Introduction [to be written]

1. How do categories appear in modern mathematics?

The question "How do categories appear in modern mathematics?" has many answers; this section is devoted to only one of them, far away from the original answer visible in the joint work of S. Eilenberg and S. Mac Lane, and our presentation is very brief of course…

First, thinking of mathematics as the study of abstract mathematical structures, such as groups, rings, topological spaces, etc., we ask: what is a mathematical structure in general? And, having Bourbaki structures in mind, we might answer:

• We begin with two finite collections of sets: *constant* sets $E_1, \dots , E_m$ and *variable* sets $X_1, \dots , X_n$.
• We build a *scale*, which is a sequence of sets obtained from the sets above by taking finite products and power sets, and by iterating these operations.
• A *type* is a *uniformly* defined subset $T(X_1,\dots,X_n)$ of a set in such a scale, and a structure of that type on the sets $X_1, \dots , X_n$ is an element s in $T(X_1,\dots,X_n)$; one then also says that $(X_1,\dots,X_n,s)$ is a structure of the type T. Making the term "uniformly" precise would be a long story, which we omit; let us only mention that considering various structures of a given type T, we will fix the sets $E_1, \dots , E_m$, but not the sets $X_1, \dots , X_n$ – which explains why we write $T(X_1,\dots,X_n)$ and not $T(E_1,\dots,E_m,X_1,\dots,X_n)$.

For the readers not familiar with Bourbaki structures it might be helpful to consider the following simple examples, where, as for most basic mathematical structures, we have $m = 0$ and $n = 1$:

Example 1.1. (a) A topology on a set X is an element of the set

$$T = T(X) = \{\tau \in PP(X) \mid \tau \text{ is closed under arbitrary unions and finite intersections}\},$$

where $P(X)$ denotes the power set of X;

(b) a binary operation on a set X is an element of the set

$$T = T(X) = \{m \in P(X\times X\times X) \mid m \text{ determines a map } X\times X \to X\}. \square$$

It turns out that every mathematical structure ever considered in mathematics can indeed be presented an $(X_1,\dots,X_n,s)$ above, and moreover, using the fact that arbitrary bijections $f_1 : X_1 \to X'_1, \dots , f_n : X_n \to X'_n$ induce a bijection $T(f_1,\dots,f_n) : T(X_1,\dots,X_n) \to T(X'_1,\dots, X'_n)$, it is easy to define a general notion of an isomorphism for structures of the same type:

Definition 1.2. Let $(X_1,\dots,X_n,s)$ and $(X'_1,\dots, X'_n,s')$ be mathematical structures of the same type T; an isomorphism

$$(f_1,\dots,f_n) : (X_1,\dots,X_n,s) \to (X'_1,\dots, X'_n,s'),$$

is a family of bijections $f_1 : X_1 \to X'_1, \dots , f_n : X_n \to X'_n$ with $T(f_1,\dots,f_n)(s) = s'$. $\square$

However, *we are not able to define structure preserving maps (=homomorphisms) in general*. The best we can do, is:

Definition 1.3. Let T be a type. For structures $(X_1,\dots,X_n,s)$ and $(X'_1,\dots, X'_n,s')$ of the same type T, a map

$$(f_1,\dots,f_n) : (X_1,\dots,X_n,s) \to (X'_1,\dots, X'_n,s'),$$

is a family of maps $f_1 : X_1 \to X'_1, \dots , f_n : X_n \to X'_n$. A class **M** of such maps is said to be a class of morphisms, if it satisfies the following conditions:

(a) If $(f_1,\dots,f_n) : (X_1,\dots,X_n,s) \to (X'_1,\dots, X'_n,s')$ and $(f'_1,\dots,f'_n) : (X'_1,\dots, X'_n,s') \to (X''_1,\dots, X''_n,s'')$ are in **M**, then so is $(f'_1f_1,\dots,f'_nf_n) : (X_1,\dots,X_n,s) \to (X''_1,\dots, X''_n,s'')$;

(b) the class of invertible morphisms in **M** coincides with the class of isomorphisms in the sense of Definition 1.2. □

Accordingly, our study of the structures of a given type T will depend on the chosen class **M** of morphisms – suggesting that it is a study of a new structure whose "elements" are structures of the type T and the elements of **M**. And such a new structure is first of all a category of course, but is it merely a category? Would not replacing our T and **M** with an abstract category *trivialize* our study? In other words, is abstract category theory powerful enough to express deep properties of classical mathematical structures and simple enough to clarify those properties and to help proving them? Answering these questions seriously, and especially saying well-motivated "yes" to the last one, is not what we can do in a few page section of these notes. But the following definition, of one of the oldest categorical definitions, due to S. Mac Lane, should give some initial indication of the remarkable power of the categorical approach:

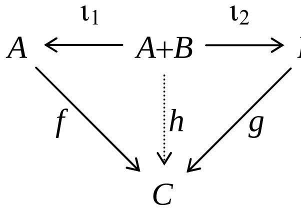
Definition 1.4. The product of two objects A and B in a category **C** is an object $A\times B$ in **C** together with two morphisms $\pi_1 : A\times B \to A$ and $\pi_2 : A\times B \to B$, such that for every object C and morphisms $f : C \to A$ and $g : C \to B$, there exists a unique morphism $h : C \to A\times B$ making the diagram

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{f} & \downarrow^{h} & \searrow^{g} & \\ A & \xleftarrow[\pi_1]{} & A\times B & \xrightarrow[\pi_2]{} & B \end{array} \qquad (1.1)$$

commute, i.e. satisfying $\pi_1h = f$ and $\pi_2h = g$. □

This so simple definition is equivalent to the familiar ones in essentially all important categories of interest in algebra and geometry/topology, and the same is true for its dual, which is:

Definition 1.5. The coproduct of two objects A and B in a category **C** is an object $A+B$ in **C** together with two morphisms $\iota_1 : A \to A+B$ and $\iota_2 : B \to A+B$, such that for every object C and morphisms $f : A \to C$ and $g : B \to C$, there exists a unique morphism $h : A+B \to C$ making the diagram



(1.2)

commute, i.e. satisfying $h\iota_1 = f$ and $h\iota_2 = g$. □

Furthermore, these categorical definitions give a new insight into our understanding of very first mathematical concepts, such as multiplication and addition of natural numbers, intersection, product, and union of sets, and conjunction and disjunction in mathematical logic. In particular they make addition dual to multiplication and make disjoint union more natural than the ordinary one. In simple words, everyone knows that, say,

$a + b = b + a$ and $ab = ba$ (for natural a and b),

but only category theory tells us that these equalities are special cases of a single result!

2. Isomorphism and equivalence of categories

The purpose of this section is to list and prove basic properties of isomorphisms and equivalences of categories. We assume that the readers are familiar with:

• Isomorphisms in general categories: they compose, they have uniquely determined inverses that are isomorphisms themselves, and they determine the isomorphism relation $\approx$ on the set of objects of the given category; and that relation is an equivalence relation.
• Isomorphisms of categories: the following condition on a functor $F : \mathbf{A} \to \mathbf{B}$ are equivalent: (a) F is an isomorphism; (b) F is bijective on objects and on morphisms; (c) F is bijective on objects and fully faithful (recall that "fully faithful" means "bijective of hom sets").
• Isomorphism of functors: a natural transformation $\tau : F \to G$ of functors $\mathbf{A} \to \mathbf{B}$ is an isomorphism if and only if the morphism $\tau_A : F(A) \to G(A)$ is an isomorphism for each object A in $\mathbf{A}$. The isomorphism relation is a *congruence* on the category of all categories, i.e. if (F,F') and (G,G') are composable pairs of functors, then $F \approx F'$ & $G \approx G' \Rightarrow FF' \approx GG'$.

Theorem 2.1. Let $F : \mathbf{A} \to \mathbf{B}$ be a functor, G_0 a map from the set $\mathbf{A}_0$ of objects in $\mathbf{A}$ to the set $\mathbf{B}_0$ of objects in $\mathbf{B}$, and $\tau = (\tau_A : F(A) \to G_0(A))_{A\in\mathbf{A}_0}$ a family of isomorphisms. Then there exists a unique functor $G : \mathbf{A} \to \mathbf{B}$, for which G_0 is the object function and $\tau : F \to G$ is an (iso)morphism.

Proof. On the one hand $\tau : F \to G$ is an isomorphism if and only if for each morphism $\alpha : A \to A'$ in $\mathbf{A}$, we have $G(\alpha) = \tau_{A'}F(\alpha)\tau_A^{-1}$, and on the other hand it is easy to check that sending $\alpha : A \to A'$ to $\tau_{A'}F(\alpha)\tau_A^{-1}$ determines a functor $\mathbf{A} \to \mathbf{B}$ whose object function is G_0. □

Remark 2.2. (a) Since G_0 above is completely determined by the family $\tau = (\tau_A)_{A\in\mathbf{A}_0}$, the assumptions of Theorem 2.1 should be understood as "given $F : \mathbf{A} \to \mathbf{B}$ and, for each object A in $\mathbf{A}$, an isomorphism τ_A from $F(A)$ to somewhere".

(b) Theorem 2.1 has an interesting application: Starting from an arbitrary isomorphism $\theta : X \to Y$ in a category $\mathbf{A}$, we apply this theorem to $\mathbf{B} = \mathbf{A}$, $F = 1_{\mathbf{A}}$, and

$$\tau_A = \begin{cases} \theta : X \to Y, \text{ if } A = X; \\ \theta^{-1} : Y \to X, \text{ if } A = Y; \\ 1_A : A \to A, \text{ if } X \neq A \neq Y; \end{cases} \tag{2.1}$$

it is easy to see that the resulting functor $G : \mathbf{A} \to \mathbf{A}$ is an isomorphism (for, use Theorem 2.3(c) below, and the fact that a functor is an isomorphism if and only it is bijective on objects and fully faithful). This in fact explains how to *interchange isomorphic objects in any categorical construction*. □

Given a functor $F : \mathbf{A} \to \mathbf{B}$ and objects A and A' in $\mathbf{A}$, let us write

$$F_{A,A'} : \hom_{\mathbf{A}}(A,A') \to \hom_{\mathbf{B}}(F(A),F(A')) \tag{2.2}$$

for the induced map between the hom sets $\hom_{\mathbf{A}}(A,A')$ and $\hom_{\mathbf{B}}(F(A),F(A'))$. As in fact already observed in the proof of Theorem 2.1, given an isomorphism $\tau : F \to G$, the diagram

$$\begin{array}{ccc} & & \hom_{\mathbf{B}}(F(A),F(A')) \\ & \overset{F_{A,A'}}{\nearrow} & \quad f \mapsto \tau_{A'} f \tau_A^{-1} \Big\downarrow \Big\uparrow g \mapsto \tau_{A'}^{-1} g \tau_A \\ \hom_{\mathbf{A}}(A,A') & & \\ & \underset{G_{A,A'}}{\searrow} & \\ & & \hom_{\mathbf{B}}(G(A),G(A')) \end{array} \tag{2.3}$$

commutes. Since its vertical arrows are bijections, we obtain:

Theorem 2.3. If F and G are isomorphic functors, then:

(a) F is faithful (=all $F_{A,A'}$'s above are injective) if and only if so is G;

(b) F is full (=all $F_{A,A'}$'s above are surjective) if and only if so is G;

(c) F is fully faithful (=all $F_{A,A'}$'s above are bijective) if and only if so is G. □

Definition 2.4. An equivalence of categories $\mathbf{A}$ and $\mathbf{B}$ is a system consisting of functors

$$\mathbf{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{B} \text{ and isomorphisms } \alpha : 1_{\mathbf{A}} \to GF \text{ and } \beta : 1_{\mathbf{B}} \to FG;$$

we will also say that $(F,G,\alpha,\beta) : \mathbf{A} \to \mathbf{B}$ is a category equivalence, and (briefly) that $F : \mathbf{A} \to \mathbf{B}$ is a category equivalence. □

Observation 2.5. (a) If $F : \mathbf{A} \to \mathbf{B}$ is a category isomorphism, then it is a category equivalence;

(b) If $(F,G,\alpha,\beta) : \mathbf{A} \to \mathbf{B}$ is a category equivalence, then so is $(G,F,\beta,\alpha,) : \mathbf{B} \to \mathbf{A}$;

(c) If $(F,G,\alpha,\beta) : \mathbf{A} \to \mathbf{B}$ and $(H,I,\gamma,\delta) : \mathbf{B} \to \mathbf{C}$ are category equivalences, then so is $(HF,GI,(G\gamma F)\alpha,(H\beta I)\delta) : \mathbf{A} \to \mathbf{C}$, where $G\gamma F : GF \to GIHF$ and $H\beta I : HI \to HFGI$ denote natural transformations defined by $(G\gamma F)_A = G(\gamma_{F(A)})$ and $(H\beta I)_C = H(\beta_{I(C)})$ respectively.

(d) As follows from the previous assertions, the category equivalence determines an equivalence relation on the collection of all categories; we will simple write $\mathbf{A} \sim \mathbf{B}$ when there exists a category equivalence $\mathbf{A} \to \mathbf{B}$.

(e) If $F : \mathbf{A} \to \mathbf{B}$ is a category equivalence and $F' \approx F$, then $F' : \mathbf{A} \to \mathbf{B}$ also is a category equivalence. □

The next definition will later help us describe the relationship between isomorphisms and equivalences of categories precisely.

Definition 2.6. A category **S** is said to be a skeleton, if for objects A and B in **S**, we have:

$$A \approx B \Rightarrow A = B;$$

for an arbitrary category **C**, we say that **S** is a (the) skeleton of **C** and write $\mathbf{S} = \mathrm{Sk}(\mathbf{C})$ if **S** is a full subcategory in **C**, and the inclusion functor $\mathbf{S} \to \mathbf{C}$ is a category equivalence. □

This definition immediately suggests to ask, if every category has a skeleton, and if the skeleton of a category is uniquely (up to an isomorphism?) determined. These questions are answered below.

Lemma 2.7. If $F : \mathbf{A} \to \mathbf{B}$ is a category equivalence, then F is fully faithful and essentially (=up to isomorphism) bijective on objects, i.e.:

(a) for objects A and A' in **A**, $F(A) \approx F(A') \Rightarrow A \approx A'$ (essential injectivity);

(b) for each object B in **B**, there exists an object A in **A** with $F(A) \approx B$ (essential surjectivity).

Proof. Let $(F,G,\alpha,\beta) : \mathbf{A} \to \mathbf{B}$ a category equivalence involving F. As follows from Theorem 2.3(c) applied to $1_\mathbf{A} \approx GF$, the functor GF is fully faithful. Therefore the composite

$$\hom_\mathbf{A}(A,A') \xrightarrow{F_{A,A'}} \hom_\mathbf{B}(F(A),F(A')) \xrightarrow{G_{F(A),F(A')}} \hom_\mathbf{A}(GF(A),GF(A'))$$

is a bijection for all objects A and A' in **A**, from which we conclude:

- F is faithful;
- since F is always faithful in such a situation, G is also faithful by 2.5(b);
- since G is faithful, $G_{F(A),F(A')}$ is always injective;
- since $F_{A,A'}$ and $G_{F(A),F(A')}$ are injective and their composite is bijective, $F_{A,A'}$ is bijective too.

That is, F is fully faithful. Essential bijectivity on objects is obvious: $F(A) \approx F(A') \Rightarrow A \approx GF(A) \approx GF(A') \approx A'$ and $F(A) \approx B$ for $A = G(B)$. □

Remark 2.8. (a) In fact the crucial properties here are fully faithful-ness and essential surjectivity, since it is easy to show that a fully faithful functor is always essentially injective on objects. Indeed, if $F : \mathbf{A} \to \mathbf{B}$ is fully faithful, and $\beta : F(A) \to F(A')$ is an isomorphism in

B, then we can choose $\alpha : A \to A'$ with $F(\alpha) = \beta$ and $\alpha' : A' \to A$ with $F(\alpha') = \beta^{-1}$ – and these chosen morphisms will be inverse to each other since so are their images under F.

(b) Proving essential injectivity of the functor F in (a) we in fact also proved another important property of a fully faithful functor, which is reflection of isomorphisms. It says: if $F(\alpha)$ is an isomorphism, then so is α. □

From Observation 2.5(a), Lemma 2.7, and Remark 2.8 we obtain:

Lemma 2.9. The following conditions on a functor F between skeletons are equivalent:

(a) F is a category equivalence;

(b) F is fully faithful and essentially bijective on objects;

(c) F is fully faithful and essentially surjective on objects;

(d) F is an isomorphism. □

Remark 2.10. (a) It is not, however, true of course that $G = F^{-1}$ for any equivalence $(F,G,\alpha,\beta) : \mathbf{A} \to \mathbf{B}$ between skeletons.

(b) As follows from 2.5(d) and 2.9(a)⇔(d), skeletons of equivalent categories are always isomorphic. In particular so are every two skeletons of the same category. □

Theorem 2.11. Every category has a skeleton.

Proof. Given a category **A**, we choose:

- an object in each isomorphism class of objects in **A**, and for any object A in **A**, the chosen object isomorphic to A will be denoted by $\Phi(A)$;
- an isomorphism $\varphi_A : A \to \Phi(A)$, assuming for simplicity that $\varphi_{\Phi(A)} = 1_{\Phi(A)}$;
- $\Phi : \mathbf{A} \to \mathbf{A}$ to be the functor obtained from the identity functor of **A** and the family $(\varphi_A)_{A\in\mathbf{A}_0}$ as in Theorem 2.1 (see also Remark 2.2(a)), making $\varphi : 1_\mathbf{A} \to \Phi$ an isomorphism;
- **S** to be the full subcategory in **A** with object all $\Phi(A)$ $(A \in \mathbf{A}_0)$;
- $F : \mathbf{S} \to \mathbf{A}$ to be the inclusion functor;
- $G : \mathbf{A} \to \mathbf{S}$ defined by $FG = \Phi$ (which indeed defines a functor since the image of Φ is inside **S**), making $GF = 1_\mathbf{S}$, since $\varphi_{\Phi(A)} = 1_{\Phi(A)}$ for all objects A in $\mathbf{A}_0$.

Here **S** is a skeleton and $(F,G,1_{1_\mathbf{S}},\varphi) : \mathbf{S} \to \mathbf{A}$ is a category equivalence. □

Theorem 2.12. (a) A functor is a category equivalence if and only if it is fully faithful and essentially surjective on objects.

(b) Two categories are equivalent if and only if they have isomorphic skeletons.

