MODULI SPACES OF COHERENT SHEAVES ON MULTIPLES CURVES

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Contents

1. Introduction

Let S be a projective smooth irreducible surface over $\mathbb{C}$. The subject of this paper is the study of coherent sheaves on multiple curves embedded in S. Coherent sheaves on singular non reduced curves and their moduli spaces have been studied (cf. [2], [3]) and some general results have been obtained in [10] by M.-A. Inaba on moduli spaces of stable sheaves on reduced varieties of any dimension. In the case of curves we may hope of course much more detailed results.

The results of this paper come mainly from [6]. We introduce new invariants for coherent sheaves on multiple curves : the *canonical filtrations*, *generalized rank* and *degree*, and prove a *Riemann-Roch theorem*. We define the *quasi locally free sheaves* which play the same role as locally free sheaves on smooth varieties. We study more precisely the coherent sheaves on double curves. In this case we can describe completely the torsion free sheaves of generalized rank 2, and give examples of moduli spaces of stable sheaves of generalized rank 3.

This work can easily be generalized to *primitive multiple curves* which have been defined and studied by C. Bănică and O. Forster in [1].

1.1. Motivations - Moduli spaces of sheaves on multiple curves behave sometimes like moduli spaces of sheaves on varieties of higher dimension. Moduli spaces can be non reduced (this can happen only for moduli spaces of non locally free sheaves), we can observe the same phenomenon in the study of unstable rank 2 vector bundles on surfaces. Moduli spaces can have multiple components with non empty intersections, this is the case also for moduli spaces of rank 2 stable sheaves on $\mathbb{P}_3$. We can hope that the study of these phenomenons on multiple curves will be simpler than in the higher dimensional cases, and will give ideas to treat them.

2. Preliminaries

2.1. multiple curves

Let S be a smooth projective irreducible surface over $\mathbb{C}$. Let $C \subset S$ be a smooth irreducible projective curve, and $n \geq 2$ be an integer. Let $s \in H^0(\mathcal{O}_S(C))$ be a section whose zero scheme is C and C_n be the curve defined by $s^n \in H^0(\mathcal{O}_S(nC))$. We have a filtration $C = C_1 \subset C_2 \subset \cdots \subset C_n$, hence a coherent sheaf on C_i with $i < n$ can be viewed as a coherent sheaf on C_n .

Let $\mathcal{I}_{C_i}$ denote the ideal sheaf of C_i in C_n . Then $L = \mathcal{I}_C/\mathcal{I}_{C_2}$ is a line bundle on C and $\mathcal{I}_{C_j}/\mathcal{I}_{C_{j+1}} \simeq L^j$. We have $L \simeq \mathcal{O}_C(-C)$.

Let $\mathcal{O}_n = \mathcal{O}_{C_n}$. If $0 < m < n$ we can view $\mathcal{O}_m$ as a sheaf of $\mathcal{O}_n$-modules.

2.2. Extension of vector bundles

2.2.1. Theorem : *If $1 \leq i \leq n$ then every vector bundle on C_i can be extended to a vector bundle on C_n .*

2.2.2. *Parametrization* - Let $\mathbb{E}$ be a vector bundle on C_n, and $\mathbb{E}_{n-1} = \mathbb{E}_{|C_{n-1}}$, $E = \mathbb{E}_{|C}$. Then we have an exact sequence

$$0 \longrightarrow \mathbb{E}_{n-1} \otimes \mathcal{O}_{n-1}(-C) \longrightarrow \mathbb{E} \longrightarrow E \longrightarrow 0 ,$$

(with $\mathcal{O}_{n-1}(-C) = \mathcal{O}_S(-C)$).

Conversely, let $\mathbb{E}_{n-1}$ be a vector bundle on C_{n-1} and $E = \mathbb{E}_{n-1|C}$. Then using suitable locally free resolutions on C_n one can find canonical isomorphisms

$$\mathcal{H}om(E, \mathbb{E}_{n-1} \otimes \mathcal{O}_{n-1}(-C)) \ \simeq \ E^* \otimes E \otimes L^{n-1}, \quad \mathcal{E}xt^1_{\mathcal{O}_n}(E, \mathbb{E}_{n-1} \otimes \mathcal{O}_{n-1}(-C)) \ \simeq \ E^* \otimes E.$$

It follows that we have an exact sequence

$$0 \longrightarrow H^1(E^* \otimes E \otimes L^{n-1}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_n}(E, \mathbb{E}_{n-1} \otimes \mathcal{O}_{n-1}(-C)) \longrightarrow \mathrm{End}(E) \longrightarrow 0.$$

Now let $0 \to \mathbb{E}_{n-1} \otimes \mathcal{O}_{n-1}(-C) \to \mathcal{E} \to E \to 0$ be an extension, associated to $\sigma \in \mathrm{Ext}^1_{\mathcal{O}_n}(E, \mathbb{E}_{n-1} \otimes \mathcal{O}_{n-1}(-C))$. Then $\mathcal{E}$ is locally free if and only the image of σ in $\mathrm{End}(E)$ is an isomorphism. In particular if E is simple then the set of vector bundles on C_n extending $\mathbb{E}_{n-1}$ can be identified with $H^1(E^* \otimes E \otimes L^{n-1})$.

