means of tensor multiplication and contraction of indices. Any permutation of indices $i_1, \ldots, i_r$ with c_p components splits into Young symmetrizations [12]. Consequently formula (5.4) can be written as follows,

$$\sigma = \sum_{\Pi\in S_r} c'_\Pi B_{l_1,\ldots,l_r}^{\langle l_1,\ldots,l_r\rangle\Pi}, \tag{5.5}$$

where S_r denotes the set of Young symmetrizations of r indices.

We recall the fundamental properties of the Young symmetrizations S_r, [12], p. 174:

$$\Pi\Pi = \Pi, \tag{i}$$

$$\Pi\Pi' = 0 \quad \text{for } \Pi' = \text{const}\cdot\Pi, \tag{ii}$$

$$1 = \sum_{\Pi\in S_r} \Pi, \tag{iii}$$

where 1 denotes the identity permutation.

Thus we obtain

$$\sigma = \sum_{\Pi,\Pi'\in S_r} c''_{\Pi\Pi'} B_{\langle l_1,\ldots,l_r\rangle\Pi'}^{\langle l_1,\ldots,l_r\rangle\Pi}.$$

The contraction of indices with different Young symmetries is by (ii) equal zero.

Hence we obtain

$$\sigma = \sum_{\Pi\in S_r} c_\Pi B_{\langle l_1,\ldots,l_r\rangle\Pi}^{\langle l_1,\ldots,l_r\rangle\Pi}, \tag{5.6}$$

We thus come to the following formula

$$\sigma_p^\Pi = g^{\langle\lambda_1\mu_1}g^{\nu_1\varrho_1}\ldots g^{\lambda_p\mu_p}g^{\nu_p\varrho_p\rangle\Pi}R_{\langle\lambda_1\mu_1\nu_1\varrho_1}\ldots R_{\lambda_p\mu_p\nu_p\varrho_p\rangle\Pi} \tag{5.7}$$

for the basic polynomial invariants of $g^{\lambda\mu}$, $R_{\varrho\mu\nu\lambda}$. According to [12], p. 188, we conclude that $p \leqslant n$. This proves Theorem 4.

Proof of Theorem 5. Denote by $(O(n, R), \mathscr{R})$ the object defined by the curvature tensor $R_{\varrho\mu\nu\lambda}$ and the orthogonal group. The number $N = \dim \mathscr{R} = n^2(n^2-1)/12$ is the number of the essential components of the curvature tensor $R_{\varrho\mu\nu\lambda}$. Consider the transformation $R_{\varrho\mu\nu\lambda} \to R_{ab}$, $a = (\varrho\mu)$, $b = (\nu\lambda)$, into the bivector space V with $\dim V = n(n-1)/2$ ([9], p. 114). From the equality $R_{\varrho\mu\nu\lambda} = R_{\nu\lambda\varrho\mu}$ it follows that $R_{ab} = R_{ba}$.

To this transformation there corresponds the homomorphism [1], p. 30,

$$h_2 : O(n, R) \to O\left(\frac{n(n-1)}{2}, R\right).$$

According to [6], $\ker h_2 \subset \{E_n, -E_n\}$, where E_n is the $n \times n$ unity matrix. Consequently the scalar product of $x = (R_{\varrho\mu\nu\lambda})$, $\bar{x} = (\bar{R}_{\varrho\mu\nu\lambda})$, defined by

$$(5.8) \qquad (x, \bar{x}) = \sum_{\varrho\mu\nu\lambda} R_{\varrho\mu\nu\lambda}\bar{R}_{\varrho\mu\nu\lambda} = \sum_{a,b} R_{ab}\bar{R}_{ab},$$

is invariant with respect to $O(n, R)$, i.e. $(gx, g\bar{x}) = (x, \bar{x})$ for $g \in O(n, R)$. Denote by $|x-y| = \sqrt{(x-y, x-y)}$ the metric induced by (5.8).

Let $d([x], [y])$ be the Hausdorff distance on the space of transitive fibers $\mathscr{R}_0$ determined by the metric $|x-y|$. Since the group $O(n(n-1)/2)$ does not change the distance between points of $\mathscr{R}$, we conclude that

$$(5.9) \qquad d([x], [y]) \leqslant |x-y| \quad \text{for } x, y \in \mathscr{R}.$$

Consequently the topology induced by Hausdorff metric on the space $\mathscr{R}_0$ is compatible to the quotient topology.

Consider the open dense subset $D \subset \mathscr{R}$ of points with different characteristic roots of the mixed Ricci tensor.

The stability group of points $x \in D$ is a discrete subgroup of $O(n, R)$. Indeed, in view of the well-known facts of elementary linear algebra the orthogonal matrix commuting to

$$(R^\alpha_\beta) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$, has the form

$$\begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix},$$

where $\varepsilon_1^2 = 1$, $i = 1, \ldots, n$.

The stability group of the curvature tensor is contained in the stability group of its Ricci mixed tensor. Consequently it is discrete. We have an isomorphism $[x] \approx O(n, R)/S_x$, where S_x is the stability group of x.

Consequently we obtain

$$\dim[x] = \dim O(n, R) - \dim S_x = \frac{n(n-1)}{2} \tag{5.10}$$

for every $x \in D$.

Now we apply the Hurewicz dimension theorem, cf. [4], p. 124, to the space $\mathscr{R}_0$. Thus we obtain,

$$\dim \mathscr{R}_0 = N - \frac{n(n-1)}{2}. \tag{5.11}$$

By Theorem 2 invariants (5.7) determine a closed embedding of $\mathscr{R}_0$ into R^b. Consequently we obtain $b \geqslant \dim \mathscr{R}_0$. Denote by $\Delta\colon \mathscr{R} \to R^b$ the mapping $x \to (\sigma_1(x), \ldots, \sigma_b(x))$, where σ_i are a basic polynomial invariants of $(O(n, R), \mathscr{R})$. According to the above considerations we have for $x \in D$. dim ker $d\Delta(x) = n(n-1)/2$. This completes the proof of Theorem 4.

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