Proof. (a): Suppose $F : \mathbf{A} \to \mathbf{B}$ is fully faithful and essentially surjective on objects. Consider the diagram

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{F} & \mathbf{B} \\ K \big\uparrow\big\downarrow L & & M \big\uparrow\big\downarrow N \\ \mathrm{Sk}(\mathbf{A}) & \underset{(NFK)^{-1}}{\overset{NFK}{\rightleftarrows}} & \mathrm{Sk}(\mathbf{B}) \end{array} \tag{2.4}$$

in which:

• the vertical arrows determine equivalences $\mathbf{A} \sim \mathrm{Sk}(\mathbf{A})$ and $\mathbf{B} \sim \mathrm{Sk}(\mathbf{B})$, which exist by Theorem 2.11.
• the composite NFK is fully faithful and essentially surjective on objects, because so are N, F, and K; therefore NFK is an isomorphism by Lemma 2.9(c)⇒(d).

Using Observation 2.5 we conclude that $MNFKL$ is a category equivalence, and then that since $MNFKL \approx 1_{\mathbf{B}}F1_{\mathbf{A}} = F$, so is F.

The "only if" part is Lemma 2.7.

(b): Again, just use Observation 2.5, Lemma 2.9, and the square diagram above (although the "only if" part has already been proved: see Remark 2.10(b)). □

3. Yoneda lemma and Yoneda embedding

The purpose of this section is to describe fully faithful functors

$$\mathbf{C} \xrightarrow{\quad Y \quad} \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \xrightarrow{\quad G \quad} (\mathbf{Cat}{\downarrow}\mathbf{C}), \tag{3.1}$$

where $\mathbf{C}$ is an arbitrary category, $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is the category of functors $\mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$, and $(\mathbf{Cat}{\downarrow}\mathbf{C})$ is the comma category of the category $\mathbf{Cat}$ of all categories over the category $\mathbf{C}$ (i.e. the category of pairs $(\mathbf{D},P)$, where $\mathbf{D}$ is a category and $P : \mathbf{D} \to \mathbf{C}$ a functor. As we will see, the fully faithful-ness of Y will follow from

Theorem 3.1("Yoneda lemma"). For any functor $T : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ and any object C in $\mathbf{C}$, the map

$$\mathrm{Nat}(\hom_{\mathbf{C}}(-,C),T) \to T(C), \quad \tau \mapsto \tau_C(1_C) \tag{3.2}$$

from the set $\mathrm{Nat}(\hom_{\mathbf{C}}(-,C),F)$, of natural transformations from $\hom_{\mathbf{C}}(-,C)$ to T, to the set $T(C)$ is bijective.

Proof. Let us denote the map above by α and define a map $\beta : T(C) \to \mathrm{Nat}(\hom_{\mathbf{C}}(-,C),T)$ by

$$\beta(t)_A(f) = T(f)(t) \text{ – for a } t \in T(C) \text{ and a morphism } f : A \to C \text{ in } \mathbf{C}.$$

We are going to show that α and β are inverse to each other. We have

$$\alpha\beta(t) = \beta(t)_C(1_C) = T(1_C)(t) = t \text{ for each } t \in T(C),$$

proving that $\alpha\beta$ is the identity map of $T(C)$. On the other hand, for $\tau : \hom_{\mathbf{C}}(-,C) \to T$ and $f : A \to C$, we have

$$\beta\alpha(\tau)_A(f) = T(f)(\alpha(\tau)) = T(f)(\tau_C(1_C)) = \tau_A(\hom_{\mathbf{C}}(f,C)(1_C)) = \tau_A(f),$$

where the last equality is visible in the naturality square

$$\begin{array}{ccc} \hom_{\mathbf{C}}(C,C) & \xrightarrow{\quad\tau_C\quad} & T(C) \\ {\scriptstyle \hom_{\mathbf{C}}(f,C)}\Big\downarrow & & \Big\downarrow{\scriptstyle T(f)} \\ \hom_{\mathbf{C}}(A,C) & \xrightarrow[\quad\tau_A\quad]{} & T(A), \end{array}$$

and the equality $\beta\alpha(\tau)_A(f) = \tau_A(f)$ (for all f) implies that $\beta\alpha$ is the identity map of $\mathrm{Nat}(\hom_{\mathbf{C}}(-,C),T)$. □

Consider the special case of this theorem in which the functor T is of the form $T = \hom_{\mathbf{C}}(-,C')$ for some C' in $\mathbf{C}$. Then the bijection of Theorem 3.1 together with its inverse become

$$\mathrm{Nat}(\hom_{\mathbf{C}}(-,C),\hom_{\mathbf{C}}(-,C')) \underset{(f \mapsto tf) \leftarrow\!\dashv\, t}{\overset{\tau \mapsto \tau_C(1_C)}{\rightleftarrows}} \hom_{\mathbf{C}}(C,C'), \tag{3.3}$$

where $(f \mapsto tf) \leftarrow\!\dashv\, t$ means that $t : C \to C'$ is sent to the natural transformation

$$\tau : \hom_{\mathbf{C}}(-,C) \to \hom_{\mathbf{C}}(-,C') \text{ defined by } \tau_A(f) = tf.$$

However this map $\hom_{\mathbf{C}}(C,C') \to \mathrm{Nat}(\hom_{\mathbf{C}}(-,C),\hom_{\mathbf{C}}(-,C'))$ is the same as $Y_{C,C'}$, where

$$Y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \text{ is the functor defined by } Y(C) = \hom_{\mathbf{C}}(-,C),$$

i.e. the functor corresponding to the functor $\hom : \mathbf{C}^{\mathrm{op}}\times\mathbf{C} \to \mathbf{Sets}$ via the canonical category isomorphism

$$\hom_{\mathbf{Cat}}(\mathbf{C}^{\mathrm{op}}\times\mathbf{C},\mathbf{Sets}) \approx \hom_{\mathbf{Cat}}(\mathbf{C},\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}). \tag{3.4}$$

Therefore Theorem 3.1 gives

Corollary 3.2. The functor

$$Y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \text{ defined by } Y(C) = \hom_{\mathbf{C}}(-,C) \tag{3.5}$$

is fully faithful. □

The functor Y above is usually called the Yoneda embedding (for $\mathbf{C}$), while the functor $G : \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \to (\mathbf{Cat}\!\downarrow\!\mathbf{C})$ we are going to introduce now has no name; a somewhat artificial name would be "the discrete form of Grothendieck construction".

For a functor $T : \mathbf{C}^{op} \to \mathbf{Sets}$, the category El($T$) is defined as the category of pairs (A,a), where A is an object in **C** and a is an element $T(A)$; in this category, a morphism

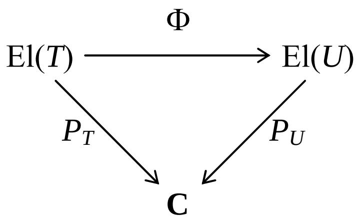
$$f : (A,a) \to (B,b) \text{ is a morphism } f : A \to B \text{ in } \mathbf{C} \text{ with } T(f)(b) = a.$$

We define the functor

$$G : \mathbf{Sets}^{\mathbf{C}^{op}} \to (\mathbf{Cat}{\downarrow}\mathbf{C}) \text{ by } G(T) = (\mathrm{El}(T),P_T),$$

where $P_T : \mathrm{El}(T) \to \mathbf{C}$ is the forgetful functor, sending $f : (A,a) \to (B,b)$ to $f : A \to B$. In order to see how exactly is G defined on morphisms, let us describe morphisms in $(\mathbf{Cat}{\downarrow}\mathbf{C})$ of the form $\Phi : (\mathrm{El}(T),P_T) \to (\mathrm{El}(U),P_U)$:

Such a morphism is a functor $\Phi : \mathrm{El}(T) \to \mathrm{El}(U)$ making the diagram



commute. At the level of objects this means that, for each (A,a) in El(T), $\Phi(A,a)$ should a pair whose first component is A. This means that to give the object function of Φ is to give a fimily of maps $\varphi = (\varphi_A : T(A) \to U(A))_{A\in\mathbf{A}_0}$ and define Φ on objects by $\Phi(A,a) = (A,\varphi_A(a))$. After that, again, since the diagram above commutes, on morphisms Φ must be defined by

$$\Phi(f : (A,a) \to (B,b)) = f : (A,\varphi_A(a)) \to (B,\varphi_B(b)).$$

This simply means that the images of morphisms are uniquely determined, but the fact that Φ is indeed defined on morphisms puts the following condition on the family φ: if f is a morphism from (A,a) to (B,b), then it also must be a morphism from $(A,\varphi_A(a))$ to $(B,\varphi_B(b))$. And since f is a morphism from (A,a) to (B,b) if and only if $a = T(f)(b)$, this means that every $f : A \to B$ must be a morphism from $(A,\varphi_A T(f)(b))$ to $(B,\varphi_B(b))$ for each b in $T(B)$. In other words, for every $f : A \to B$ in **A**, we must have $\varphi_A T(f) = U(f)\varphi_B$, which is the same as to say that φ is a natural transformation from T to U.

That is, we can define

$$G : \mathbf{Sets}^{\mathbf{C}^{op}} \to (\mathbf{Cat}{\downarrow}\mathbf{C}) \text{ by } G(\varphi : T \to U) = \Phi : (\mathrm{El}(T),P_T) \to (\mathrm{El}(U),P_U) \qquad (3.6)$$

In the notation above (omitting routine verification of preservation of composition and identity morphisms), and this makes it fully faithful.

4. Representable functors and discrete fibrations

Definition 4.1. (a) A functor $T : \mathbf{C}^{op} \to \mathbf{Sets}$ is said to be representable if it is isomorphic to a functor of the form $Y(C) = \hom_{\mathbf{C}}(-,C)$ for some object C in **C**.

(b) A functor $P : \mathbf{D} \to \mathbf{C}$ is said to be a discrete fibration, if the diagram

$$\begin{array}{ccc} \mathbf{D}_1 & \longrightarrow & \mathbf{D}_0 \\ {\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_0} \\ \mathbf{C}_1 & \longrightarrow & \mathbf{C}_0, \end{array} \tag{4.1}$$

in which the horizontal arrows are the codomain maps of **D** and **C**, and the vertical arrows are the morphism function and the object function of P respectively, is a pullback. □

This section is devoted to the following two theorems:

Theorem 4.2. A functor $T : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ is representable if and only if the category El(T) has a terminal object. Moreover, a natural transformation $\tau : \hom_{\mathbf{C}}(-,C) \to T$ is an isomorphism if and only if the pair (C,t), in which t is the image of τ under the map (3.2), is a terminal object in El(T).

Proof. For the assertions (a) – (f) below we obviously have (a)⇔(b)⇔(c)⇔(d)⇔(e)⇔(f):

(a) $\tau : \hom_{\mathbf{C}}(-,C) \to T$ is an isomorphism;

(b) $\tau_A : \hom_{\mathbf{C}}(A,C) \to T(A)$ is a bijection for each object A in **C**;

(c) for every object A in **C** and every $a \in T(A)$ there exists a unique morphism $f : A \to C$ with $\tau_A(f) = a$;

(d) for every object A in **C** and every $a \in T(A)$ there exists a unique morphism $f : A \to C$ with $T(f)\tau_C(1_C) = a$;

(e) for every object (A,a) in El(T) there exists a unique morphism from (A,a) to $(C,\tau_C(1_C))$;

(f) $(C,\tau_C(1_C))$ is a terminal object in El(T).

And since $(C,\tau_C(1_C))$ is exactly the image of τ under the map (3.2), this completes the proof. □

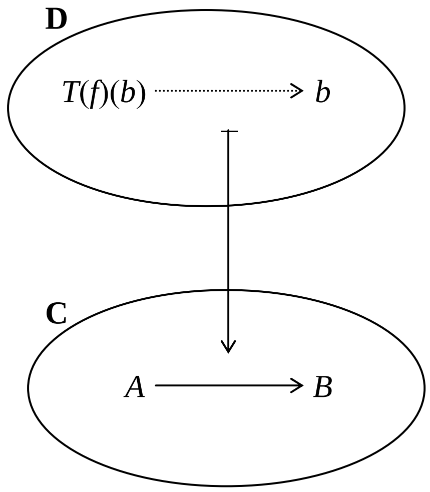
Theorem 4.3. A functor $P : \mathbf{D} \to \mathbf{C}$ is a discrete fibration, if and only if the object $(\mathbf{D},P)$ of $(\mathbf{Cat}{\downarrow}\mathbf{C})$ is isomorphic to $G(T) = (\mathrm{El}(T),P_T)$, for some functor $T : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$. □

Proof. “If”: We have to prove that $(\mathrm{El}(T),P_T)$ is always a discrete fibration. This means to prove that for every morphism $f : A \to B$ in **C** and every $b \in T(B)$, there exists a unique $a \in T(A)$ for which f is a morphism from (A,a) to (B,b). However this is trivial since f is a morphism from (A,a) to (B,b) if and only if $a = T(f)(b)$.

“Only if”: Assuming that $P : \mathbf{D} \to \mathbf{C}$ is a discrete fibration, we define a functor $T : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ as follows:

• For an object C in **C**, we take $T(C)$ to be the set of objects D in **D** with $P(D) = C$.
• For a morphism $f : A \to B$ in **C**, and an element b in $T(B)$, which in fact an object in **D** with $P(b) = B$, we take g to be the morphism g in **D**, with $P(g) = f$ and codomain of g equal to b. The existence and uniqueness of such a g follows from the fact that the diagram (4.1) is a pullback. We then take $T(f)(b)$ to be the domain of g.

Accordingly the procedure of defining $T(f)$’s (for all f) displays as



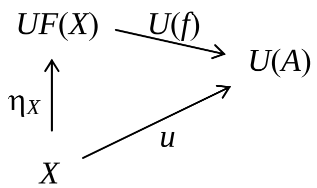
(4.2)

and it is easy to see that it indeed defines a functor $T : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ in such a way that $(\mathrm{El}(T),P_T)$ becomes isomorphic to $(\mathbf{D},P)$. □

5. Adjoint functors

Adjoint functors will be defined at the end of this section via several equivalent kinds of data that will be described before.

Definition 5.1. Let $U : \mathbf{A} \to \mathbf{X}$ be a functor and X an object in $\mathbf{X}$. A universal arrow $X \to U$ is a pair $(F(X),\eta_X)$ in which $F(X)$ is an object in $\mathbf{A}$ and $\eta_X : X \to UF(X)$ a morphism in $\mathbf{X}$ with the following universal property: for every object A in $\mathbf{A}$ and every morphism $u : X \to U(A)$ in $\mathbf{X}$ there exists a unique morphism $f : F(X) \to A$ making the diagram



(5.1)

commute. □

Theorem 5.2. Let $U : \mathbf{A} \to \mathbf{X}$ be a functor and $((F(X),\eta_X))_{X\in\mathbf{X}_0}$ a family of universal arrows $X \to U$ given for each object X in $\mathbf{X}$. Then there exists a unique functor $F : \mathbf{X} \to \mathbf{A}$ for which the family $((F(X),\eta_X))_{X\in\mathbf{X}_0}$ determines a natural transformation $\eta : 1_{\mathbf{X}} \to UF$.

Proof. Given a morphism $h : X \to Y$ in $\mathbf{X}$, we can define $F(h) : F(X) \to F(Y)$ as the unique morphism making the diagram (5.1) commute for $A = F(Y)$ and $u = \eta_Y h$. Since the commutativity of (5.1) in this case is equivalent to the commutativity of the naturality square

$$\begin{array}{ccc} UF(X) & \xrightarrow{UF(h)} & UF(Y) \\ {\scriptstyle \eta_X}\uparrow & & \uparrow{\scriptstyle \eta_Y} \\ X & \xrightarrow[h]{} & Y, \end{array} \qquad (5.2)$$

this proves the theorem. □

Observation 5.3. (a) The universal property given in Definition 5.1 can be equivalently reformulated as: the map

$$\varphi_{X,A} : \hom_{\mathbf{A}}(F(X),A) \to \hom_{\mathbf{X}}(X,U(A)), \text{ defined by } \varphi_{X,A}(f) = U(f)\eta_X, \qquad (5.3)$$

is a bijection for each object A in $\mathbf{A}$. Moreover, since this map is obviously natural in A, that universal property can also be reformulated as: the natural transformation

$$\varphi_{X,-} : \hom_{\mathbf{A}}(F(X),-) \to \hom_{\mathbf{X}}(X,U(-)), \text{ defined by } \varphi_{X,A}(f) = U(f)\eta_X, \qquad (5.4)$$

is an isomorphism. Furthermore, let

$$\varphi_{X,-} : \hom_{\mathbf{A}}(F(X),-) \to \hom_{\mathbf{X}}(X,U(-)) \qquad (5.5)$$

be an arbitrary isomorphism. Then, for any $f : F(X) \to A$, using the naturality square

$$\begin{array}{ccc} \hom_{\mathbf{A}}(F(X),F(X)) & \xrightarrow{\varphi_{X,F(X)}} & \hom_{\mathbf{X}}(X,U(F(X))) \\ {\scriptstyle \hom_{\mathbf{A}}(F(X),f)}\downarrow & & \downarrow{\scriptstyle \hom_{\mathbf{X}}(X,U(f))} \\ \hom_{\mathbf{A}}(F(X),A) & \xrightarrow[\varphi_{X,A}]{} & \hom_{\mathbf{X}}(X,U(A)), \end{array} \qquad (5.6)$$

we obtain $\varphi_{X,A}(f) = \varphi_{X,A}\hom_{\mathbf{A}}(F(X),f)(1_{F(X)}) = \hom_{\mathbf{X}}(X,U(f))\varphi_{X,F(X)}(1_{F(X)}) = U(f)\varphi_{X,F(X)}(1_{F(X)})$. Therefore we have one more reformulation of the universal property given in Definition 5.1, namely: there exists an isomorphism (5.5); and with this reformulation η_X and $\varphi_{X,-}$ determine each other by

$$\varphi_{X,A}(f) = U(f)\eta_X \text{ and } \eta_X = \varphi_{X,F(X)}(1_{F(X)}). \qquad (5.7)$$

(b) The relationship between η_X and $\varphi_{X,-}$ can be seen of course as a special case of the statement dual to Theorem 4.2, but we omit details here.