2.3. Picard group

It follows from 2.2 that we have an exact sequence of abelian groups

$$0 \longrightarrow H^1(L^{n-1}) \longrightarrow \mathrm{Pic}(C_n) \xrightarrow{r_n} \mathrm{Pic}(C_{n-1}) \longrightarrow 0,$$

where r_n is the restriction morphism. Let $\mathbf{P}_n \subset \mathrm{Pic}(C_n)$ be the subgroup consisting of line bundles whose restriction to C is $\mathcal{O}_C$. Then we have a filtration of abelian groups $O = G_0 \subset G_1 \subset \cdots G_{n-1} = \mathbf{P}_n$ such that $G_i/G_{i-1} \simeq H^1(L^i)$ for $1 \leq i \leq n-1$. Here G_i is the subgroup of $\mathbf{P}_n$ of line bundles whose restriction to C_{n-i} is trivial. It follows from this filtration that $\mathbf{P}_n$ is isomorphic to a product of groups $\mathbb{G}_a$, i.e. to a finite dimensional vector space.

3. Canonical filtrations - Generalized rank and degree

Let $P \in C$ and $z \in \mathcal{O}_{n,P}$ be a local equation of C. Let $x \in \mathcal{O}_{n,P}$ be such that x and z generate the maximal ideal of $\mathcal{O}_{n,P}$. Let M be a $\mathcal{O}_{n,P}$-module of finite type and $\mathcal{E}$ a coherent sheaf on C_n.

3.1. Canonical filtrations

3.1.1. *First canonical filtration* - For $1 \leq i \leq n+1$, let $M_i = z^{i-1}M$. The *first canonical filtration* (or simply the canonical filtration) of M is

$$M_{n+1} = \{0\} \subset M_n \subset \cdots \subset M_2 \subset M_1 = M.$$

We have

$$M_i/M_{i+1} \simeq M_i \otimes_{\mathcal{O}_{n,P}} \mathcal{O}_{C,P}, \qquad M/M_{i+1} \simeq M \otimes_{\mathcal{O}_{n,P}} \mathcal{O}_{i,P}.$$

Let $Gr(M) = \bigoplus_{i=1}^n M_i/M_{i+1}$. It is a $\mathcal{O}_{C,P}$-module.

Similarly one can define the first canonical filtration

$$0 = \mathcal{E}_{n+1} \subset \mathcal{E}_n \subset \cdots \subset \mathcal{E}_2 \subset \mathcal{E}_1 = \mathcal{E}$$

where the $\mathcal{E}_i$ are defined inductively : $\mathcal{E}_{i+1}$ is the kernel of the restriction $\mathcal{E}_i \to \mathcal{E}_{i|C}$. Let $Gr(\mathcal{E}) = \bigoplus_{i=1}^n \mathcal{E}_i/\mathcal{E}_{i+1}$. It is concentrated on C.

3.1.2. *Second canonical filtration* - The *second canonical filtration* of M

$$M^{n+1} = \{0\} \subset M^n \subset \cdots \subset M^2 \subset M^1 = M$$

is defined by $M^i = \{u; z^{n+1-i}u = 0\}$. In the same way we can define the second canonical filtration of $\mathcal{E}$

$$0 = \mathcal{E}^{n+1} \subset \mathcal{E}^n \subset \cdots \subset \mathcal{E}^2 \subset \mathcal{E}^1 = \mathcal{E}.$$

3.1.3. *Basic properties* - 1 - We have $\mathcal{E}_i = 0$ if and only if $\mathcal{E}$ is concentrated on C_{i-1} .

2 - $\mathcal{E}_i$ is concentrated on C_{n+1-i} and its first canonical filtration is $0 = \mathcal{E}_{n+1} \subset \mathcal{E}_n \subset \cdots \subset \mathcal{E}_{i+1} \subset \mathcal{E}_i$; $\mathcal{E}^{(i)}$ is concentrated on C_{n+1-i} and its second canonical filtration is $0 = \mathcal{E}^{(n+1)} \subset \mathcal{E}^{(n)} \subset \cdots \subset \mathcal{E}^{(i+1)} \subset \mathcal{E}^{(i)}$.

3 - Canonical filtrations are preserved by morphisms of sheaves.

3.1.4. *Examples* - 1 - If $\mathcal{E}$ is locally free and $E = \mathcal{E}_{|C}$, then $\mathcal{E}_i = \mathcal{E}^{(i)}$ and $\mathcal{E}_i/\mathcal{E}_{i+1} = E \otimes L^{i-1}$ for $1 \leq i \leq n$.

2 - If $\mathcal{E}$ is the ideal sheaf of a finite subscheme T of C then $\mathcal{E}_i/\mathcal{E}_{i+1} = (\mathcal{O}_C(-T) \otimes L^{i-1}) \oplus \mathcal{O}_T$ if $1 \leq i < n$, $\mathcal{E}_n = \mathcal{O}_C(-T) \otimes L^{n-1}$, $\mathcal{E}^{(i)}/\mathcal{E}^{(i+1)} = L^{i-1}$ if $2 \leq i \leq n$ and $\mathcal{E}^{(1)}/\mathcal{E}^{(2)} = \mathcal{O}_C(-T)$.

3.2. Generalized rank and degree and Riemann-Roch theorem

The integer $R(M)=rk(Gr(M))$ is called the *generalized rank* of M.

The integer $R(\mathcal{E})=rk(Gr(\mathcal{E}))$ is called the *generalized rank* of $\mathcal{E}$, and $\mathrm{Deg}(\mathcal{E})=\deg(Gr(\mathcal{E}))$ is called the *generalized degree* of $\mathcal{E}$.

3.2.1. *Example* - If $\mathcal{E}$ is locally free and $E=\mathcal{E}_{|C}$, then $R(\mathcal{E})=n.rk(E)$, and $\mathrm{Deg}(\mathcal{E})=n.\deg(E)+\frac{n(n-1)}{2}rk(E)\deg(L)$.

3.2.2. Riemann-Roch theorem : *We have* $\chi(\mathcal{E})=\mathrm{Deg}(\mathcal{E})+R(E)(1-g_C)$.

This follows immediately from the first canonical filtration. The generalized rank can be computed as follows

3.2.3. Theorem : *We have*

$$R(M) \ = \ \lim_{p\to\infty}\Big(\frac{1}{p}\dim_{\mathbb{C}}(M\otimes_{\mathcal{O}_{n,P}}\mathcal{O}_{n,P}/(x^p))\Big).$$

This result can be used to prove that the generalized rank and degree are *additive* :

3.2.4. Corollary : *1 - Let* $0\longrightarrow M'\longrightarrow M\longrightarrow M''\longrightarrow 0$ *be an exact sequence of* $\mathcal{O}_{n,P}$*-modules of finite type. Then we have* $R(M)=R(M')+R(M'')$.

2 - Let $0\longrightarrow\mathcal{E}'\longrightarrow\mathcal{E}\longrightarrow\mathcal{E}''\longrightarrow 0$ *be an exact sequence of coherent sheaves on* C_n*. Then we have* $R(\mathcal{E})=R(\mathcal{E}')+R(\mathcal{E}'')$ *and* $\mathrm{Deg}(\mathcal{E})=\mathrm{Deg}(\mathcal{E}')+\mathrm{Deg}(\mathcal{E}'')$.

Proof. This follows from Theorem 3.2.3. The assertion on degrees follows from the one on ranks and from the Riemann-Roch theorem. □

The generalized rank and degree are invariant by deformation.

3.2.5. *Hilbert polynomial and (semi-)stability* - Let $\mathcal{D}$ be a line bundle on C_n and $D=\mathcal{D}_{|C}$. Then for every coherent sheaf $\mathcal{E}$ on C_n we have

$$R(\mathcal{E}\otimes\mathcal{D})=R(\mathcal{E}),\qquad Deg(\mathcal{E}\otimes\mathcal{D})=\mathrm{Deg}(\mathcal{E})+R(\mathcal{E})\deg(D).$$

Hence if $\mathcal{O}(1)$ is a very ample line bundle on C_n then the Hilbert polynomial of $\mathcal{E}$ with respect to $\mathcal{O}(1)$ is

$$P_{\mathcal{E}}(m) \ = \ \chi(\mathcal{E})+R(\mathcal{E})\deg(\mathcal{O}(1)_{|C}).m \quad .$$

It follows that a coherent sheaf $\mathcal{E}$ of positive rank is *semi-stable* (resp. *stable*) if and only if it is pure of dimension 1 (i.e. it has no subsheaf with finite support) and if for every proper subsheaf $\mathcal{F}\subset\mathcal{E}$ we have

$$\frac{\mathrm{Deg}(\mathcal{F})}{R(\mathcal{F})} \ \leq \ \frac{\mathrm{Deg}(\mathcal{E})}{R(\mathcal{E})} \qquad\qquad (\text{resp. } <\ ).$$

4. Quasi locally free sheaves

Let $P \in C$ and $z \in \mathcal{O}_{n,P}$ be a local equation of C.