(c) Suppose η_X, or, equivalently, $\varphi_{X,-}$ is given for every object X in $\mathbf{X}$. Then, by Theorem 5.2, there is a unique way to make F a functor $\mathbf{X} \to \mathbf{A}$, so that the family $((F(X),\eta_X))_{X\in\mathbf{X}_0}$ determines a natural transformation $\eta : 1_{\mathbf{X}} \to UF$. And it is easy to check that this will also make $\varphi_{X,-}$ natural in X, yielding a natural isomorphism

$$
\begin{array}{ccccc}
 & & \mathbf{A}^{op}\times\mathbf{A} & & \\
 & {\scriptstyle F^{op}\times 1}\nearrow & & \searrow{\scriptstyle \hom_{\mathbf{A}}} & \\
\mathbf{X}^{op}\times\mathbf{A} & & \varphi\big\downarrow & & \mathbf{Sets} \\
 & {\scriptstyle 1\times U}\searrow & & \nearrow{\scriptstyle \hom_{\mathbf{X}}} & \\
 & & \mathbf{X}^{op}\times\mathbf{X} & &
\end{array}
\qquad (5.8)
$$

Moreover, the "φ approach" shows that the unique functoriality of F is actually a consequence of the fact that the Yoneda embedding $\mathbf{C}^{op} \to \mathbf{Sets}^{\mathbf{C}}$ is fully faithful. Indeed, given a morphism $h : X \to Y$ in $\mathbf{X}$, the naturality square

$$
\begin{array}{ccc}
\hom_{\mathbf{A}}(F(Y),-) & \xrightarrow{\varphi_{Y,-}} & \hom_{\mathbf{X}}(Y,U(-)) \\
{\scriptstyle \hom_{\mathbf{A}}(F(h),-)}\big\downarrow & & \big\downarrow{\scriptstyle \hom_{\mathbf{X}}(h,U(-))} \\
\hom_{\mathbf{A}}(F(X),-) & \xrightarrow[\varphi_{X,-}]{} & \hom_{\mathbf{X}}(X,U(-)),
\end{array}
\qquad (5.9)
$$

determines $\hom_{\mathbf{A}}(F(h),-)$, and since the Yoneda embedding $\mathbf{C}^{op} \to \mathbf{Sets}^{\mathbf{C}}$ is fully faithful, $\hom_{\mathbf{A}}(F(h),-)$ determines $F(h)$. □

From Observation 5.3 we obtain

Theorem 5.4. For a functor $U : \mathbf{A} \to \mathbf{X}$, the following kinds of data uniquely determine each other:

(a) a family $((F(X),\eta_X))_{X\in\mathbf{X}_0}$ of universal arrows $X \to U$ given for each object X in $\mathbf{X}$;

(b) a functor $F : \mathbf{X} \to \mathbf{A}$ and a natural transformation $\eta : 1_{\mathbf{X}} \to UF$ such that $(F(X),\eta_X)$ is a universal arrow $X \to U$ for each object X in $\mathbf{X}$;

(c) a family $(F(X))_{X\in\mathbf{X}_0}$ of objects in $\mathbf{A}$ and a family

$$(\varphi_{X,-} : \hom_{\mathbf{A}}(F(X),-) \to \hom_{\mathbf{X}}(X,U(-)))_{X\in\mathbf{X}_0}$$

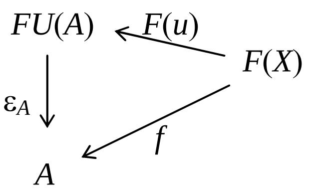
of isomorphisms given for each object X in $\mathbf{X}$;

(d) a functor $F : \mathbf{X} \to \mathbf{A}$ and an isomorphism (5.8).

Moreover, the η_X of (a) corresponds to the η_X of (b), the $\varphi_{X,-}$ of (c) corresponds to (the X-component) of φ of (d), and these η_X and $\varphi_{X,-}$ corresponding to each other via (5.7). □

The data 5.4(d) shows certain dual symmetry between U and F, and suggests to dualize Definition 5.1 and Theorem 5.4 as follows:

Definition 5.5. Let $F : \mathbf{X} \to \mathbf{A}$ be a functor and A an object in $\mathbf{A}$. A universal arrow $F \to A$ is a pair $(U(A),\varepsilon_A)$ in which $U(A)$ is an object in $\mathbf{X}$ and $\varepsilon_A : FU(A) \to A$ a morphism in $\mathbf{A}$ with the following universal property: for every object X in $\mathbf{X}$ and every morphism $f : F(X) \to A$ in $\mathbf{A}$ there exists a unique morphism $u : X \to U(A)$ making the diagram



(5.10)

commute. □

Theorem 5.6. For a functor $F : \mathbf{X} \to \mathbf{A}$, the following kinds of data uniquely determine each other:

(a) a family $((U(A),\varepsilon_A))_{A\in\mathbf{A}_0}$ of universal arrows $F \to A$ given for each object A in $\mathbf{A}$;

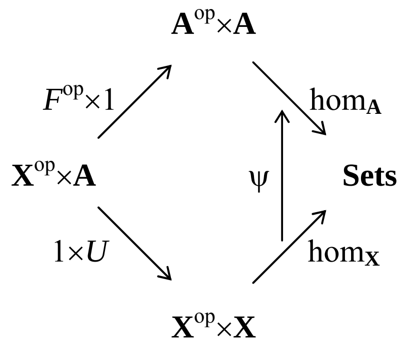
(b) a functor $U : \mathbf{A} \to \mathbf{X}$ and a natural transformation $\varepsilon : FU \to 1_{\mathbf{A}}$ such that $(U(A),\varepsilon_A)$ is a universal arrow $F \to A$ for each object A in $\mathbf{A}$;

(c) a family $(U(A))_{A\in\mathbf{A}_0}$ of objects in $\mathbf{X}$ and a family

$$(\psi_{-,A} : \hom_{\mathbf{X}}(-,U(A)) \to \hom_{\mathbf{A}}(F(-),A))_{A\in\mathbf{A}_0}$$

of isomorphisms given for each object A in $\mathbf{A}$;

(d) a functor $U : \mathbf{A} \to \mathbf{X}$ and an isomorphism



(5.11)

Moreover, the ε_A of (a) corresponds to the ε_A of (b), the $\psi_{-,A}$ of (c) corresponds to (the A-component) of ψ of (d), and these ε_A and $\psi_{-,A}$ corresponding to each other via

$$\psi_{X,A}(u) = \varepsilon_A F(u) \text{ and } \varepsilon_A = \psi_{U(A),A}(1_{U(A)}). \ \square \qquad (5.12)$$

Remark 5.7. The data described in Theorem 5.4(d) is obviously identical to the data described in Theorem 5.6(d): just take φ and ψ inverse to each other. Therefore these two theorems actually describe eight equivalent kinds of data. □

Remark 5.7 is not the end of this story: although eight is a large number, it is good to add at least one more, which is purely equational. For, we observe:

• Having functors $U : \mathbf{A} \to \mathbf{X}$ and $F : \mathbf{X} \to \mathbf{A}$, and merely natural transformations $\eta : 1_{\mathbf{X}} \to UF$ and $\varepsilon : FU \to 1_{\mathbf{A}}$, we can still define natural transformations φ and ψ as in (5.8) and in (5.11) respectively.
• Under no conditions on η and ε, those φ and ψ will also be merely natural transformations independent from each other. But requiring them to be each other's inverses and reformulating this requirement in terms of η and ε will give us a new equivalent form of the desired data, which is purely equational.
• Requiring φ and ψ to be each other's inverses means to require $\psi_{X,A}\varphi_{X,A}(f) = f$ and $\varphi_{X,A}\psi_{X,A}(u) = u$ for each $f : F(X) \to A$ in $\mathbf{A}$ and each $u : X \to U(A)$ in $\mathbf{X}$. But then Yoneda lemma (Theorem 3.1) tells us that it suffices to have these equalities for $f = 1_{F(X)} : F(X) \to F(X)$ and $u = 1_{U(A)} : U(A) \to U(A)$.
• Thus, we are interested in $\psi_{X,F(X)}\varphi_{X,F(X)}(1_{F(X)}) = 1_{F(X)}$ and $\varphi_{U(A),A}\psi_{U(A),A}(1_{U(A)}) = 1_{U(A)}$. Translated into the language of η and ε, these equations become

$$\varepsilon_{F(X)}F(\eta_X) = 1_{F(X)} \text{ and } U(\varepsilon_A)\eta_{U(A)} = 1_{U(A)}, \tag{5.13}$$

and we obtain:

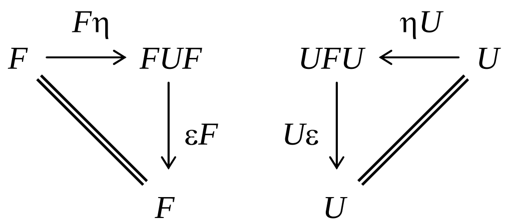
Theorem 5.8. Let $U : \mathbf{A} \to \mathbf{X}$ and $F : \mathbf{X} \to \mathbf{A}$ be functors and $\eta : 1_{\mathbf{X}} \to UF$ and $\varepsilon : FU \to 1_{\mathbf{A}}$ natural transformations. The following conditions are equivalent:

(a) $(F(X),\eta_X)$ is a universal arrow $X \to U$ for each object X in $\mathbf{X}$, and ε is the corresponding family of morphisms, i.e. $U(\varepsilon_A)\eta_{U(A)} = 1_{U(A)}$ for every object A in $\mathbf{A}$;

(b) $(U(A),\varepsilon_A)$ is a universal arrow $F \to A$ for each object A in $\mathbf{A}$, and η is the corresponding family of morphisms, i.e. $\varepsilon_{F(X)}F(\eta_X) = 1_{F(X)}$ for every object X in $\mathbf{X}$;

(c) the equalities $\varepsilon_{F(X)}F(\eta_X) = 1_{F(X)}$ and $U(\varepsilon_A)\eta_{U(A)} = 1_{U(A)}$ hold for every object X in $\mathbf{X}$ and every object A in $\mathbf{A}$. □

Remark 5.9. Using the standard notation for composing functors and natural transformations, the equalities (5.13) (for all X and A) are displayed as commutative diagrams



(5.14)

and called triangular identities. □

Definition 5.10. Let $U : \mathbf{A} \to \mathbf{X}$ and $F : \mathbf{X} \to \mathbf{A}$ be functors, $\eta : 1_{\mathbf{X}} \to UF$ and $\varepsilon : FU \to 1_{\mathbf{A}}$ be natural transformations satisfying the triangular identities, and φ and ψ be as in Theorems 5.4 and 5.6 respectively. We will say that:

(a) $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ is an adjunction; however, we might also omit either η or ε, or replace them with either φ or ψ;

(b) F is the left adjoint (of U), U is the right adjoint (of F), η is the unit of adjunction, and ε is the counit of adjunction. □

6. Monoidal categories

In this section we introduce monoidal categories with some examples and related concepts.

Definition 6.1. A monoidal category is a system $(\mathbf{C},I,\otimes,\alpha,\lambda,\rho)$ in which:

(a) $\mathbf{C}$ is a category;

(b) I is an object in $\mathbf{C}$;

(c) $\otimes : \mathbf{C}\times\mathbf{C} \to \mathbf{C}$ is a functor, written as $\otimes(A,B) = A\otimes B$;

(d) $\alpha = (\alpha_{A,B,C} : A\otimes(B\otimes C) \to (A\otimes B)\otimes C)_{A,B,C\in\mathbf{C}}$, $\lambda = (\lambda_A : A \to I\otimes A)_{A\in\mathbf{C}}$, and $\rho = (\rho_A : A \to A\otimes I)_{A\in\mathbf{C}}$ are natural isomorphisms making the diagrams commute:

$$\begin{array}{ccc} A\otimes(I\otimes B) & \xrightarrow{\ \alpha\ } & (A\otimes I)\otimes B \\ {\scriptstyle 1\otimes\lambda}\uparrow & & \uparrow{\scriptstyle \rho\otimes 1} \\ A\otimes B & = & A\otimes B, \end{array} \tag{6.1}$$

$$\begin{array}{ccccc} A\otimes(B\otimes(C\otimes D)) & \xrightarrow{\ \alpha\ } & (A\otimes B)\otimes(C\otimes D) & \xrightarrow{\ \alpha\ } & ((A\otimes B)\otimes C)\otimes D \\ {\scriptstyle 1\otimes\alpha}\downarrow & & & & \uparrow{\scriptstyle \alpha\otimes 1} \\ A\otimes((B\otimes C)\otimes D) & & \xrightarrow[\ \alpha\ ]{} & & (A\otimes(B\otimes C))\otimes D. \end{array} \tag{6.2}$$

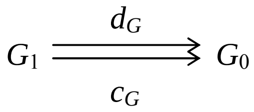
commute. Here and below we write just α instead of $\alpha_{A,B,C}$ for short; it is also often useful to write $(\mathbf{C},I,\otimes,\alpha,\lambda,\rho) = (\mathbf{C},I,\otimes) = (\mathbf{C},\otimes) = \mathbf{C}$. A monoidal category $(\mathbf{C},I,\otimes,\alpha,\lambda,\rho)$ is said to be strict if $A\otimes(B\otimes C) = (A\otimes B)\otimes C$ for all A, B, C; $I\otimes A = A = A\otimes I$ for all A; and α, λ, and ρ are the identity morphisms. □

Example 6.2. Any monoid $\mathbf{M} = (\mathbf{M},e,m)$ can be regarded as a strict monoidal category $(\mathbf{C},I,\otimes)$, in which $\mathbf{C}$ is the underlying set $\mathbf{M}$ regarded as a discrete category (i.e. a category with no non-identity arrows), $I = e$, and $\otimes = m$.

Example 6.3. Any category $\mathbf{X}$ yields the strict monoidal category $\mathrm{End}(\mathbf{X}) = (\mathrm{End}(\mathbf{X}),1_{\mathbf{X}},\bullet)$ of functors $\mathbf{X} \to \mathbf{X}$, where $1_{\mathbf{X}}$ is the identity functor $\mathbf{X} \to \mathbf{X}$ and $\bullet$ is the composition of functors.

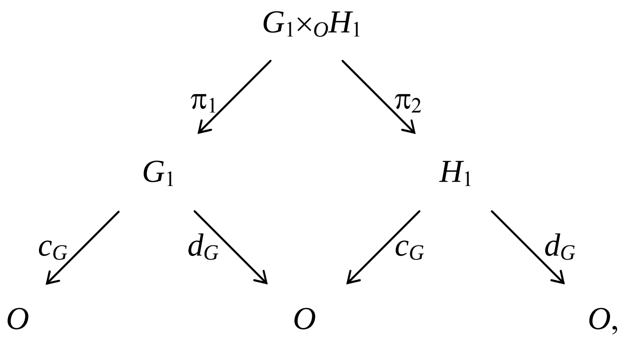
Example 6.4. If $\mathbf{C}$ is a category with finite products, then $(\mathbf{C},I,\otimes,\alpha,\lambda,\rho)$, in which $I = \mathbf{1}$ is a terminal object in $\mathbf{C}$, $\otimes = \times$ is a (chosen) binary product operation, and α, λ, ρ arise from the canonical isomorphisms $A\times(B\times C) \cong (A\times B)\times C$, $A \cong \mathbf{1}\times A$, $A \cong A\times\mathbf{1}$ respectively, is a monoidal category. Such a monoidal structure is said to be *cartesian*. □

Example 6.5. An internal graph G in a category $\mathbf{C}$ is a diagram of the form



in **C**. For a fixed object O, the internal graphs G in **C** with $G_0 = O$ are called internal O-graphs in **C**, and their category will be denoted by **Graphs**(**C**,O); a morphism $f : G \to H$ in **Graphs**(**C**,O) is a morphism $f : G_1 \to H_1$ in **C** with $d_H f = d_G$ and $c_H f = c_G$. When **C** has chosen pullbacks, this category becomes a monoidal category (**Graphs**(**C**,O),I,$\otimes$,α,λ,ρ) as follows:

- I has $I_0 = I_1 = O$ and $d_I = c_I = 1_O$;
- $\otimes$ is defined as the span composition, i.e. for G and H in **Graphs**(**C**,O), $G\otimes H$ is defined by $(G\otimes H)_1 = G_1\times_O H_1$, $d_{G\otimes H} = d_H\pi_2$, and $c_{G\otimes H} = c_G\pi_1$ via the diagram



(6.3)

in which diamond part is the chosen pullback of the pair (d_G,c_G).
- α, λ, and ρ arise from the appropriate canonical isomorphisms.

In the special case in which $O = \mathbf{1}$ is a terminal object in **C**, the pullbacks we need become binary products, and the monoidal category we obtain coincides with the one from Example 6.4. □

Example 6.6. Dualizing Example 6.4, if **C** is a category with finite coproducts, then (**C**,I,$\otimes$,α,λ,ρ), in which $I = \mathbf{0}$ is an initial object in **C**, $\otimes = +$ is a (chosen) binary coproduct operation, and α, λ, ρ arise from the canonical isomorphisms $A+(B+C) \cong (A+B)+C$, $A \cong \mathbf{0}+A$, $A \cong A+\mathbf{0}$ respectively, is a monoidal category. □

Example 6.7. Let R be a commutative ring, and **C** the category of R-modules. Then (**C**,I,$\otimes$,α,λ,ρ), in which $I = R$, $\otimes$ the usual tensor product over R, and α, λ, ρ the usual natural isomorphisms, forms a monoidal category. □

Definition 6.8. Let **C** = (**C**,I,$\otimes$,α,λ,ρ) and **C**' = (**C**',I,$\otimes$,α,λ,ρ) be monoidal categories (we use the prime sign ' only for **C**, although the I, $\otimes$, etc. in **C** and in **C**' are not, of course, supposed to be the same). A monoidal functor $F = (F,\theta,\phi) : \mathbf{C} \to \mathbf{C}'$ consists of

(a) an ordinary functor $F : \mathbf{C} \to \mathbf{C}'$;

(b) a morphism $\theta : I \to F(I)$ in **C**';

(c) a natural transformation $\phi = (\phi_{A,B} : F(A)\otimes F(B) \to F(A\otimes B))_{A,B\in\mathbf{C}}$ making the diagrams

$$\begin{array}{ccc}
F(A)\otimes(F(B)\otimes F(C)) & \xrightarrow{\alpha} & (F(A)\otimes F(B))\otimes F(C) \\
\downarrow{\scriptstyle 1\otimes\phi} & & \downarrow{\scriptstyle \phi\otimes 1} \\
F(A)\otimes(F(B\otimes C)) & & (F(A\otimes B))\otimes F(C) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\
F(A\otimes(B\otimes C)) & \xrightarrow[F(\alpha)]{} & F((A\otimes B)\otimes C),
\end{array} \tag{6.4}$$

$$\begin{array}{ccc}
I\otimes F(\mathrm{A}) & \xleftarrow{\lambda} & F(A) \\
\downarrow{\scriptstyle \theta\otimes 1} & & \downarrow{\scriptstyle F(\lambda)} \\
F(I)\otimes F(\mathrm{A}) & \xrightarrow[\phi]{} & F(I\otimes A),
\end{array} \tag{6.5}$$

$$\begin{array}{ccc}
F(A)\otimes I & \xleftarrow{\rho} & F(A) \\
\downarrow{\scriptstyle 1\otimes\theta} & & \downarrow{\scriptstyle F(\rho)} \\
F(A)\otimes F(I) & \xrightarrow[\phi]{} & F(A\otimes I),
\end{array} \tag{6.6}$$

commute. A monoidal functor $F = (F,\theta,\phi)$ is said to be *strong* if θ and ϕ are isomorphisms, and *strict* if moreover $F(I) = I$, $F(A)\otimes F(B) = F(A\otimes B)$ for all A and B, and θ and ϕ are the identity morphisms. □

Definition 6.9. Let $F_i = (F_i,\theta_i,\phi_i) : \mathbf{C} \to \mathbf{C}'$ $(i = 1,2)$ be monoidal functors. A monoidal natural transformation $\tau : F_1 \to F_2$ is an ordinary natural transformation $\tau : F_1 \to F_2$ such that the diagrams

$$\begin{array}{ccc}
I & \xrightarrow{\theta_1} & F_1(I) \\
\| & & \downarrow{\scriptstyle \tau} \\
I & \xrightarrow[\theta_2]{} & F_2(I),
\end{array} \tag{6.7}$$

$$
\begin{array}{ccc}
F_1(A)\otimes F_1(B) & \xrightarrow{\ \phi_1\ } & F_1(A\otimes B) \\
{\scriptstyle \tau\otimes\tau}\downarrow & & \downarrow{\scriptstyle \tau} \\
F_2(A)\otimes F_2(B) & \xrightarrow[\ \phi_2\ ]{} & F_2(A\otimes B)
\end{array}
\qquad (6.8)
$$

commute. □

Several examples of monoidal functors are used as definitions of important concepts. Two of them will be given here with further cases considered in the next sections.

Definition 6.10. Let **C** be monoidal category and **X** a category. A **C**-action on **X** is a monoidal functor $\mathbf{C} \to \mathrm{End}(\mathbf{X})$, where $\mathrm{End}(\mathbf{X})$ is as in Example 6.3. □

Equivalently such a **C**-action can be defined as a functor $\mathbf{C}\times\mathbf{X} \to \mathbf{X}$, which we will write as $(C,X) \mapsto C\otimes X$, equipped with natural transformations $\theta = (\theta_X : X \to I\otimes X))_{X\in\mathbf{X}}$ and $\phi = (\phi_{A,B,X} : A\otimes(B\otimes X) \to (A\otimes B)\otimes X)_{A,B\in\mathbf{C};\, X\in\mathbf{X}}$ making the diagrams

$$
\begin{array}{ccc}
A\otimes(B\otimes(C\otimes X)) & = & A\otimes(B\otimes(C\otimes X)) \\
{\scriptstyle 1\otimes\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\
A\otimes((B\otimes C)\otimes X) & & (A\otimes B)\otimes(C\otimes X) \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\
(A\otimes(B\otimes C))\otimes X & \xrightarrow[\ \alpha\otimes 1\ ]{} & ((A\otimes B)\otimes C)\otimes X,
\end{array}
\qquad (6.9)
$$

$$
\begin{array}{ccc}
A\otimes X & = & A\otimes X \\
{\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle \lambda\otimes 1} \\
I\otimes(A\otimes X) & \xrightarrow[\ \phi\ ]{} & (I\otimes A)\otimes X,
\end{array}
\qquad (6.10)
$$

$$
\begin{array}{ccc}
A\otimes X & = & A\otimes X \\
{\scriptstyle 1\otimes\theta}\downarrow & & \downarrow{\scriptstyle \rho\otimes 1} \\
A\otimes(I\otimes X) & \xrightarrow[\ \phi\ ]{} & (A\otimes I)\otimes X,
\end{array}
\qquad (6.11)
$$

commute.

Observation and Definition 6.11. Let $\underline{\mathbf{1}}$ be the trivial monoid considered as a monoidal category. A monoidal functor from it to an arbitrary monoidal category **C** can be presented as a triple $M = (M,e,m)$, in which M is an object in **C** and $e : I \to M$ and $m : M\otimes M \to M$ morphisms in **C** making the diagram

$$\begin{array}{ccccccc}
M\otimes(M\otimes M) & \xrightarrow{\alpha} & (M\otimes M)\otimes M & \xrightarrow{m\otimes 1} & M\otimes M & \xleftarrow{(e\otimes 1)\lambda} & M \\
\downarrow{\scriptstyle 1\otimes m} & & & & \downarrow{\scriptstyle m} & = & \downarrow{\scriptstyle (1\otimes e)\rho} \\
M\otimes M & & \xrightarrow{\quad m \quad} & & M & \xleftarrow{\quad m \quad} & M\otimes M
\end{array} \tag{6.12}$$

commute. Such a triple is called a monoid in **C**. □

Moreover, a monoidal natural transformation $\tau : (M_1,e_1,m_1) \to (M_2,e_2,m_2)$ being a morphism $\tau : M_1 \to M_2$ in **C** with $\tau e_1 = e_2$ and $\tau m_1 = m_2(\tau\otimes\tau)$, is nothing but a monoid homomorphism in **C**. So, the monoids in **C** form a category Mon(**C**), which is the category **MonCat**(**1**,**C**) of monoidal functors **1** → **C**. In particular this immediately tells us that every monoidal functor $F = (F,\theta,\phi) : \mathbf{C} \to \mathbf{C}'$ induces a functor Mon(F) : Mon(**C**) → Mon(**C**'), which sends (M,e,m) to the composite

$$\underline{\mathbf{1}} \xrightarrow{(M,e,m)} \mathbf{C} \xrightarrow{(F,\theta,\phi)} \mathbf{C}'$$

considered as a monoid in **C**'.

7. Monads and algebras

In this section we introduce monads, algebras over monads, and free algebras; we also introduce a very general notion of a monoid action as a "general example".

Definition 7.1. A monad on a category **X** is a monoid in the monoidal category End(**X**) of Example 6.3. Explicitly, a monad on **X** is a triple $T = (T,\eta,\mu)$, in which $T : \mathbf{X} \to \mathbf{X}$ is a functor and $\eta : 1_{\mathbf{X}} \to T$ and $\mu : T^2 \to T$ natural transformations making the diagram

$$\begin{array}{ccccc}
T^3 & \xrightarrow{\mu T} & T^2 & \xleftarrow{\eta T} & T \\
\downarrow{\scriptstyle T\mu} & & \downarrow{\scriptstyle \mu} & = & \downarrow{\scriptstyle T\eta} \\
T^2 & \xrightarrow{\mu} & T & \xleftarrow{\mu} & T^2
\end{array} \tag{7.1}$$

commute. □

Definition 7.2. Let $T = (T,\eta,\mu)$ be a monad on a category **X**. A T-algebra (or an algebra over T) is a pair (X,ξ), in which X is an object in **X** and $\xi : T(X) \to X$ a morphism making the diagram

$$\begin{array}{ccccc} T^2(X) & \xrightarrow{\mu_X} & T(X) & \xleftarrow{\eta_X} & X \\ {\scriptstyle T(\xi)}\downarrow & & \downarrow{\scriptstyle \xi} & \diagup\!\!\diagup & \\ T(X) & \xrightarrow[\xi]{} & X & & \end{array} \tag{7.2}$$

commute. A morphism $h : (X,\xi) \to (X',\xi')$ of T-algebras is a morphism $h : X \to X'$ making the diagram

$$\begin{array}{ccc} T(X) & \xrightarrow{T(h)} & T(X') \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \xi'} \\ X & \xrightarrow[h]{} & X' \end{array} \tag{7.3}$$

commute. The category of T-algebras will be denoted by $\mathbf{X}^T$. □

Theorem 7.3. Let $T = (T,\eta,\mu)$ be a monad on a category $\mathbf{X}$, and let $U^T : \mathbf{X}^T \to \mathbf{X}$ be the forgetful functor defined by $U^T(X,\xi) = X$. Then:

(a) for each object X in $\mathbf{X}$, the pair $(T(X),\mu_X)$ is a T-algebra;

(b) the functor $F^T : \mathbf{X} \to \mathbf{X}^T$, defined by $F^T(X) = (T(X),\mu_X)$ is a left adjoint of U^T. The unit and counit of the adjunction are $\eta : 1_{\mathbf{X}} \to T = U^T F^T$ and $\varepsilon : F^T U^T \to 1_{\mathbf{X}^T}$ defined by $\varepsilon_{(T(X),\mu_X)} = \mu_X$ respectively.

Proof. (a): We have to prove the commutativity of

$$\begin{array}{ccccc} T^3(X) & \xrightarrow{\mu_{T(X)}} & T^2(X) & \xleftarrow{\eta_{T(X)}} & T(X) \\ {\scriptstyle T(\mu_X)}\downarrow & & \downarrow{\scriptstyle \mu_X} & \diagup\!\!\diagup & \\ T^2(X) & \xrightarrow[\mu_X]{} & T(X) & & \end{array} \tag{7.4}$$

but it follows from the commutativity of (7.1).

(b): The square part of (7.4) insures that putting

$$\varepsilon_{(T(X),\mu_X)} = \mu_X$$

determines a natural transformation $\varepsilon : F^T U^T \to 1_{\mathbf{X}^T}$, and it is easy to see that η and ε satisfy the triangular identities. □

Example 7.4. Let $\mathbf{X}$ be a category equipped with an action of a monoidal category **C.** According to Definition 6.10, such an action is simply a monoidal functor $F : \mathbf{C} \to \mathrm{End}(\mathbf{X})$, and, like every monoidal functor, it induces a functor Mon(F) : Mon(**C**) → Mon(End(**X**)). Therefore every monoid $M = (M,e,m)$ in **C** determines a monad on **X**; the algebras over that monad are called M-actions, and their category is denoted by $\mathbf{X}^M$. Explicitly, such an M-action is a pair (X,ξ), in which $\xi : M\otimes X \to X$ is a morphism in **X** making the diagram

$$\begin{array}{ccccccc} M\otimes(M\otimes X) & \xrightarrow{\phi} & (M\otimes M)\otimes X & \xrightarrow{m\otimes 1} & M\otimes X & \xleftarrow{\quad(e\otimes 1)\theta\quad} & X \\ {\scriptstyle 1\otimes\xi}\downarrow & & & & {\scriptstyle \xi}\downarrow & \swarrow\!\!\!= & \\ M\otimes X & & \xrightarrow{\qquad\quad\xi\qquad\quad} & & X & & \end{array}$$

commute. Here θ and ϕ are as in (6.9)–(6.11). □

Remark 7.5. (a) According to G. M. Kelly, an "M-action" is the right name not for a pair (X,ξ) above, but just for its structure morphism ξ.

(b) Example 7.4 is at the same time a "generalization". Indeed, starting from an arbitrary monad T on **X**, we can consider T-algebras as T-actions in the sense of Example 7.4, putting **C** = End(**X**) and considering the identity momoidal functor End(**X**) → End(**X**) as the action of End(**X**) on **X**. □

8. More on adjoint functors and category equivalences

This section contains additional observations on adjoint functors and category equivalence; some them will be explicitly used later, while others simply help to understand the concepts involved. We begin with

Observation 8.1. (a) It is easy to see that $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ is an adjunction if and only if so is $(U^{op},F^{op},\varepsilon^{op},\eta^{op}) : \mathbf{X}^{op} \to \mathbf{A}^{op}$ (in the obvious notation). Therefore every general property of adjoint functors has its dual, where the left and the right adjoints exchange their roles (see e.g. Theorems 8.5 and 8.6 below).

(b) Since in an adjunction $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$, $\eta_X : X \to UF(X)$ is a universal arrow $X \to U$ for each object X in **X**, the functor U alone determines such an adjunction uniquely up to an isomorphism; dually, the same is true for F.

(c) It is easy to see that adjunctions compose: if $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ and $(G,V,\vartheta,\zeta) : \mathbf{Y} \to \mathbf{X}$ are adjunctions, then so is $(FG,VU,(V\eta G)\vartheta,\varepsilon(F\zeta U)) : \mathbf{Y} \to \mathbf{A}$ (cf. 2.5(c)).

(d) Let

$$\begin{array}{ccc} \mathbf{A} & \underset{M}{\overset{K}{\rightleftarrows}} & \mathbf{B} \\ {\scriptstyle K'}\downarrow\uparrow{\scriptstyle M'} & & {\scriptstyle N}\uparrow\downarrow{\scriptstyle L} \\ \mathbf{B}' & \underset{L'}{\overset{N'}{\leftrightarrows}} & \mathbf{C} \end{array} \tag{8.1}$$

a diagram of functors in which M, M', N, and N' are the left adjoints of K, K', L, and L' respectively. Then, as easily follows from (b) and (c), we have $LK \approx L'K' \Leftrightarrow MN \approx M'N'$. □

Lemma 8.2. Every fully faithful functor *reflects isomorphisms*, i.e. under such a functor only isomorphisms are sent to isomorphisms.

Proof. Let $U : \mathbf{A} \to \mathbf{X}$ be a fully faithful functor with $U(f : A \to B)$ being an isomorphism. Since U is full, $U(f)^{-1} = U(g)$ for some $g : B \to A$ in $\mathbf{A}$. Then since $U(gf) = 1_{U(A)}$, $U(fg) = 1_{U(B)}$, and U is faithful, we obtain $gf = 1_A$ and $fg = 1_B$, which shows that $f : A \to B$ is an isomorphism.

Definition 8.3. An adjunction $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ is said to be an adjoint equivalence if η and ε are isomorphisms. □

Theorem 8.4. Let $U : \mathbf{A} \to \mathbf{X}$ be a category equivalence, F_0 a map from the set $\mathbf{X}_0$ of objects in $\mathbf{X}$ to the set $\mathbf{A}_0$ of objects in $\mathbf{A}$, and $\eta = (\eta_X : X \to UF_0(X))_{X\in\mathbf{X}_0}$ a family of isomorphisms. Then there exists a unique functor $F : \mathbf{X} \to \mathbf{A}$ and a unique natural transformation $\varepsilon : FU \to 1_{\mathbf{A}}$, for which F_0 is the object function of F and $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ is an adjunction. Moreover, that adjunction is always an adjoint equivalence.

Proof. Since U is fully faithful (by Lemma 2.7) and each $\eta_X : X \to UF_0(X)$ is an isomorphism, it is easy to see that $\eta_X : X \to UF_0(X)$ is a universal arrow $X \to U$ for each object X in $\mathbf{X}$. After that the first assertion of the theorem follows from Remark 5.7 (see also Definition 5.10). Next, since η_X's are isomorphisms, so are $U(\varepsilon_A)$'s (by the second identity in (5.13)), and by Lemma 8.2 this implies that ε is an isomorphism. □

Theorem 8.5. Let $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ be an adjunction. Then:

(a) U is faithful if and only if ε is an epimorphism;

(b) U is full if and only if ε is a split monomorphism;

(c) and therefore U is fully faithful if and only if ε is an isomorphism.

Proof. For two arbitrary objects A and B in $\mathbf{A}$, consider the diagram

$$\begin{array}{ccc} & & \hom_{\mathbf{X}}(U(A),U(B)) \\ & \overset{U_{A,B}}{\nearrow} & u \mapsto \varepsilon_B F(u) \Big\downarrow \Big\uparrow f \mapsto U(f)\eta_{U(A)} \\ \hom_{\mathbf{A}}(A,B) & & \\ & \underset{\hom_{\mathbf{A}}(\varepsilon_A,B)}{\searrow} & \\ & & \hom_{\mathbf{A}}(FU(A),B), \end{array} \tag{8.2}$$

where the vertical arrows are bijections inverse to each other (since they are $\psi_{U(A),B}$ and $\varphi_{U(A),B}$ respectively: see (5.7) and (5.12)). Since the left-hand vertical arrow is bijective and makes the triangle commute (by naturality of ε), we have:

$U_{A,B}$ is injective $\Leftrightarrow$ $\hom_{\mathbf{A}}(\varepsilon_A,B)$ is injective;
$U_{A,B}$ is surjective $\Leftrightarrow$ $\hom_{\mathbf{A}}(\varepsilon_A,B)$ is surjective;

$U_{A,B}$ is bijective ⇔ $\hom_{\mathbf{A}}(\varepsilon_A, B)$ is bijective.

Since $\hom_{\mathbf{A}}(\varepsilon_A,B)$ is injective, surjective, or bijective if and only if ε_A is an epimorphism, split monomorphism, or isomorphism respectively, this completes the proof. □

Dually, we obtain:

Theorem 8.6. Let $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ be an adjunction. Then:

(a) F is faithful if and only if η is a monomorphism;

(b) F is full if and only if η is a split epimorphism;

(c) and therefore F is fully faithful if and only if η is an isomorphism. □

– which helps to prove the following:

Theorem 8.7. The following conditions on an adjunction $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ are equivalent:

(a) $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ is an adjoint equvalence;

(b) F and U are fully faithful;

(c) F is fully faithful and U reflects isomorphisms;

(d) η is an isomorphism and U reflects isomorphisms;

(e) U is fully faithful and F reflects isomorphisms;

(f) ε is an isomorphism and F reflects isomorphisms.

Proof. (a)⇔(b), (c)⇔(d), and (e)⇔(f) follow from 8.6(a)⇔(c) and 8.5(a)⇔(c). (b)⇒(c) and (b)⇒(e) follow from Lemma 8.1. Therefore it suffices to prove the implications (d)⇒(a) and (f)⇒(a). Moreover, since these implications are dual to each other, it suffices to prove only one of them, say, (d)⇒(a). For, consider the second identity $U(\varepsilon_A)\eta_{U(A)} = 1_{U(A)}$ in (5.13). Assuming that η is an isomorphism, we conclude that so is $U(\varepsilon_A)$ for each A, and, when U reflects isomorphisms, this implies that ε is an isomorphism, as desired. □

9. Remarks on coequalizers

The remarks on coequalizers we make in this section are presented as a definition and an example:

Definition 9.1. (a) A coequalizer diagram in a given category is a diagram in of the form

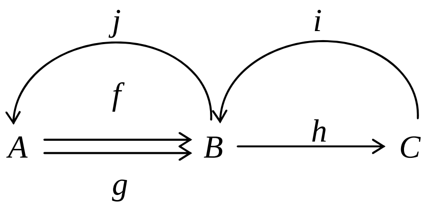
$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C \qquad (9.1)$$

in which $hf = hg$, and for every morphism $h' : B \to C'$ with $h'f = h'g$, there exists a unique morphism $k : C \to C'$ with $kh = h'$. We will then also say that h is the coequalizer of the pair (f,g).

(b) A morphism that occurs in a coequalizer diagram as the morphism *h* occurs in (9.1) is called a regular epimorphism.