Let M be a $\mathcal{O}_{n,P}$-module of finite type. Then M is called *quasi free* if there exist non negative integers $m_1, \ldots, m_n$ and an isomorphism $M \simeq \oplus_{i=1}^{n} m_i \mathcal{O}_{i,P}$. The integers $m_1, \ldots, m_n$ are uniquely determined : it is easy to recover them from the first canonical filtration of M. We say that $(m_1, \ldots, m_n)$ is the *type* of M.

Let $\mathcal{E}$ be a coherent sheaf on C_n. We say that $\mathcal{E}$ is *quasi free at* P if $\mathcal{E}_P$ is quasi free, and that $\mathcal{E}$ is *quasi locally free* if it is quasi free at every point of C.

4.0.6. Theorem : *The $\mathcal{O}_{n,P}$-module M is quasi free if and only if $Gr(M)$ is a free $\mathcal{O}_{C,P}$-module, if and only if all the M_i/M_{i+1} are free $\mathcal{O}_{C,P}$-modules.*

It follows that the set of points $P \in C$ such that $\mathcal{E}$ is quasi free at P is open and nonempty, and that $\mathcal{E}$ is quasi locally free if and only if $Gr(\mathcal{E})$ is a vector bundle on C, if and only if all the $\mathcal{E}_i/\mathcal{E}_{i+1}$ are vector bundles on C.

5. Coherent sheaves on double curves

We work in this section on C_2, that we call a *double curve*. If $\mathcal{E}$ is a coherent sheaf on C_2, let $E_{\mathcal{E}} \subset \mathcal{E}$ (resp. $G_{\mathcal{E}} \subset \mathcal{E}$) be its first (resp. second) canonical filtration. Let $F_{\mathcal{E}} = \mathcal{E}/E_{\mathcal{E}}$.

For $P \in C$, let z be an equation of C in $\mathcal{O}_{2,P}$. Let $x \in \mathcal{O}_{2,P}$ such that x, z generate the maximal ideal of $\mathcal{O}_{2,P}$.

5.1. Quasi locally free sheaves

5.1.1. *Locally free resolutions of vector bundles on C* - Let F be a vector bundle on C. Using theorem 2.2.1 we find a locally free sheaf $\mathbb{F}$ on C_2 such that $\mathbb{F}_{|C} = F$, and a free resolution of F on C_2 :

$$\cdots \mathbb{F} \otimes \mathcal{O}_2(-2C) \longrightarrow \mathbb{F} \otimes \mathcal{O}_2(-C) \longrightarrow \mathbb{F} \longrightarrow F \longrightarrow 0.$$

From this it follows that for every vector bundle E on C we have

$$\mathcal{E}xt^i_{\mathcal{O}_2}(F, E) \ \simeq \ \mathcal{H}om(F \otimes L^i, E)$$

for $i \geq 1$.

5.1.2. *Construction of quasi locally free sheaves* - Let $\mathcal{F}$ be a quasi locally free coherent sheaf on C_2. Let $E = E_{\mathcal{F}}$, $F = F_{\mathcal{F}}$. We have an exact sequence

$$(*) \qquad\qquad 0 \longrightarrow E \longrightarrow \mathcal{F} \longrightarrow F \longrightarrow 0$$

and E, F are vector bundles on C_2. The canonical morphism $\mathcal{F} \otimes \mathcal{I}_C \to \mathcal{F}$ comes from a surjective morphism $\Phi_{\mathcal{F}} : F \otimes L \to E$.

Conversely suppose we want to construct the quasi locally free sheaves $\mathcal{F}$ whose first canonical filtration gives the exact sequence $(*)$. For this we need to compute $\mathrm{Ext}^1_{\mathcal{O}_2}(F,E)$. The Ext spectral sequence gives the exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}_C}(F,E) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}_2}(F,E) & \xrightarrow{\beta} & \mathrm{Hom}(F\otimes L,E) & \longrightarrow & 0 \\
 & & \| & & & & \| & & \\
 & & H^1(\mathcal{H}om(F,E)) & & & & H^0(\mathcal{E}xt^1_{\mathcal{O}_2}(F,E)) & &
\end{array}$$

Let $\sigma\in\mathrm{Ext}^1_{\mathcal{O}_2}(F,E)$ and $0\to\ E\to\mathcal{E}\to F\to 0$ the corresponding extension. Then it is easy to see that this exact sequence comes from the canonical filtration of $\mathcal{E}$ if and only if $\beta(\sigma)$ is surjective. Moreover in this case we have $\Phi_{\mathcal{E}}=\beta(\sigma)$.

5.1.3. *Second canonical filtration* - Let $\Gamma_{\mathcal{F}}=\Gamma$ be the kernel of the surjective morphism $\Phi_{\mathcal{F}}\otimes I_{L^*}:F\to E\otimes L^*$ and G the kernel of the composition

$$\mathcal{F}\longrightarrow F\xrightarrow{\Phi_{\mathcal{F}}\otimes I_{L^*}} E\otimes L^* \quad ,$$

which is also surjective. Then G in the maximal subsheaf of $\mathcal{F}$ which is concentrated on C. In other words, $G\subset\mathcal{F}$ is the second canonical filtration of $\mathcal{F}$, and $G=G_{\mathcal{F}}$.

5.1.4. *Duality and tensor products* - If M is a $\mathcal{O}_{2,P}$-module of finite type, let $M^\vee$ be the dual of M : $M^\vee=\mathrm{Hom}(M,\mathcal{O}_{2,P})$. If $\mathcal{F}$ is a coherent sheaf on C_2 let $\mathcal{E}^\vee$ denote the dual sheaf of $\mathcal{E}$, i.e. $\mathcal{E}^\vee=\mathcal{H}om(\mathcal{E},\mathcal{O}_2)$. If N is a $\mathcal{O}_{C,P}$-module of finite type, let N^* be the dual of N : $N^*=\mathrm{Hom}(N,\mathcal{O}_{C,P})$. If E is a coherent sheaf on C let E^* be the dual of E on C. We use different notations on C and C_2 because $E^\vee\neq E^*$, we have $E^\vee=E^*\otimes L$.

Let $\mathcal{F}$ be a quasi locally free sheaf on C_2. Then $\mathcal{F}^\vee$ is also quasi locally free, and we have

$$E_{\mathcal{F}^\vee}\ \simeq\ E^*_{\mathcal{F}}\otimes L^2,\qquad F_{\mathcal{F}^\vee}\ \simeq\ G^*_{\mathcal{F}}\otimes L,\qquad G_{\mathcal{F}^\vee}\ \simeq\ F^*_{\mathcal{F}}\otimes L.$$

Let $\mathcal{E}$, $\mathcal{F}$ be quasi locally free sheaves on C_2. Then $\mathcal{E}\otimes\mathcal{F}$ is quasi locally free and we have $E_{\mathcal{E}\otimes\mathcal{F}}=E_{\mathcal{E}}\otimes E_{\mathcal{F}}\otimes L^*$, $F_{\mathcal{E}\otimes\mathcal{F}}=F_{\mathcal{E}}\otimes F_{\mathcal{F}}$, $G_{\mathcal{E}\otimes\mathcal{F}}=G_{\mathcal{E}}\otimes G_{\mathcal{F}}\otimes L^*$.

If $\mathcal{E}$, $\mathcal{F}$ are quasi locally free sheaves on C_2 then the canonical morphism $\mathcal{E}^\vee\otimes\mathcal{F}\to\mathcal{H}om(\mathcal{E},\mathcal{F})$ is not in general an isomorphism. For instance $\mathcal{O}_C^\vee\otimes\mathcal{O}_C=L$ but $\mathcal{H}om(\mathcal{O}_C,\mathcal{O}_C)=\mathcal{O}_C$. We have an exact sequence

$$0\longrightarrow\Gamma^*_{\mathcal{E}}\otimes\Gamma_{\mathcal{F}}\otimes L\longrightarrow\mathcal{E}^\vee\otimes\mathcal{F}\longrightarrow\mathcal{H}om(\mathcal{E},\mathcal{F})\longrightarrow\Gamma^*_{\mathcal{E}}\otimes\Gamma_{\mathcal{F}}\longrightarrow 0\ .$$

The sheaves canonically associated to $\mathcal{H}=\mathcal{H}om(\mathcal{E},\mathcal{F})$ are : $E_{\mathcal{H}}=\mathcal{H}om(E_{\mathcal{E}},E_{\mathcal{F}}\otimes L)$,
$G_{\mathcal{H}}=\mathcal{H}om(F_{\mathcal{E}},G_{\mathcal{F}})$, and we have exact sequences

$$0\longrightarrow\mathcal{H}om(\Gamma_{\mathcal{E}},E_{\mathcal{F}})\oplus\mathcal{H}om(E_{\mathcal{E}},\Gamma_{\mathcal{F}}\otimes L)\longrightarrow F_{\mathcal{H}}\longrightarrow\mathcal{H}om(\Gamma_{\mathcal{E}},\Gamma_{\mathcal{F}})\oplus\mathcal{H}om(E_{\mathcal{E}},E_{\mathcal{F}})\longrightarrow 0,$$

$$0\longrightarrow\mathcal{H}om(\Gamma_{\mathcal{E}},E_{\mathcal{F}})\oplus\mathcal{H}om(E_{\mathcal{E}},\Gamma_{\mathcal{F}}\otimes L)\longrightarrow\Gamma_{\mathcal{H}}\longrightarrow\mathcal{H}om(\Gamma_{\mathcal{E}},\Gamma_{\mathcal{F}})\longrightarrow 0.$$

5.2. Torsion free sheaves

A coherent sheaf on C_2 is called *torsion free* if it is pure of dimension 1, i.e. if it has no subsheaf with a zero dimensional support.