(c) A coequalizer diagram is said to be absolute, if it is preserved by any functor, i.e. if its image under any functor is a coequalizer diagram. □

Example 9.2. (a) Consider a *split fork*, i.e. a diagram of the form



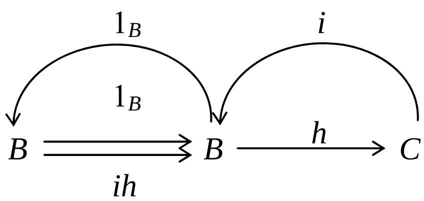
(9.2)

in which $hf = hg$, $hi = 1_C$, $fj = 1_B$, and $gj = ih$. In each such diagram *f*, *g*, and *h* form a coequalizer diagram. Indeed, given $h' : B \to C'$ with $h'f = h'g$, it is easy to see that there is a unique morphism $k : C \to C'$ with $kh = h'$: just take $k = h'i$, which gives

$$kh = h'ih = h'gj = h'fj = h',$$

and the uniqueness follows from the fact that *h* is a split epimorphism. Since the conditions imposed on the diagram (9.2) were purely equational and therefore are "preserved" by every functor, this also proves that *f*, *g*, and *h* form an absolute coequalizer diagram.

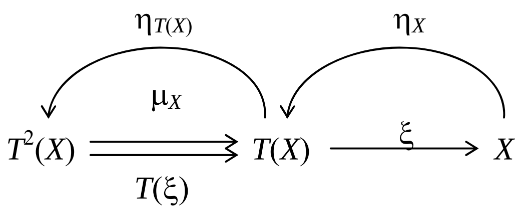
(b) An arbitrary split epimorphism $h : B \to C$ can be involved in a split fork, namely in



(9.3)

where *i* is a splitting, i.e. a morphism from *C* to *B* with $hi = 1_C$. Therefore every split epimorphism is a regular epimorphism.

(c) For a monad $T = (T,\eta,\mu)$ on a category **X**, any *T*-algebra (X,ξ) determines the following split fork in **X**:



□ (9.4)

10. Monadicity

In this section we discuss the relationship between adjunctions and monad.

Theorem 10.1. For every adjunction $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$, the triple $T = (T,\eta,\mu)$ defined by

• $T = UF$,
• η of (T,η,μ) is the same as η of (F,U,η,ε),
• $\mu = U\varepsilon F$, i.e. $\mu = (\mu_X : T^2(X) \to T(X))_{X\in\mathbf{X}_0}$ is defined by $\mu_X = U(\varepsilon_{F(X)})$,

is a monad on **X**.

Proof. For the triple above, and any object X in **X**, the X-component of the diagram (7.1) becomes

$$\begin{array}{ccccc} UFUFUF(X) & \xrightarrow{U(\varepsilon_{FUF(X)})} & UFUF(X) & \xleftarrow{\eta_{UF(X)}} & UF(X) \\ {\scriptstyle UFU(\varepsilon_{F(X)})}\downarrow & & {\scriptstyle U(\varepsilon_{F(X)})}\downarrow & // & \downarrow{\scriptstyle UF(\eta_X)} \\ UFUF(X) & \xrightarrow[U(\varepsilon_{F(X)})]{} & UF(X) & \xleftarrow[U(\varepsilon_{F(X)})]{} & UFUF(X) \end{array} \tag{10.1}$$

and its left-hand square commutes by the naturality of ε while the triangles commute by the triangular identities (5.13). □

Example 10.2. (a) Starting from an arbitrary monad $T = (T,\eta,\mu)$ on a category **X**, we obtain the *forgetful-free* adjunction $(F^T,U^T,\eta^T,\varepsilon^T) : \mathbf{X} \to \mathbf{X}^T$ described in Theorem 7.3. It is easy to see that the corresponding monad on **X** is the same as the original monad $T = (T,\eta,\mu)$. This tells us that every monad can be obtained from an adjunction as in Theorem 10.1. Since this result is originally due to S. Eilenberg and J. Moore, the category $\mathbf{X}^T$ is often called the *Eilenberg-Moore category* (of algebras over T). Note also, that using only free T-algebras, i.e. the T-algebras of the form $F^T(X) = (T(X),\mu_X)$ we could also obtain an adjunction whose corresponding monad is $T = (T,\eta,\mu)$. Furthermore, since such an algebra $(T(X),\mu_X)$ is fully determined by its underlying object X, the full subcategory in $\mathbf{X}^T$ with objects all free T-algebras can be described as the so-called *Kleisli category* of T, whose objects are the same as the objects in X. In detail:

• The category Kleisli(T) is defined as the category with the same objects as the in X, and a morphism $f : X \to Y$ being a morphism $f : X \to T(Y)$ in **X**; the composite of morphisms $f : X \to Y$ and $g : Y \to Z$ in Kleisli(T) is the composite

$$X \xrightarrow{f} T(Y) \xrightarrow{T(g)} T^2(Z) \xrightarrow{\mu_Z} T(Z)$$

in **X**.
• The forgetful functor U : Kleisli(T) $\to$ **X** is defined by $U(f : X \to Y) = \mu_Z T(f) : T(X) \to T(Y)$, and free functor $F : \mathbf{X} \to$ Kleisli(T) is defined by $F(f : X \to Y) = \eta_Y f : X \to T(Y)$, considered as a morphism from X to Y in Kleisli(T).
• And the monad obtained from adjunction as in Theorem 10.1 is again the same as the original monad $T = (T,\eta,\mu)$ (a result due to H. Kleisli). □

It is now natural ask, to what extend is it possible to recover the adjunction $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ from the monad $T = (T,\eta,\mu)$ in the situation of Theorem 10.1? In order to formulate this question properly, we need:

Theorem 10.3. $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ and $T = (T,\eta,\mu)$ be as in Theorem 10.1. Then there exists a unique functor $K : \mathbf{A} \to \mathbf{X}^T$ with $U^T K = U$ and $KF = F^T$.

Proof. *Existence:* Simply define K by

$$K(A) = (U(A),U(\varepsilon_A)). \tag{10.2}$$

To prove that $(U(A),U(\varepsilon_A))$ is indeed a T-algebra is to prove that the diagram

$$\begin{array}{ccccc} UFUFU(A) & \xrightarrow{U(\varepsilon_{FU(A)})} & UFU(A) & \xleftarrow{\eta_{U(A)}} & U(A) \\ {\scriptstyle UFU(\varepsilon_A)}\downarrow & & \downarrow{\scriptstyle U(\varepsilon_A)} & \swarrow\!\!= & \\ UFU(A) & \xrightarrow[U(\varepsilon_A)]{} & U(A) & & \end{array} \tag{10.3}$$

commutes, which, for the left-hand square, follows from the naturality of ε, and, for the triangle, follows from the second identity in (5.13) (cf. (10.1)). Defining K by (10.2), we also obviously have $U^T K = U$, and $KF = F^T$ since $KF(X) = (UF(X),U(\varepsilon_{F(X)})) = (T(X),\mu_X) = F^T(X)$.

Uniqueness: Let $H : \mathbf{A} \to \mathbf{X}^T$ be a functor satisfying $U^T H = U$ and $HF = F^T$. Since $U^T H = U$, such a functor must be given by $H(A) = (U(A),\xi_A)$ for some natural transformation $\xi : UFU \to U$. On the other hand, since $HF = F^T$, we must have $\xi_{F(X)} = \mu_X = U(\varepsilon_{F(X)})$. After that, comparing the naturality square from (10.3) with the naturality square

$$\begin{array}{ccc} UFUFU(A) & \xrightarrow{U(\varepsilon_{FU(A)}) = \xi_{FU(A)}} & UFU(A) \\ {\scriptstyle UFU(\varepsilon_A)}\downarrow & & \downarrow{\scriptstyle U(\varepsilon_A)} \\ UFU(A) & \xrightarrow[\xi_A]{} & U(A) \end{array}$$

we obtain $\xi_A UFU(\varepsilon_A) = U(\varepsilon_A)UFU(\varepsilon_A)$, which implies $\xi_A = U(\varepsilon_A)$, since $UFU(\varepsilon_A)$ is a split epimorphism by the second identity in (5.13). □

Definition 10.4. Let $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ and $T = (T,\eta,\mu)$ be as in Theorems 10.1 and 10.3. Then:

(a) the functor $K : \mathbf{A} \to \mathbf{X}^T$ as in Theorem 10.3 is called the comparison functor;

(b) the functor $U : \mathbf{A} \to \mathbf{X}$ is said to be monadic if the functor $K : \mathbf{A} \to \mathbf{X}^T$ above is a category equivalence. □

Accordingly, saying that an adjunction $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ can be recovered from the corresponding monad $T = (T,\eta,\mu)$ on $\mathbf{X}$ should be understood as saying that the functor U is monadic. In order to formulate some of the monadicity results, we will need the following construction containing long calculations:

Construction 10.5. Let $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ and $T = (T,\eta,\mu)$ be as above, and suppose that for every T-algebra (X,ξ), the pair $(\varepsilon_{F(X)},F(\xi))$ has a coequalizer in $\mathbf{A}$. Then the comparison functor $K : \mathbf{A} \to \mathbf{X}^T$ has a left adjoint forming an adjunction $(L,K,\acute{\eta},\acute{\varepsilon}) : \mathbf{X} \to \mathbf{A}$ that can be described as follows:

(a) For a T-algebra (X,ξ), the object $L(X,\xi)$ is defined via the coequalizer diagram

$$FUF(X) \underset{F(\xi)}{\overset{\varepsilon_{F(X)}}{\rightrightarrows}} F(X) \xrightarrow{\pi_{(X,\xi)}} L(X,\xi) \tag{10.4}$$

(b) For a morphism $h : (X,\xi) \to (X',\xi')$ of T-algebras, we form the diagram

$$\begin{array}{ccccc}
FUF(X) & \underset{F(\xi)}{\overset{\varepsilon_{F(X)}}{\rightrightarrows}} & F(X) & \xrightarrow{\pi_{(X,\xi)}} & L(X,\xi) \\
{\scriptstyle FUF(h)}\downarrow & & \downarrow{\scriptstyle F(h)} & & \vdots\,{\scriptstyle L(h)} \\
FUF(X') & \underset{F(\xi')}{\overset{\varepsilon_{F(X')}}{\rightrightarrows}} & F(X') & \xrightarrow{\pi_{(X',\xi')}} & L(X',\xi')
\end{array} \tag{10.5}$$

of solid arrows, in which

$$\pi_{(X',\xi')}F(h)\varepsilon_{F(X)} = \pi_{(X',\xi')}\varepsilon_{F(X')}FUF(h) = \pi_{(X',\xi')}F(\xi')FUF(h) = \pi_{(X',\xi')}F(h)F(\xi)$$

implies the existence and uniqueness of the dotted arrow making the right-hand square commute. This determines a functor $L : \mathbf{X}^T \to \mathbf{A}$.
(c) We then define $\acute{\eta}_{(X,\xi)} : (X,\xi) \to KL(X,\xi) = (UL(X,\xi),U(\varepsilon_{L(X,\xi)}))$ as the composite $U(\pi_{(X,\xi)})\eta_X$, which we can do since the diagram

$$\begin{array}{ccccc}
UF(X) & \xrightarrow{UF(\eta_X)} & UFUF(X) & \xrightarrow{UFU(\pi_{(X,\xi)})} & UFUL(X,\xi) \\
{\scriptstyle \xi}\downarrow & & & & \downarrow{\scriptstyle U(\varepsilon_{L(X,\xi)})} \\
X & \xrightarrow[\eta_X]{} & UF(X) & \xrightarrow[U(\pi_{(X,\xi)})]{} & UL(X,\xi)
\end{array} \tag{10.6}$$

commutes. Indeed, we have

$U(\varepsilon_{L(X,\xi)})UFU(\pi_{(X,\xi)})UF(\eta_X) = U(\varepsilon_{L(X,\xi)}FU(\pi_{(X,\xi)})F(\eta_X))$ (by functoriality of U)
$= U(\pi_{(X,\xi)}\varepsilon_{F(X)}F(\eta_X))$ (by naturality of ε)
$= U(\pi_{(X,\xi)})$ (by the first identity in (5.13))
$= U(\pi_{(X,\xi)})U(\varepsilon_{F(X)})\eta_{UF(X)}$ (by the second identity in (5.13) applied to $A = F(X)$)
$= U(\pi_{(X,\xi)}\varepsilon_{F(X)})\eta_{UF(X)}$ (by functoriality of U)
$= U(\pi_{(X,\xi)}F(\xi))\eta_{UF(X)}$ (since (10.4) is a coequalizer diagram)

$= U(\pi_{(X,\xi)})UF(\xi)\eta_{UF(X)}$ (by functoriality of U)
$= U(\pi_{(X,\xi)})\eta_X\xi$ (by naturality of η).

(d) To show that $\acute{\eta}_{(X,\xi)}$ is a universal arrow $(X,\xi) \to K$ is to show that for every morphism $k : (X,\xi) \to (U(A),U(\varepsilon_A))$ there exists a unique morphism $l : L(X,\xi) \to A$ with

$$U(l)U(\pi_{(X,\xi)})\eta_X = k.$$

Since $U(l)U(\pi_{(X,\xi)}) = U(l\pi_{(X,\xi)})$ and (F,U,η,ε) is an adjunction, this is the same as to show that there exists a unique morphism $l : L(X,\xi) \to A$ with $l\pi_{(X,\xi)} = \varepsilon_A F(k)$. Since (10.4) is a coequalizer diagram this simply means to show that

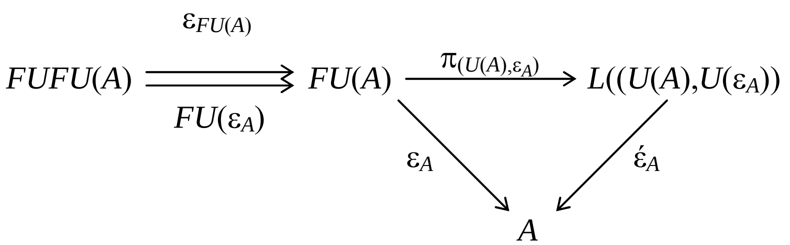
$$\varepsilon_A F(k)\varepsilon_{F(X)} = \varepsilon_A F(k)F(\xi). \tag{10.7}$$

For, we have

$\varepsilon_A F(k)F(\xi) = \varepsilon_A F(k\xi)$ (by functoriality of F)
$= \varepsilon_A F(U(\varepsilon_A)UF(k))$ (since $k : (X,\xi) \to (U(A),U(\varepsilon_A))$ is a morphism of T-algebras)
$= \varepsilon_A FU(\varepsilon_A F(k))$ (by functoriality of U)
$= \varepsilon_A F(k)\varepsilon_{F(X)}$ (by naturality of ε),

as desired.

(e) In particular, for an object A in **A**, the morphism $\acute{\varepsilon}_A : LK(A) \to A$ is the unique morphism $L((U(A),U(\varepsilon_A)) \to A$ making the diagram



(10.8)

commute. □

Remark 10.6. As an intermediate result of the calculation in 10.5(c), we have

$$U(\pi_{(X,\xi)})\eta_X\xi = U(\pi_{(X,\xi)}) \tag{10.9}$$

for every T-algebra $(T(X),\xi)$. Since $\acute{\eta}_{(X,\xi)} : (X,\xi) \to KL(X,\xi)$ was defined (in 10.5(c)) as $U(\pi_{(X,\xi)})\eta_X$, this equality together with Example 9.2 tell us that $\acute{\eta}_{(X,\xi)}$ considered as a morphism in **X** is the unique morphism making the diagram

$$\begin{array}{ccccc} UFUF(X) & \overset{U(\varepsilon_{F(X)}) = \mu_X}{\underset{UF(\xi) = T(\xi)}{\rightrightarrows}} & UF(X) & \xrightarrow{\quad\xi\quad} & X \\ & & & {\scriptstyle U(\pi_{(X,\xi)})}\searrow \quad \swarrow{\scriptstyle \acute{\eta}_{(X,\xi)} = U(\pi_{(X,\xi)})\eta_X} & \\ & & & UL(X,\xi) & \end{array} \qquad (10.10)$$

commute. □

Theorem 10.7. For $(F,U,\eta,\varepsilon) : \mathbf{X} \to \mathbf{A}$ and $T = (T,\eta,\mu)$ as above the following conditions are equivalent:

(a) the functor $U : \mathbf{A} \to \mathbf{X}$ is monadic;

(b) the functor U preserves the coequalizer diagram (10.4) for every T-algebra (X,ξ), and, for every object A in $\mathbf{A}$, the morphism ε_A is the coequalizer of the pair $(\varepsilon_{FU(A)},FU(\varepsilon_A))$;

(c) the functor U reflects isomorphisms and preserves the coequalizer diagram (10.4) for every T-algebra (X,ξ);

(d) the functor U reflects isomorphisms, and every pair (f,g) of parallel morphisms in $\mathbf{A}$, for which the pair $(U(f),U(g))$ has an absolute coequalizer, has a coequalizer preserved by U.

Proof. We observe:
• Since $U^TK = U$, and $U^T : \mathbf{X}^T \to \mathbf{X}$ obviously reflects isomorphisms, U reflects isomorphisms if and only if K does.
• As follows from Remark 10.6 and the fact that the top part of the diagram (10.10) is a coequalizer diagram (see Example 9.2), the functor U preserves the coequalizer diagram (10.4) if and only if $\acute{\eta}_{(X,\xi)} : (X,\xi) \to KL(X,\xi)$ is an isomorphism.
• As follows from 10.5(e), the morphism ε_A is the coequalizer of the pair $(\varepsilon_{FU(A)},FU(\varepsilon_A))$ if and only if $\acute{\varepsilon}_A : LK(A) \to A$ is an isomorphism.
• This proves (a)⇔(b) and makes (b)⇔(c) a consequence of Theorem 8.7 (in fact a consequence of the last argument in its proof).
• Since the pair $(U(\varepsilon_{F(X)}),UF(\xi)) = (\mu_X,T(\xi))$ involved in (10.10) is a part of a split fork (9.4), (d) implies (c).
• After this all we need to prove is that if (f,g) of parallel morphisms in $\mathbf{X}^T$, for which the pair (f,g) has an absolute coequalizer in $\mathbf{X}$, then (f,g) has a coequalizer in $\mathbf{X}^T$ preserved by U^T. For, consider the diagram

$$\begin{array}{ccccc} T(X) & \overset{T(f)}{\underset{T(g)}{\rightrightarrows}} & T(Y) & \xrightarrow{T(h)} & T(Z) \\ \downarrow & & \downarrow & & \vdots \\ X & \overset{f}{\underset{g}{\rightrightarrows}} & Y & \xrightarrow{h} & Z \end{array} \qquad (10.11)$$

where: h is the coequalizer of (f,g) in $\mathbf{X}$; the left-hand and the middle vertical arrow are the domain and the codomain of f (and of g) respectively in the category $\mathbf{X}^T$; and the dotted arrow is determined by the fact that the top row in (10.11) is a coequalizer diagram (since h is the absolute coequalizer of (f,g) in **X**). Using the fact that not only T but also T^2 preserves the equalizer of (f,g), it is easy to check that the dotted arrow determines a T-algebra structure on Z and then makes h is the coequalizer of (f,g) in $\mathbf{X}^T$ – and this coequalizer is trivially preserved by U^T. □

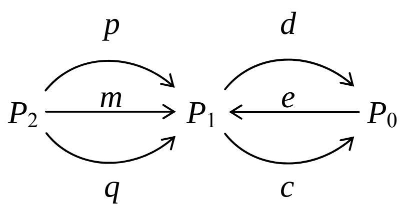
Remark 10.8. (a) Condition 10.7(d) can modified by asking the pair $(U(f),U(g))$ to be a split coequalizer (i.e. to be a part of a split fork) instead of an absolute one. As one can see from the argument proving (d)⇒(c) of Theorem 10.7, this follows from the fact that the diagram (9.4) is a split fork.