5.2.1. *First properties* - Let $\mathcal{E}$ be a coherent sheaf on C_2, $E \subset \mathcal{E}$ its canonical filtration. Then $\mathcal{E}$ is torsion free if and only if E is a vector bundle on C. The quotient $\mathcal{E}/E$ may be non locally free. Let $\mathcal{E}/E \simeq F \oplus T$, where F is locally free on C and T supported on a finite subset of C. The kernel of the morphism $\mathcal{E} \to T$ deduced from this isomorphism is a quasi locally free subsheaf $\mathcal{F}$ of $\mathcal{E}$ containing E, and $E \subset \mathcal{F}$ is its canonical filtration. Note that $\mathcal{F}$ may not be unique, it depends on the above isomorphism. The morphism $\Phi_{\mathcal{F}} : F \otimes L \to E$ does not depend on $\mathcal{F}$ since it comes from the canonical morphism $\mathcal{E} \otimes \mathcal{I}_C \to \mathcal{E}$. So we will note $\Phi_{\mathcal{E}} = \Phi_{\mathcal{F}}$.

If T is a torsion sheaf on C, let $\widetilde{T} = \mathcal{E}xt^1_{\mathcal{O}_C}(T, \mathcal{O}_C)$, which is (non canonically) isomorphic to T.

5.2.2. Lemma : *Let T a torsion sheaf on C and $\mathbb{F}$ a vector bundle on C_2. Let $F = \mathbb{F}_{|C}$. Then*

1 - The canonical morphism $\mathrm{Ext}^1_{\mathcal{O}_2}(T, \mathbb{F}) \to \mathrm{Ext}^1_{\mathcal{O}_2}(T, F)$ *vanishes.*

2 - We have a canonical isomorphism $\mathrm{Ext}^1_{\mathcal{O}_2}(T, \mathbb{F}) \simeq \mathrm{Ext}^1_{\mathcal{O}_2}(T, F \otimes L)$*, and* $\mathrm{Ext}^i_{\mathcal{O}_2}(T, \mathbb{F}) = \{0\}$ *if $i \geq 2$.*

3 - If $j \geq 1$ we have $\mathrm{Ext}^j_{\mathcal{O}_2}(T, F) \simeq \mathrm{Ext}^1_{\mathcal{O}_2}(T, F \otimes L^{1-j}) \simeq \mathrm{Hom}(F^* \otimes L^{j-1}, \widetilde{T})$.

Let $\sigma_{\mathcal{E}}$ be the element of $\mathrm{Ext}^1_{\mathcal{O}_2}(T, E)$ coming from the exact sequence $0 \to E \to \mathcal{E} \to F \oplus T \to 0$. From the preceding lemma we can view $\sigma_{\mathcal{E}}$ as a morphism $E^* \to \widetilde{T}$. *This morphism is surjective.*

5.2.3. *Construction of torsion free sheaves* - We start with the following data : two vector bundles E, F on C, a torsion sheaf T on C and surjective morphisms $\Phi : F \otimes L \to E$ and $\sigma : E^* \to \widetilde{T}$.

Let $\mathcal{F}$ a quasi locally free sheaf on C_2 such that $\Phi_{\mathcal{F}} = \Phi$ (see 5.1.2). From $\mathcal{F}$ and σ we get an element of $\mathrm{Ext}^1_{\mathcal{O}_2}(F \oplus T, E)$ corresponding to an extension $0 \to E \to \mathcal{E} \to F \oplus T \to 0$. It is then easy to see that $E \subset \mathcal{E}$ is the canonical filtration of $\mathcal{E}$ and that $\sigma_{\mathcal{E}} = \sigma$.

5.2.4. *Second canonical filtration* - Let G be the kernel of the morphism

$$\mathcal{E} \longrightarrow F \xrightarrow{\Phi_{\mathcal{E}} \otimes I_{L^*}} E \otimes L^*.$$

Then G in the maximal subsheaf of $\mathcal{E}$ which is concentrated on C. In other words, $G \subset \mathcal{E}$ is the second canonical filtration of $\mathcal{F}$.

5.2.5. Proposition : *There exist a quasi locally free sheaf $\mathcal{V}$ and a surjective morphism $\mathcal{V} \to T$ such that $\mathcal{E} \simeq \ker(\alpha)$.*

5.2.6. Proposition : *1 - A $\mathcal{O}_{2,P}$-module is reflexive if and only it is torsion free.*

2 - A coherent sheaf on C_2 is reflexive if and only if it is torsion free.

If $m \geq 1$, let $I_{m,P} =\subset \mathcal{O}_{2,P}$ be the ideal generated by x^m and z.

5.2.7. *Local structure of torsion free sheaves* - Let M be a torsion free $\mathcal{O}_{2,P}$-module. Then there exist integers m, q, $n_1, \cdots, n_p$ such that

$$M \;\simeq\; \left(\bigoplus_{i=1}^{p} I_{n_i,P}\right) \oplus m\mathcal{O}_{2,P} \oplus q\mathcal{O}_{C,P}.$$

5.3. Deformations of sheaves

If E is a coherent sheaf on C then the canonical morphism

$$\mathrm{Ext}^1_{\mathcal{O}_n}(E,E) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_S}(E,E)$$

is an isomorphism.

Let M be a $\mathcal{O}_{2,P}$-module, $M_2 \subset M$ its canonical filtration. Let $r_0(M) = rk(M_2)$. Then we have $R(M) \geq 2r_0(M)$. If M is quasi free then we have $R(M) = 2r_0(M)$ if and only if M is free.

5.3.1. Proposition : *Let M be a quasi free $\mathcal{O}_{2,P}$-module, and r_0 an integer such that $0 < 2r_0 \leq R(M)$. Then M can be deformed in quasi free modules N such that $r_0(N) = r_0$ if and only if $r_0 \geq r_0(M)$.*

It follows that if a quasi locally free sheaf $\mathcal{E}$ on C_2 can be deformed in quasi locally free sheaves $\mathcal{F}$ such that $r_0(\mathcal{F}) = r_0$ then we must have $r_0 \geq r_0(\mathcal{E})$. The converse is not true : if $R(\mathcal{E})$ is even, and if $\mathcal{E}$ can be deformed in locally free sheaves, then we have $\mathrm{Deg}(\mathcal{E}) \equiv \frac{R(\mathcal{E})}{2}\deg(L) \pmod 2$. Hence if this equality is not true it is impossible to deform $\mathcal{E}$ in locally free sheaves.

5.4. Rank 2 sheaves

There are 3 kinds of torsion free sheaves of generalized rank 2 on C_2 :

– the rank 2 vector bundles on C,
– the line bundles on C_2,
– the sheaves of the form $\mathcal{I}_Z \otimes \mathcal{L}$, where Z is a nonempty finite subscheme of C, $\mathcal{I}_Z$ is its ideal sheaf on C_2 and $\mathcal{L}$ a line bundle on C_2.

Line bundles can be deformed only in line bundles. The sheaves $\mathcal{I}_Z \otimes \mathcal{L}$ can only be deformed in sheaves of the same type $\mathcal{I}_{Z'} \otimes \mathcal{L}'$, with $h^0(\mathcal{O}_{Z'}) = h^0(\mathcal{O}_Z)$, $\mathrm{Deg}(\mathcal{L}') = \mathrm{Deg}(\mathcal{L})$. Only the rank-2 vector bundles can be deformed in sheaves of another type.

Let Z be a finite subscheme of C and $\mathcal{L}$ a line bundle on C_2. Let $\mathcal{E} = \mathcal{I}_Z \otimes \mathcal{L}$. Then Z is uniquely determined by $\mathcal{E}$, but $\mathcal{L}$ need not be unique. The integer $i_0(\mathcal{E}) = h^0(\mathcal{O}_Z)$ is called the *index* of $\mathcal{E}$ (in particular the index of a line bundle on C_2 is 0). It is invariant by deformation of $\mathcal{E}$.

5.4.1. Proposition : *Let $p \geq 0$, d be integers and E a rank 2 vector bundle of degree d on C. Let $q = \frac{1}{2}(d + \deg(L) + p)$. Then*

1 - If E can be deformed in torsion free sheaves on C_2 that are not concentrated on C and of index p, then there exist a line bundle V on C of degree q and a non zero morphism $\alpha : V \to E$ such that $\operatorname{Hom}((E/\operatorname{im}(\alpha)) \otimes L, V) \neq \{0\}$.

2 - If E has a sub-line bundle V of degree q such that $\operatorname{Hom}((E/\operatorname{im}(\alpha)) \otimes L, V) \neq \{0\}$ then E can be deformed in torsion free sheaves on C_2, non concentrated on C and of index p.