(b) The pair $(\varepsilon_{F(X)},F(\xi))$ involved in (10.4) is *reflexive*, which means that $\varepsilon_{F(X)}$ and $F(\xi)$ are split epimorphisms with a common splitting – which is $F(\eta_X)$. Therefore using the same arguments as in the proof of Theorem 10.7, we can prove the following: if a functor admits a left adjoint, reflects isomorphisms, and preserves coequalizers of reflexive pairs, then it is monadic. □

11. Internal precategory actions

This section presents generalized versions of very first concepts of internal category theory need for the purposes of categorical Galois theory.

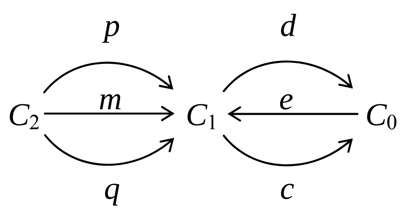
Definition 11.1. An internal precategory in a category **X** is a diagram
$P = (P_0,P_1,P_2,d,c,e,m) =$



(11.1)

in **X** with $de = 1 = ce$, $dp = cq$, $dm = dq$, and $cm = cp$. An internal precategory in **Sets** is simply called a precategory. □

Example 11.2. Any (small) category C can be regarded as a precategory; it is then to be displayed as



(11.2)

where:

- C_0 is the set of objects in C;
- C_1 is the set of morphisms in C;
- d and c are the domain map and the codomain map respectively, i.e. $d(f) = x$ and $c(f) = y$ if and only if f is a morphism from x to y;
- $C_2 = \{(g,f) \mid d(g) = c(f)\}$ is the set of composable pairs of morphisms in C;

• p and q are the projection maps, i.e. $p(g,f) = g$ and $q(g,f) = f$. □

Example 11.2 suggests:

Definition 11.3. An internal category in a category **X** with pullbacks is an internal precategory C in **X**, in which the diagram formed by d, c, p, q is a pullback (yielding $C_2 = C_1{\times}_{(d,c)}C_1$) and the diagram

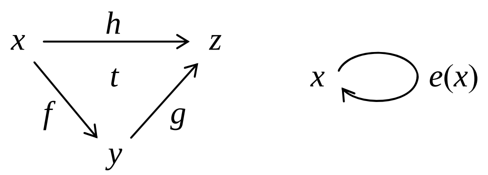
$$\begin{array}{ccccc} C_1{\times}_{(d,c)}C_1{\times}_{(d,c)}C_1 & \xrightarrow{1\times m} & C_1{\times}_{(d,c)}C_1 & \xleftarrow{\langle ec,1\rangle} & C_1 \\ {\scriptstyle m\times 1}\downarrow & & {\scriptstyle m}\downarrow & \diagup\!\!\diagup & \downarrow{\scriptstyle \langle 1,ed\rangle} \\ C_1{\times}_{(d,c)}C_1 & \xrightarrow[m]{} & C_1 & \xleftarrow[m]{} & C_1{\times}_{(d,c)}C_1 \end{array} \qquad (11.3)$$

commutes. □

Observation 11.4. (a) Comparing diagrams (11.3) and (6.12) makes clear that an internal category C in **X** is nothing but a monoid in the monoidal category (**Graphs**(**X**,O),I,⊗,α,λ,ρ), described in Example 6.5, for $O = C_0$.

(b) An internal category in **Sets** is of course the same as an ordinary (small) category. □

The readers familiar with simplicial sets might prefer to consider precategories as truncated simplicial sets, and present Example 11.2 via the notion of *nerve* of a category. According to this approach, but also independently of it, given a precategory P, it is convenient to use displays like

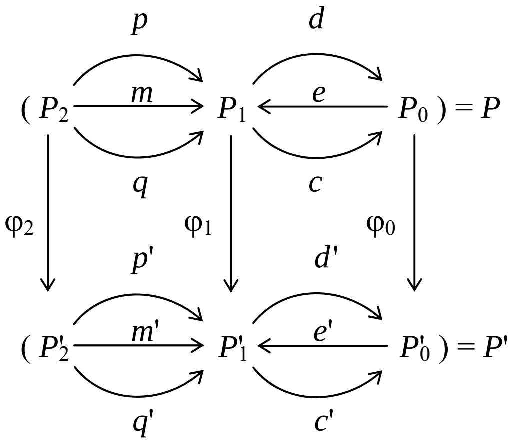


(11.4)

for t in P_2, $g = p(t)$, $f = q(t)$, $h = m(t)$, $x = d(f) = d(h)$, $y = d(g) = c(f)$, and $z = c(g) = c(h)$. Note that these displays "remember" all identities required in Definition 11.1.

Thinking of internal precategories as generalized categories, we are going now to generalize functors. In fact there are several concepts to be introduced, and the first obvious step is to define precategory morphisms as the corresponding diagram morphisms, which brings us to

Definition 11.5. Let P and P' be internal precategories in **X**. A morphism $\varphi : P \to P'$ is a diagram in **X** of the form



(11.5)

which *reasonably commutes*, i.e. has $\varphi_0 d = d'\varphi_1$, $\varphi_0 c = c'\varphi_1$, $\varphi_1 e = e'\varphi_0$, $\varphi_1 p = p'\varphi_2$, $\varphi_1 q = q'\varphi_2$, and $\varphi_1 m = m'\varphi_2$. A morphism $\varphi : P \to P'$ above is said to be

(a) a discrete fibration if the squares $\varphi_0 c = c'\varphi_1$ and $\varphi_1 p = p'\varphi_2$ in (11.5) are pullbacks;

(b) a discrete opfibration if the squares $\varphi_0 d = d'\varphi_1$ and $\varphi_1 q = q'\varphi_2$ in (11.5) are pullbacks. □

Remark 11.6. It is easy to show that if $\varphi : P \to P'$ is a discrete fibration and P' is an internal category, then P also is an internal category. On the other hand, if P and P' were internal categories, then $\varphi : P \to P'$ is a discrete fibration whenever just the square $\varphi_0 c = c'\varphi_1$ in (11.5) is a pullback. Therefore discrete fibrations of internal categories in **Sets** are the same as ordinary ones defined in 4.1(b). □

Next, we need "functors" $P \to \mathbf{X}$, and since this concept is less obvious, let us begin with the case $\mathbf{X} = \mathbf{Sets}$:

Definition 11.7. Let P be a precategory. Then:

(a) For a category C, a prefunctor $P \to C$ is a precategory morphism $P \to C$, where C is regarded as a precategory in the same way as in Example 11.2.

(b) A P-action is a diagram $A = (A_0,\pi,\xi) =$

$$P_1\times_{P_0}A_0 \xrightarrow{\ \xi\ } A_0 \xrightarrow{\ \pi\ } P_0, \tag{11.6}$$

where $P_1\times_{P_0}A_0 = P_1\times_{(d,\pi)}A_0$ is the pullback of d and π, ξ is written as $\xi(f,a) = fa$, and

$$\pi(fa) = c(f),\ e(x)a = a,\ ha = g(fa) \tag{11.7}$$

in the situation (11.4) whenever $\pi(a) = x$. □

Remark 11.8. (a) When P is a category (see Example 11.2), the equalities (11.7) are to be rewritten as

$$\pi(fa) = c(f),\ 1_x a = a,\ (gf)a = g(fa). \tag{11.8}$$

That is, when P is a category, a P-action is nothing but a functor from P to **Sets**. To be absolutely precise, we should say that in that case there is a canonical equivalence between the category of P-actions and the category of functors from P to **Sets**.

(b) The general case reduces to the case of categories. Indeed: Let

$$L : \text{Precategories} \to \text{Categories} \tag{11.9}$$

be the left adjoint of the inclusion functor from the category of categories to the category of precategories. Explicitly, for a precategory $P = (P_0,P_1,P_2,d,c,e,m)$, the category $L(P)$ is the quotient category $\mathrm{Pa}(G)/\sim$, where:

• $\mathrm{Pa}(G)$ is the free category ("the category of paths") on the underlying graph $G = (P_0,P_1,d,c)$ of P; that is, the objects of $\mathrm{Pa}(G)$ are the elements of P_0, and a morphism $x \to y$ is a finite (possibly empty) sequence $(f_0,\ldots,f_n)$, in which $d(f_n) = x$, $c(f_i) = d(f_{i-1})$ (for $i = 1,\ldots,n$), and $c(f_0) = y$.
• $\sim$ is the smallest *congruence* on $\mathrm{Pa}(G)$, for which $e(x) \sim 1_x$ and $m(t) \sim p(t)q(t)$ for each x in P_0 and t in P_2.

Requiring $m(t) \sim p(t)q(t)$ here is of course the same to require $h \sim gf$ in the situation (11.4), and the category of P-actions can be identified with the category of $L(P)$-actions.

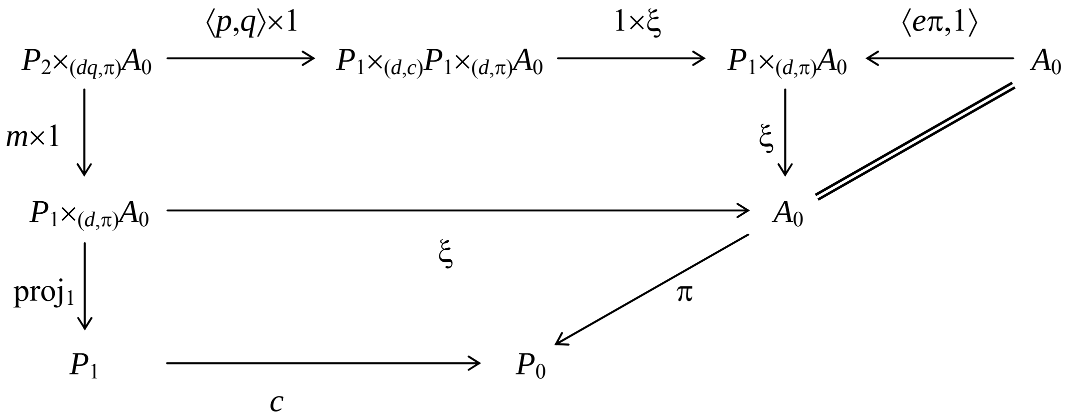
(c) The category of P-actions is canonically equivalent to the category of prefunctors $P \to$ **Sets**. This can be either shown directly, or deduced from (a) and (b), since the category of prefunctors $P \to$ **Sets** is obviously canonically isomorphic to the category of functors $L(P) \to$ **Sets**. □

Internalizing now Definition 11.7(b) we arrive at:

Definition 11.9. Let $P = (P_0,P_1,P_2,d,c,e,m)$ be an internal precategory in a category **X** with pullbacks. A P-action is a diagram $A = (A_0,\pi,\xi) =$

$$P_1\times_{P_0}A_0 \xrightarrow{\ \xi\ } A_0 \xrightarrow{\ \pi\ } P_0, \tag{11.10}$$

where $P_1\times_{P_0}A_0 = P_1\times_{(d,\pi)}A_0$ is the pullback of d and π, and the diagram



(11.11)

commutes. The category of P-actions will be denoted by $\mathbf{X}^P$. □

Remark 11.10. When P is an internal category, the diagram (11.11) becomes

$$\begin{array}{ccccc}
P_1\times_{(d,c)}P_1\times_{(d,\pi)}A_0 & \xrightarrow{1\times\xi} & P_1\times_{(d,\pi)}A_0 & \xleftarrow{\langle e\pi,1\rangle} & A_0 \\
{\scriptstyle m\times 1}\downarrow & & \downarrow{\scriptstyle \xi} & \swarrow\!\!= & \\
P_1\times_{(d,\pi)}A_0 & \xrightarrow[\xi]{} & A_0 & & \\
{\scriptstyle \mathrm{proj}_1}\downarrow & & \downarrow{\scriptstyle \pi} & & \\
P_1 & \xrightarrow[c]{} & P_0 & &
\end{array} \qquad (11.12)$$

This makes a P-action a special case of an M-action in the sense of Example 7.4. Specifically:

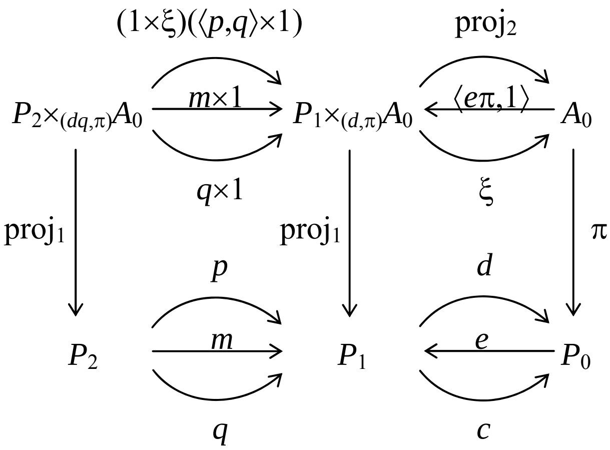
- we take the monoidal category **C** of Example 7.4 to be $(\mathbf{Graphs}(\mathbf{X},P_0),I,\otimes,\alpha,\lambda,\rho)$;
- the role of **X** in Example 7.4 will be played by the comma category $(\mathbf{X}\downarrow P_0)$ (of pairs $A = (A_0,\pi)$, where $\pi : A_0 \to P_0$ is a morphism in **X**);
- the **C**-action on $(\mathbf{Graphs}(\mathbf{X},P_0),I,\otimes,\alpha,\lambda,\rho)$ is defined in the obvious way using $P\otimes A = (P_1\times_{(d,\pi)}A_0,c(\mathrm{proj}_1))$ defined via

$$\begin{array}{ccccc}
 & & P_1\times_{(d,\pi)}A_0 & & \\
 & {\scriptstyle \mathrm{proj}_1}\swarrow & & \searrow{\scriptstyle \mathrm{proj}_2} & \\
 & P_1 & & A_0 & \\
{\scriptstyle c}\swarrow & & {\scriptstyle d}\searrow & & \swarrow{\scriptstyle \pi} \\
P_0 & & & P_0; &
\end{array} \qquad (11.13)$$

- then since P becomes a monoid in $(\mathbf{Graphs}(\mathbf{X},P_0),I,\otimes,\alpha,\lambda,\rho)$, we have the category $(\mathbf{X}\downarrow P_0)^P$ of P-actions in the sense of Example 7.4, and it coincides with the category $\mathbf{X}^P$ of P-actions in the sense of Definition 11.9. □

We end this section with a natural (dual) internal-precategorical version of the results of Section 4 concerning discrete fibrations:

Theorem 11.11. Let $P = (P_0,P_1,P_2,d,c,e,m)$ be an internal precategory in a category **X** with pullbacks and $A = (A_0,\pi,\xi)$ a P-action. Then the diagram



(11.14)

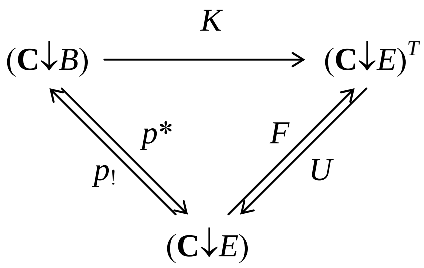
is a discrete opfibration. Moreover, sending A to the so defined opfibration determines an equivalence between the category $\mathbf{X}^P$ of P-actions and the category DisOpfib(P) of discrete opfibrations over P (i.e. the category of discrete opfibrations ? $\to P$ considered as a full subcategory of the comma category ((Precategories in $\mathbf{X}$)$\downarrow P$)).

Proof is a routine calculation. □

12. Descent via monadicity and internal actions

In this section we develop a simplified approach to Grothendieck descent theory suitable for our purposes.

Let $p : E \to B$ be a fixed morphism in a category $\mathbf{C}$ with pullbacks. Consider the diagram



(12.1)

in which:

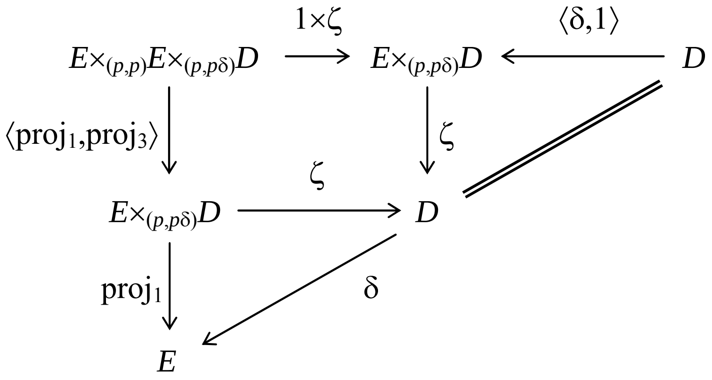
• $p_!$ is defined as the composition with p, i.e. by $p_!(D,\delta) = (D,p\delta)$;
• p^* is the pullback-along-p (change-of-base functor determined by p), and we will write $p^*(A,\alpha) = (E\times_{(p,\alpha)}A,\mathrm{proj}_1) = (E\times_B A,\mathrm{proj}_1)$;
• it is to see that $p_!$ is the left adjoint p^*, and T denotes the corresponding monad on $(\mathbf{C}\downarrow E)$;
• $(\mathbf{C}\downarrow E)^T$ is the category of T-algebras and U, F, and K the corresponding forgetful functor, free functor, and comparison functor respectively.