5.5. Rank 3 sheaves

Let $l = -\deg(L)$. We suppose that $l = C^2 \geq 1$. Let $\mathcal{E}$ be a quasi locally free sheaf on C_2 not concentrated on C and of generalized rank 3. Then $\mathcal{E}$ is locally isomorphic to $\mathcal{O}_2 \oplus \mathcal{O}_C$. We have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \Gamma_{\mathcal{E}} & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & E_{\mathcal{E}} & \longrightarrow & \mathcal{E} & \longrightarrow & F_{\mathcal{E}} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & G_{\mathcal{E}} & \longrightarrow & \mathcal{E} & \longrightarrow & E_{\mathcal{E}} \otimes L^* & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \Gamma_{\mathcal{E}} & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

where the rows come from the first and second canonical filtrations. We have $rk(E_{\mathcal{E}}) = rk(\Gamma_{\mathcal{E}}) = 1$ and $rk(F_{\mathcal{E}}) = rk(G_{\mathcal{E}}) = 2$.

The rank and degree of $E_{\mathcal{E}}$, $F_{\mathcal{E}}$, $G_{\mathcal{E}}$ and $\Gamma_{\mathcal{E}}$ are invariant by deformation of $\mathcal{E}$.

5.5.1. Proposition : *The sheaf $\mathcal{E}$ is (semi-)stable if and only if*

(i) *For every sub-line bundle D' of $G_{\mathcal{E}}$ we have $\deg(D') \leq \mu(\mathcal{E})$ (resp. $<$).*
(ii) *For every quotient line bundle D'' of $F_{\mathcal{E}}$ we have $\mu(\mathcal{E}) \leq \deg(D'')$ (resp. $<$).*

It follows that if $F_{\mathcal{E}}$ and $G_{\mathcal{E}}$ are stable then so is $\mathcal{E}$. Let $\epsilon = \deg(E_{\mathcal{E}})$, $\gamma = \deg(\Gamma_{\mathcal{E}})$. We have then $\operatorname{Deg}(\mathcal{E}) = 2\epsilon + \gamma + l$. By considering the subsheaves $E_{\mathcal{E}}$, $G_{\mathcal{E}}$ of $\mathcal{E}$ we find that if $\mathcal{E}$ is semi-stable (resp. stable) then $\gamma - 2l \leq \epsilon \leq l + \gamma$ (resp $<$).

We suppose now that $\gamma - l < \epsilon < \gamma$, which is equivalent to $\mu(E_{\mathcal{E}}) < \mu(G_{\mathcal{E}})$ and $\mu(\Gamma_{\mathcal{E}}) < \mu(F_{\mathcal{E}})$. We impose these conditions to allow $F_{\mathcal{E}}$ and $G_{\mathcal{E}}$ to be stable. In this case we get distinct components of the moduli spaces of stable sheaves corresponding to sheaves $\mathcal{E}$ such that $E_\mathcal{E}$ and $\Gamma_\mathcal{E}$ have fixed degrees. In the description of these components two moduli spaces of *Brill-Noether pairs* on C will appear : the one corresponding to $E_\mathcal{E} \subset G_\mathcal{E}$, and the one corresponding to $\Gamma_\mathcal{E} \subset F_\mathcal{E}$.

Let $M(3, 2\epsilon + \gamma + l)$ denote the moduli space of semi-stable sheaves on C_2 of generalized rank 3 and generalized degree $2\epsilon + \gamma + l$. Let $\mathcal{M}_s(\epsilon, \gamma)$ be the open subscheme of $M(3, 2\epsilon + \gamma + l)$ corresponding to quasi locally free sheaves $\mathcal{E}$ not concentrated on C, such that $\deg(E_\mathcal{E}) = \epsilon$, $\deg(\Gamma_\mathcal{E}) = \gamma$ and such that $F_\mathcal{E}$, $G_\mathcal{E}$ are stable.

5.5.2. Proposition : *The variety $\mathcal{M}_s(\epsilon, \gamma)$ is irreducible of dimension $5g + 2l - 4$. The associated reduced variety is smooth.*

The varieties $\mathcal{M}_s(\epsilon, \gamma)$ are not reduced. Let $\mathcal{M}_s^{red}(\epsilon, \gamma)$ be the reduced variety corresponding to $\mathcal{M}_s(\epsilon, \gamma)$. Then if $\mathcal{E} \in \mathcal{M}_s(\epsilon, \gamma)$ then the cokernel of the canonical map $T\mathcal{M}_s^{red}(\epsilon, \gamma)_\mathcal{E} \to T\mathcal{M}_s(\epsilon, \gamma)_\mathcal{E}$ is isomorphic to $H^0(L^*)$.

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