Explicitly:

• a T-algebra is a diagram $(D,\delta,\zeta) =$

$$E\times_{(p,\,p\delta)}D \xrightarrow{\ \zeta\ } D \xrightarrow{\ \delta\ } E, \qquad (12.2)$$

for which the diagram



(12.3)

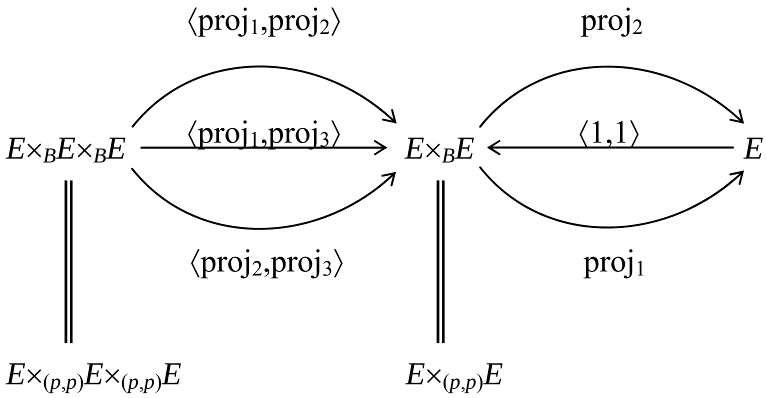
commutes;

• the functor U is defined by $U(D,\delta,\zeta) = (D,\delta)$;

• the functor F is defined by $F(D,\delta) = (E\times_{(p,p\delta)}D, \mathrm{proj}_1, \langle \mathrm{proj}_1, \mathrm{proj}_3\rangle)$, where (here and below) proj_i ($i = 1, 2, 3$) are suitable projections;

• the functor K is defined by $K(A,\alpha) = (E\times_{(p,\alpha)}A, \mathrm{proj}_1, \langle \mathrm{proj}_1, \mathrm{proj}_3\rangle)$.

The diagrams (12.2) and (12.3) look almost similar to the diagrams (11.10) and (11.11) (see also (11.12)), and in fact they are special cases of those. For, let us take ($\mathbf{X} = \mathbf{C}$ and) P to be the internal category Eq(p) =



(12.4)

and write (D,δ) instead of (A_0,π) in (11.10) and (11.11). Then (11.10) becomes

$$(E\times_{(p,p)}E)\times_{(\mathrm{proj}_2,\delta)}D \xrightarrow{\ \xi\ } D \xrightarrow{\ \delta\ } E, \tag{12.5}$$

and a straightforward calculation proves:

Theorem 12.1. For an object (D,δ) in $(\mathbf{C}\downarrow E)$, the morphism

$$\bar{\delta} = \langle \mathrm{proj}_1, \mathrm{proj}_3\rangle : (E\times_{(p,p)}E)\times_{(\mathrm{proj}_2,\delta)}D \to E\times_{(p,p\delta)}D \tag{12.6}$$

is an isomorphism and (D,δ,ζ) is a T-algebra if and only if $(D,\delta,\zeta\bar{\delta})$ is an Eq(p)-action. Moreover, sending (D,δ,ζ) to $(D,\delta,\zeta\bar{\delta})$ determines a category isomorphism

$$(\mathbf{C}\downarrow E)^T \approx \mathbf{C}^{\mathrm{Eq}(p)}. \ \square \tag{12.7}$$

Remark 12.2. (a) As the notation obviously suggests, Eq(p) is nothing but the right internal version of the equivalence on E determined by p. Moreover, of course there are suitable notions of an internal groupoid, an internal preorder, an internal equivalence relation, and the opposite internal category to a given one, for which:

• every internal groupoid is isomorphic to its opposite internal groupoid;
• a morphism of internal groupoids is a discrete fibration if and only it is a discrete opfibration;
• an internal preorder is the same as an internal category whose domain morphism and codomain morphism are jointly monic;
• an internal equivalence relation is the same as an internal groupoid that is an internal preorder.

In particular we do not need to be too careful in distinguishing Eq(p) from its opposite internal equivalence relation.

(b) Every morphism $\varphi : P \to P'$ of internal precategories in **C** obviously determines an induced functor $\mathbf{C}^{\varphi} : \mathbf{C}^{P'} \to \mathbf{C}^{P}$, and this determines a *pseudofunctor* (where "pseudo" refers to preservation of composition and identities only up to "good" isomorphisms; omitting details let us just mention that this is similar to "preservation" of ⊗ by monoidal functors)

$$\mathbf{C}^{?} : \mathrm{Precat}(\mathbf{C})^{\mathrm{op}} \to \mathbf{Cat}, \tag{12.8}$$

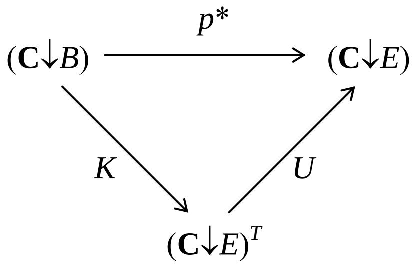
where Precat(**C**) and **Cat** denote the category of internal precategories in **C** and the category of categories respectively. In particular applying this pseudofunctor to the commutative diagram

$$\begin{array}{ccc} \mathrm{Eq}(1_E) = E & \xrightarrow{\;p\;} & B = \mathrm{Eq}(1_B) \\ & \searrow \qquad \nearrow & \\ & \mathrm{Eq}(p), & \end{array} \tag{12.9}$$

(in the obvious notation) and identifying $\mathbf{C}^{\mathrm{Eq}(1_E)}$ and $\mathbf{C}^{\mathrm{Eq}(1_B)}$ with $(\mathbf{C}\downarrow E)$ and $(\mathbf{C}\downarrow B)$ respectively, we obtain a diagram

$$\begin{array}{ccc} (\mathbf{C}\downarrow B) & \longrightarrow & (\mathbf{C}\downarrow E) \\ & \searrow \qquad \nearrow & \\ & \mathbf{C}^{\mathrm{Eq}(p)} & \end{array} \tag{12.10}$$

that can be identified with



(12.11)

via the isomorphism (12.7). □

Definition 12.3. A morphism $p : E \to B$ in a category **C** with pullbacks is said to be:

(a) a descent morphism if the comparison functor $K : (\mathbf{C}{\downarrow}B) \to (\mathbf{C}{\downarrow}E)^T$ is fully faithful;

(b) an effective descent morphism if the functor p^* is monadic, i.e. if K is an equivalence of categories; in this and in the more general situation considered in later sections we will also say that $p : E \to B$ is a monadic extension. □

13. Galois structures and admissibility

Admissible Galois structures introduced in this section are the basic categorical structures for Galois theory in general categories.

Definition 13.1. A Galois structure is a system $(\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$, in which

$$(I,H,\eta,\varepsilon) : \mathbf{C} \to \mathbf{X} \tag{13.1}$$

is an adjunction, and **F** and **Φ** class of morphisms in **C** and in **X** respectively, satisfying the following conditions:

(a) $I(\mathbf{F}) \subseteq \mathbf{\Phi}$ and $H(\mathbf{\Phi}) \subseteq \mathbf{F}$.

(b) The category **C** admits pullbacks along morphisms from **F**, and the class **F** is pullback stable; similarly, the category **X** admits pullbacks along morphisms from **Φ**, and the class **Φ** is pullback stable. Furthermore, the classes **F** and **Φ** contain all isomorphisms in **C** and **X** respectively. □

Given a Galois structure $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ and an object B in **C**, there is an induced adjunction

$$(I^B,H^B,\eta^B,\varepsilon^B) : \mathbf{F}(B) \to \mathbf{\Phi}(I(B)), \tag{13.2}$$

in which:

- $\mathbf{F}(B)$ is the full subcategory in $(\mathbf{C}{\downarrow}B)$ with objects all pairs (A,α) with $\alpha : A \to B$ in **F**;
- similarly $\mathbf{\Phi}(I(B))$ is the full subcategory in $(\mathbf{X}{\downarrow}I(B))$ with objects all pairs (X,φ) with $\varphi : X \to I(B)$ in **Φ**;
- $I^B(A,\alpha) = (I(A),I(\alpha))$;
- $H^B(X,\varphi) = (B\times_{HI(B)}H(X),\mathrm{proj}_1)$ is defined via the pullback

$$\begin{array}{ccc} B\times_{HI(B)}H(X) & \xrightarrow{\text{proj}_2} & H(X) \\ \text{proj}_1\downarrow & & \downarrow H(\varphi) \\ B & \xrightarrow[\eta_B]{} & HI(B); \end{array} \tag{13.3}$$

• $(\eta^B)_{(A,\alpha)} = \langle\alpha,\eta_A\rangle : A \to B\times_{HI(B)}HI(A)$;

• $(\varepsilon^B)_{(X,\varphi)}$ is the composite

$$I(B\times_{HI(B)}H(X)) \xrightarrow{I(\text{proj}_2)} IH(X) \xrightarrow{\varepsilon_X} X, \tag{5.4}$$

where proj_2 is as in (5.3).

Using the notation above we introduce

Definition 13.2. An object B in $\mathbf{C}$ is said to be admissible if $\varepsilon^B : I^BH^B \to 1_{\Phi(I(B))}$ is an isomorphism. If this is the case for each B in $\mathbf{C}$, then we say that the Galois structure $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ is admissible. □

Obvious but important:

Proposition 13.3. If $\varepsilon : IH \to 1_{\mathbf{X}}$ is an isomorphism, then the following conditions on an object B in $\mathbf{C}$ are equivalent:

(a) B is admissible;

(b) the functor $H^B : \mathbf{\Phi}(I(B)) \to \mathbf{F}(B)$ is fully faithful;

(c) the functor I preserves all pullbacks of the form (5.3). □

Convention 13.4. From now on $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ will denote a fixed admissible Galois structure in which $\varepsilon : IH \to 1_{\mathbf{X}}$ is an isomorphism, and so the equivalent conditions of Proposition 5.3 hold. □

More precisely, we will freely use this convention in Sections 14 and 19, and it will hold true in all examples of Sections 15-18, which we will prove there.

14. Monadic extensions and coverings

In this section we introduce the main notions of categorical Galois theory (using Convention 13.4).

Given a morphism $p : E \to B$ in $\mathbf{C}$, pulling back along p determines a functor

$$p^* : \mathbf{F}(B) \to \mathbf{F}(E), \tag{14.1}$$

and the composition with p determines a functor

$$p_! : \mathbf{F}(E) \to \mathbf{F}(B), \tag{14.2}$$

which is the left adjoint of p^*.

Definition 14.1. A pair (E,p), in which $p : E \to B$ is morphism in **C**, or a morphism $p : E \to B$ itself, is said to be a monadic extension of B if the following conditions hold:

(a) If (D,δ) is in $\mathbf{F}(E)$, then $(D,p\delta)$ is in $\mathbf{F}(B)$;

(b) the functor $p^* : \mathbf{F}(B) \to \mathbf{F}(E)$ is monadic. □

We are now ready to introduce our main definition:

Definition 14.2. (a) An object (A,α) in $\mathbf{F}(B)$ is said to be a trivial covering (of B) if the morphism $(\eta^B)_{(A,\alpha)} : (A,\alpha) \to H^B I^B(A,\alpha)$ is an isomorphism, or, equivalently, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HI(A) \\ \alpha\big\downarrow & & \big\downarrow HI(\alpha) \\ B & \xrightarrow[\eta_B]{} & HI(B) \end{array} \tag{14.3}$$

is a pullback.

(b) An object (A,α) in $\mathbf{F}(B)$ is said to be split over a monadic extension (E,p) of B if $p^*(A,\alpha)$ is a trivial covering.

(c) An object (A,α) in $\mathbf{F}(B)$ is said to be a covering of B if there exists a monadic extension (E,p) of B such that (A,α) is split over (E,p). We will then also say that $\alpha : A \to B$ is a covering morphism. □

According to this definition we have

$$\mathrm{TrivCov}(B) = \mathrm{Spl}(B,1_B) \subseteq \mathrm{Cov}(B) = \bigcup_{(E,p)} \mathrm{Spl}(E,p) \subseteq \mathbf{F}(B), \tag{14.4}$$

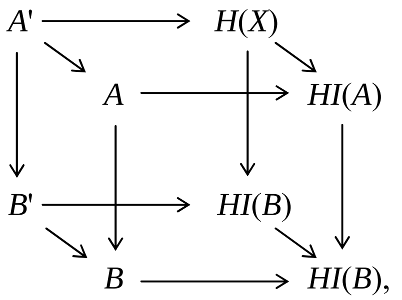
where:

- $\mathrm{TrivCov}(B)$ is the full subcategory in $\mathbf{F}(B)$ with objects all trivial coverings of B;
- $\mathrm{Spl}(E,p)$ is the full subcategory in $\mathbf{F}(B)$ of all objects split over (E,p);
- $\mathrm{Cov}(B)$ is the full subcategory in $\mathbf{F}(B)$ with objects all coverings of B;
- the union of $\mathrm{Spl}(E,p)$'s in (6.4) is taken over all monadic extensions (E,p) of B.

Remark 14.3. The following simple properties of coverings are useful:

(a) Since $\varepsilon^B : I^B H^B \to 1_{\Phi(I(B))}$ is always an isomorphism, an object (A,α) in $\mathbf{F}(B)$ is a trivial covering if and only if $(A,\alpha) \approx H^B(X,\varphi)$ for some (X,φ) in $\Phi(I(B))$.

(b) For every morphism $\beta : B' \to B$, the functor $\beta^* : \mathbf{F}(B) \to \mathbf{F}(B')$ sends trivial coverings to trivial coverings, and the functor I preserves pullbacks along trivial coverings. To see this, consider the cube diagram



(14.5)

where the left-hand face is a pullback, $A \to B$ is a trivial covering, the right-hand face is the H-image of the pullback formed by the I-images of $B' \to B$ and $A \to B$, and the arrows connecting the left-hand and right-hand faces are canonical morphisms. In this diagram all vertical faces are pullbacks, and, by the admissibility, X can be identified with $I(A')$. This implies our assertions above.

(c) As follows from (b), for monadic extensions (E,p) and (E',p'), we have $\mathrm{Spl}(E,p) \subseteq \mathrm{Spl}(E',p')$ whenever p' factors through p.

(d) Using some further arguments one can show that the union in (14.4) is in fact directed. □

15. Categories of abstract families

In this section we present an example of an admissible Galois structure, which will later help us to present the classical Galois theory as a special case of the categorical one. We take $\mathbf{X}$ to be a full subcategory of the category of sets, closed under finite limits, and $\mathbf{A}$ an arbitrary category that has a terminal object 1.

Definition 15.1. The category $\mathrm{Fam}_{\mathbf{X}}(\mathbf{A})$ of families of objects in $\mathbf{A}$ with index sets in $\mathbf{X}$ has:

(a) its objects all families $A = (A_i)_{i\in I(A)}$ of objects A_i in $\mathbf{A}$ with $I(A)$ in $\mathbf{X}$;

(b) a morphism $A \to B$ in $\mathrm{Fam}_{\mathbf{X}}(\mathbf{A})$ is a pair (f,α), in which $f : I(A) \to I(B)$ is a map of sets and α is a family of morphisms $\alpha = (\alpha_i : A_i \to B_{f(i)})_{i\in I(A)}$ in $\mathbf{A}$. □

Sending $(f,\alpha) : A \to B$ to $f : I(A) \to I(B)$ determines a functor $I : \mathrm{Fam}_{\mathbf{X}}(\mathbf{A}) \to \mathbf{X}$, with the right adjoint H defined by $H(X) = (A_i)_{i\in I(A)}$, where $I(A) = X$ and $A_i = 1$ for all i. This can easily be checked either directly, or using the following obvious facts

• Sending $\mathbf{A}$ to $\mathrm{Fam}_{\mathbf{X}}(\mathbf{A})$ determines a 2-functor

$$\mathrm{Fam}_{\mathbf{X}} : \mathbf{CAT} \to \mathbf{CAT}, \tag{15.1}$$

where $\mathbf{CAT}$ is the 2-category of all categories.

• $\mathrm{Fam}_{\mathbf{X}}(\mathbf{1})$ is canonically isomorphic to $\mathbf{X}$, where $\mathbf{1}$ denotes a (the) one-morphism category (=the terminal object in the category of all categories).

• The unique functor $\mathbf{A} \to \mathbf{1}$ has the right adjoint sending the unique object of $\mathbf{1}$ to the terminal 1 object in $\mathbf{A}$, and the functor $I : \mathrm{Fam}_{\mathbf{X}}(\mathbf{A}) \to \mathbf{X}$ above is nothing but the composite $\mathrm{Fam}_{\mathbf{X}}(\mathbf{A}) \to \mathrm{Fam}_{\mathbf{X}}(\mathbf{1}) \approx \mathbf{X}$.

It is then easy to prove:

Theorem 15.2. Let $\Gamma = (\mathrm{Fam}_{\mathbf{X}}(\mathbf{A}),\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ be a Galois structure, in which $I : \mathrm{Fam}_{\mathbf{X}}(\mathbf{A}) \to \mathbf{X}$ and $H : \mathbf{X} \to \mathrm{Fam}_{\mathbf{X}}(\mathbf{A})$ are as above, with suitable η and ε, $\mathbf{\Phi}$ the class of all morphisms in $\mathbf{X}$, and $\mathbf{F}$ an arbitrary class of morphisms in $\mathrm{Fam}_{\mathbf{X}}(\mathbf{A})$ containing $H(\mathbf{\Phi})$ and satisfying 5.1(b). Then $\varepsilon : IH \to 1_{\mathbf{X}}$ is an isomorphism and Γ is admissible. □

Theorem 15.3. Let $\Gamma = (\mathrm{Fam}_{\mathbf{X}}(\mathbf{A}),\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ be as in Theorem 7.2, and $(f,\alpha) : A \to B$ be in $\mathbf{F}$. Then $(A,(f,\alpha))$ is a trivial covering of B if and only if $\alpha_i : A_i \to B_{f(i)}$ is an isomorphism for each $i \in I(A)$. □

16. Coverings in classical Galois theory

In this section we describe the relationship between the separable/Galois extensions in classical Galois theory and covering morphisms of categorical Galois theory.

Here, K denotes a field, $\mathbf{C}$ the opposite category of commutative unitary K-algebras that are finite-dimensional as K-vector spaces, and $\mathbf{X}$ the category of finite sets. We define here $I : \mathbf{C} \to \mathbf{X}$ by

$$I(A) = \text{the set of minimal (non-zero) idempotents in } A; \qquad (16.1)$$

that is $I(A)$ consists of all $e \in A$ such that $e^2 = e \neq 0$ and $e'^2 = e' \neq 0 \neq ee'$ implies $ee' = e$. Sending A to the family $(Ae)_{e \in I(A)}$ determines a category equivalence

$$\mathbf{C} \sim \mathrm{Fam}_{\mathbf{X}}(\mathbf{A}), \qquad (16.2)$$

where $\mathbf{A}$ is the full subcategory in $\mathbf{C}$ with objects all (commutative unitary) K-algebras with no non-trivial idempotents, i.e. no elements e with $e^2 = e$ and $0 \neq e \neq 1$. Moreover, the functor $I : \mathbf{C} \to \mathbf{X}$ above is a special case of the one defined in the previous section up to the equivalence (16.2). Using this fact and Theorem 15.2 we obtain:

Theorem 16.1. Let $I : \mathbf{C} \to \mathbf{X}$ be as above, $H : \mathbf{X} \to \mathbf{C}$ the right adjoint of I defined therefore by

$$H(X) = \underbrace{K + \ldots + K}_{\substack{\text{coproduct}\\ \text{in } \mathbf{C} \text{ of } K\\ \text{with itself}\\ \text{“}X\text{-times”}}} = \text{the } K\text{-algebra of all maps from } X \text{ to } K, \qquad (16.3)$$

η and ε the unit and counit of adjunction, and $\mathbf{F}$ and $\mathbf{\Phi}$ the classes of all morphisms in $\mathbf{C}$ and in $\mathbf{X}$ respectively. Then $\varepsilon : IH \to 1_{\mathbf{X}}$ is an isomorphism and $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ is an admissible Galois structure. □

Next, using Theorem 10.7, we easily prove:

Theorem 16.2. A morphism $p : E \to B$ in **C**, in which B is a field, is a monadic extension if and only if E is a non-zero ring. In particular this is the case whenever E is a field. □

Proof. The functor $p^* : \mathbf{F}(B) \to \mathbf{F}(E)$, whose monadicity we have to prove for a non-zero E, is the same as the functor

$$E\otimes_B(-) : (\text{Commutative unitary } B\text{-algebras})^{\text{op}} \to (\text{Commutative unitary } E\text{-algebras})^{\text{op}}. \quad (16.4)$$

According to Theorem 10.7 it suffices to prove that this functor reflects isomorphisms and preserves coequalizers. Moreover, since the coequalizers in the categories involved are the same as equalizers of algebras, and since those are calculated via the corresponding equalizers of underlying modules, we only need to prove that the functor

$$E\otimes_B(-) : B\text{-modules} \to E\text{-modules} \quad (16.5)$$

reflects isomorphisms and is (left) exact, which is obvious since B is a field. □

Now we are ready to prove:

Theorem 16.3. Let $K \subseteq B \subseteq E$ be finite (=finite-dimensional over B) field extensions and $A = (A,\alpha)$ a B-algebra (in particular $\alpha : B \to A$ is a ring homomorphism and B acts on A via $ba = \alpha(b)a$). Out of the following three conditions, the first two are always equivalent, and the third always follows from them and implies them when $B \subseteq E$ is a Galois extension:

(a) (A,α) belongs to $\mathrm{Spl}(E,p)$ (where $\mathrm{Spl}(E,p)$ is defined with respect to the Galois structure described in Theorem 16.1) with p being the inclusion map $B \to E$ considered as a morphism $E \to B$ in **C**;

(b) $E\otimes_B A \approx E\times\ldots\times E$ (a finite product of K-algebras = a finite coproduct in **C**);

(c) $A \approx E_1\times\ldots\times E_n$ for some natural n (0 is not excluded), where $B \subseteq E_i \subseteq E$ ($i = 1,\ldots,n$) (and therefore $E_1,\ldots,E_n$ are field extensions of B).

Proof. (a)⇔(b) easily follows, using the equivalence (16.2), from Theorem 16.3 and the fact that $E\otimes_B A$ considered as an object in **C** is the same as $p^*(A,\alpha)$.

(b)⇔(c) ((b)⇒(c) always, and (b)⇐(c) when $B \subseteq E$ is a Galois extension) is well known in classical algebra, and we only sketch the proof here:

(b)⇒(c): (b) implies that A has no nilpotent elements. Therefore $A \approx E_1\times\ldots\times E_n$ as B-algebras, for some field extensions $E_1,\ldots,E_n$ of B, say, by the Wedderburn Theorem. After that in order to show that $E_1,\ldots,E_n$ can be chosen among the subextensions of $B \subseteq E$, it suffices to show that each of $E_1,\ldots,E_n$ admits a B-algebra homomorphism into E. This, however, immediately follows from $E\otimes_B A \approx E\times\ldots\times E$ and $A \approx E_1\times\ldots\times E_n$.

(c)⇒(b) when $B \subseteq E$ is a Galois extension: Since a finite product of B-algebras satisfying (b) itself obviously satisfies (b), we can assume from the beginning that A is a B-subalgebra in E. Moreover, since $B \subseteq E$ is a Galois extension, there is a polynomial $u \in B[x]$ that splits into linear factors $u = \prod_{i=1}^{m}(x - a_i)$ with $a_i = a_j \Rightarrow i = j$, and has $B[x]/uB[x]$. Therefore

$$E\otimes_B A \approx E\otimes_B(B[x]/uB[x]) \approx E[x]/uE[x] \approx E[x]/(\prod_{i=1}^{m}(x-a_i))$$
$$\approx \prod_{i=1}^{m}(E[x]/(x-a_i)E[x]) \approx E\times\ldots\times E \text{ (m times)},$$

as desired. □

In fact the connection with classical Galois theory goes much further, and provides categorical proofs for many of its results. Let us mention just two of them that are "almost corollaries" of Theorem 8.3:

Theorem 16.4. Let $K \subseteq B \subseteq E$ be finite field extensions and p the inclusion map $B \to E$ considered as a morphism $E \to B$ in **C**. Then the following conditions are equivalent:

(a) (E,p) belongs to Spl(E,p);

(b) $B \subseteq E$ is a Galois extension. □

Theorem 16.5. Let $K \subseteq B$ be a finite field extensions and $A = (A,\alpha)$ a B-algebra as above. Then the following conditions are equivalent:

(a) (A,α) is a covering of B;

(b) there exists a finite field extension $B \subseteq E$, such that (A,α) belongs to Spl(E,p), where Spl(E,p) is as in Theorem 8.3;

(c) there exists a finite Galois field extension $B \subseteq E$, such that (A,α) belongs to Spl(E,p), where Spl(E,p) is as in Theorem 8.3;

(d) $A = (A,\alpha)$ is a commutative separable B-algebra;

(e) $A = (A,\alpha)$ is a finite product of finite separable field extensions of B. □

17. Covering spaces in algebraic topology

The purpose of this section is to present classical covering maps of locally connected topological spaces as covering morphisms in the sence of categorical Galois theory.

Therefore we take here **C** to be the category of locally connected topological spaces and **X** the category of sets. And we define the functor $I : \mathbf{C} \to \mathbf{X}$ by

$$I(A) = \pi_0(A) \text{ the set of connected components of } A. \qquad (17.1)$$

Sending spaces to the families of their connected components determines a category equivalence

$$\mathbf{C} \sim \mathrm{Fam}_{\mathbf{X}}(\mathbf{A}), \qquad (17.2)$$

where **A** is the category of connected locally connected topological spaces. Moreover, the functor $I : \mathbf{C} \to \mathbf{X}$ above is a special case of the one defined in Section 15. Using this fact and Theorem 15.2 we easily obtain:

Theorem 17.1. Let $I : \mathbf{C} \to \mathbf{X}$ be as above, $H : \mathbf{X} \to \mathbf{C}$ the inclusion functor, η and ε the unit and counit of adjunction, **F** = **Étale** the class of local homeomorphisms (=étale maps) of locally connected topological spaces, and Φ the class of all morphisms in **X**. Then $\varepsilon : IH \to 1_X$ is an isomorphism and $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\Phi)$ is an admissible Galois structure. □

A brief story of monadic extensions and coverings with respect to this Galois structure is:

Theorem 17.2. A morphism $p : E \to B$ in **C** is a monadic extension if and only if it is a surjective local homeomorphism.

Proof. "If": Assuming that $p : E \to B$ is a surjective local homeomorphism, we have to prove that the functor (6.1), which we write here as

$$p^* : \mathbf{\acute{E}tale}(B) \to \mathbf{\acute{E}tale}(E), \tag{17.3}$$

is monadic. We observe:

(i) Since the class of local homeomorphisms is closed under composition the functor (17.3) has a left adjoint.

(ii) A morphism $f : (A,\alpha) \to (A',\alpha')$ in **Étale**(B) is an isomorphism if and only if the map $f : A \to A'$ is bijective; this easily implies that, for a surjective p, the functor (17.3) reflects isomorphisms.

(iii) When $\alpha : A \to B$ is a local homeomorphism, the local connectedness of B implies the local connectedness of A. Therefore **Étale**(B) can be identified, up to a category equivalence, with the topos of sheaves (of sets) over the space B. The same is true for E, and the functor (17.3) can be identified with the *inverse image functor*

$$p^* : \mathrm{Shv}(B) \to \mathrm{Shv}(E) \tag{17.4}$$

between the toposes of sheaves. Since the functor (17.4) has a (well-known) right adjoint, namely the *direct image functor*

$$p_* : \mathrm{Shv}(E) \to \mathrm{Shv}(B), \tag{17.5}$$

we conclude that it preserves all coequalizers. Indeed, it is easy to show that any left adjoint functor preserves all colimits, and in particular all coequalizers.

(iv) The desired monadicity follows from (i), (ii), (iii), and Theorem 10.7.

"Only if": When (E,p) is a monadic extension, p must be a local homeomorphism by Definition 6.1(a) (applied to $\delta = 1_E$). Therefore we only need to prove that p is surjective. For, consider the objects $(B,1_B)$ and ($p(E)$, inclusion) in **F**(B) = **Étale**(B); note that ($p(E)$, inclusion) is indeed in **F**(B) since p is open. Since the functor p^* reflects isomorphisms and sends the canonical map ($p(E)$, inclusion) $\to (B,1_B)$ to an isomorphism, we must have $p(E) = B$, as desired. □

Lemma 17.3. Suppose B (in **C**) is connected. Then the following conditions on an object (A,α) in **F**(B) = **Étale**(B) are equivalent:

(a) (A,α) is a trivial covering of B (in the sense of Definition 14.2(a));

(b) A is a disjoint union of open subsets, each of which is mapped homeomorphically on B by α.

Proof. This is an easy corollary of Theorem 15.3. □

Theorem 17.4. The following conditions on an object (A,α) in $\mathbf{F}(B)$ = **Étale**(B) are equivalent:

(a) (A,α) is a covering of B (in the sense of Definition 14.2(c));

(b) every element b in B has an open neighbourhood U for which the pair

$$(\alpha^{-1}(U), \text{the map } \alpha^{-1}(U) \to U \text{ induced by } \alpha) \tag{17.6}$$

is a trivial covering of U (in the sense of Definition 14.2(a));

(c) the same as (b), but with U required to be connected;

(d) (A,α) is a covering space over B in the classical sense, i.e. every element in B has an open neighbourhood whose inverse image is a disjoint union of open subsets, each of which is mapped homeomorphically on it by α.

Proof. (a)⇒(b) easily follows from the "only if" part of Theorem 17.2, and (b)⇒(a) can easily be deduced from the same theorem and the following simple observation:

For each b in B, let U_b be a chosen open neighbourhood of b, let E be the topological coproduct of all these neighbourhoods, and let $p : E \to B$ be the map induced by the family of inclusion maps $U_b \to B$ (for all b in B). Then p is a local homeomorphism.

(b)⇒(c) follows from the local connectedness of B and (c)⇒(b) is trivial.

(c)⇔(d) follows from the local connectedness of B and Lemma 17.3. □

18. Central extensions of groups

The purpose of this section is to present central extensions of groups as covering morphisms in the sence of categorical Galois theory.

Accordinly $\mathbf{C}$ will denote now the category of groups, $\mathbf{X}$ the category of abelian groups, and $I : \mathbf{C} \to \mathbf{X}$ the left adjoint of the inclusion $\mathbf{X} \to \mathbf{C}$, which will plays the role of H. That is:

• From the viewpoint of *universal algebra I* is the *abelianization functor* sending groups to their quotients determined by the identity $xy = yx$, we could write

$$I(A) = A/R, \text{ where } R \text{ is the congruence generated by } \{(a,b) \in A\times A \mid ab = ba\}. \tag{18.1}$$

• From the viewpoint of *group theory I* is to be defined by

$$I(A) = A/[A,A], \tag{18.2}$$

where $[A,A]$ is the commutator of A with itself.
• From the viewpoint of *homological algebra I* is to be defined by

$$I(A) = H_1(A,\mathbf{Z}), \tag{18.3}$$

where $H_1(A,\mathbf{Z})$ is the first homology group of A with coefficients in the additive group of integers, on which A acts trivially.

The Galois structure $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ that we fix in this section will have **C**, **X**, I, H as above, with the canonical η and ε, and **F** and **Φ** being the classes of surjective homomorphisms of groups and of abelian groups respectively. The morphism $\varepsilon : IH \to 1_{\mathbf{X}}$ is obviously an isomorphism here, but the admissibility needs a little proof:

Theorem 18.1. $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ is admissible.

Proof. Consider the pullback (13.3), which now becomes

$$\begin{array}{ccc} B\times_{B/[B,B]}X & \xrightarrow{\ \mathrm{proj}_2\ } & X \\ {\scriptstyle \mathrm{proj}_1}\downarrow & & \downarrow{\scriptstyle \varphi} \\ B & \xrightarrow[\ \eta_B\ ]{} & B/[B,B]. \end{array} \tag{18.4}$$

We need to prove that $\mathrm{proj}_2 : B\times_{B/[B,B]}X \to X$ has the universal property of the abelianization of $B\times_{B/[B,B]}X$, or, equivalently, that the kernel $\mathrm{Ker}(\mathrm{proj}_2)$ of this morphism is contained in $[B\times_{B/[B,B]}X, B\times_{B/[B,B]}X]$. We observe that any element k in $\mathrm{Ker}(\mathrm{proj}_2)$ is of the form $k = (b,1)$, where b is in $[B,B]$, and so we can present it as

$$k = ([b_1,b'_1]\ldots[b_n,b'_n],1) = ([b_1,b'_1],1)\ldots([b_n,b'_n],1). \tag{18.5}$$

Since φ is surjective, there exist $x_1,\ldots,x_n,x'_1,\ldots,x'_n$ in X with $\varphi(x_1) = b_1[B,B]$, …, $\varphi(x_n) = b_n[B,B]$, $\varphi(x'_1) = b'_1[B,B]$, …, $\varphi(x'_n) = b'_n[B,B]$; and since X is abelian, we have $[x_1,x'_1] =\ldots= [x_n,x'_n] = 1$. Therefore

$$k = ([b_1,b'_1],[x_1,x'_1])\ldots([b_n,b'_n],[x_n,x'_n]) = [(b_1,x_1),(b'_1,x'_1)]\ldots[(b_n,x_n),(b'_n,x'_n)], \tag{18.6}$$

which shows that k is in $[B\times_{B/[B,B]}X, B\times_{B/[B,B]}X]$, as desired. □

Remark 18.2. The surjectivity of φ played a crucial role in the proof of Theorem 10.1. Indeed, taking $X = 0$ in (10.4), we would obtain $\mathrm{Ker}(\mathrm{proj}_2) \approx [B,B]$, but at the same time $[B\times_{B/[B,B]}X, B\times_{B/[B,B]}X] \approx [[B,B],[B,B]]$ (canonically). □

Next, the monadic extensions:

Theorem 18.3. A morphism $p : E \to B$ in **C** is a monadic extension if and only if it is surjective.

Proof. "If": According to Remark 10.8(b), it suffices to prove that, for a surjective p, the functor $p^* : \mathbf{F}(B) \to \mathbf{F}(E)$ reflects isomorphisms and preserves coequalizers of reflexive pairs. However, it is an easy exercise to show that the coequalizers of reflexive pairs of group homomorphisms are calculated as in the category of sets – which reduces the problem to the case of sets, where the proof becomes another easy exercise.

The "only if" part follows from Definition 14.1(a) (applied to $\delta = 1_E$). □

In order to characterize coverings we will also need the following almost obvious fact:

Lemma 18.4. For a pullback diagram

$$\begin{array}{ccc} D & \xrightarrow{u} & A \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ E & \xrightarrow[v]{} & B \end{array} \tag{18.7}$$

with α and δ surjective, the conditions (a) and (b) below are related as follows: (a) always implies (b), and (b) implies (a) whenever v is surjective.

(a) (A,α) is a central extension of B (i.e. $ka = ak$ for all k in $\mathrm{Ker}(\alpha)$ and all a in A);

(b) (D,δ) is a central extension of E. □

– after which we are ready to prove

Theorem 18.5. The following conditions on an object (A,α) in $\mathbf{F}(B)$ are equivalent:

(a) (A,α) is a covering of B;

(b) (A,α) is a central extension of B.

Proof. (a)$\Rightarrow$(b) follows from (the "only if" part of) Theorem 18.3 and Lemma 18.4.

(b)$\Rightarrow$(a): As follows from the "if" part of Theorem 18.3, (A,α) is a monadic extension of B. Consider the object

$$\alpha^*(A,\alpha) = (A\times_B A,\mathrm{proj}_1) \tag{18.8}$$

in $\mathbf{F}(A)$. It has $\mathrm{Ker}(\mathrm{proj}_1)$ canonically isomorphic to $\mathrm{Ker}(\alpha)$, and proj_1 is a split epimorphism. Being central by Lemma 18.4(a)$\Rightarrow$(b), it is therefore isomorphic to

$$\alpha^*(A,\alpha) = (A\times \mathrm{Ker}(\alpha), \text{the first projection}), \tag{18.9}$$

after which we only need to observe:

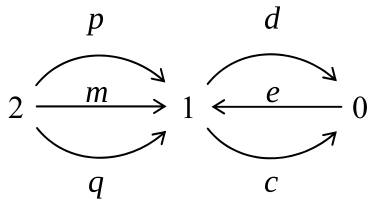
The object (18.9) is a trivial covering of A since $(\mathrm{Ker}(\alpha), \mathrm{Ker}(\alpha) \to 0)$ is a trivial covering of 0, and the class of trivial coverings is pullback stable by Remark 14.3(b). □

19. The fundamental theorem of Galois theory

In this section we formulate and prove the fundamental theorem of categorical Galois theory.

Let $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ be a fixed abstract Galois structure satisfying Convention 5.4. We begin by considering various induced adjunctions:

We can obviously look at the category of internal precategories in $\mathbf{C}$ as the functor category $\mathbf{C}^{\tau}$, where τ is the free category determined by the graph



(19.1)

and the identities $de = 1 = ce$, $dp = cq$, $dm = dq$, and $cm = cp$ as in Definition 11.1. And then the category of internal precategories in **C** becomes nothing but the functor category $\mathbf{C}^{\tau}$. Our adjoint functors between **C** and **X** induce adjoint functors between **C** and **X**, which we will display as

$$(I^{\tau},H^{\tau},\eta^{\tau},\varepsilon^{\tau}) : \mathbf{C}^{\tau} \to \mathbf{X}^{\tau}. \tag{19.2}$$

Using also $\mathbf{F}^{\tau}$ = the class of all κ in $\mathbf{C}^{\tau}$ with κ_0, κ_1, and κ_2 in **F**, and the similarly defined $\mathbf{\Phi}^{\tau}$, we obtain the induced Galois structure

$$\Gamma^{\tau} = (\mathbf{C}^{\tau},\mathbf{X}^{\tau},I^{\tau},H^{\tau},\eta^{\tau},\varepsilon^{\tau},\mathbf{F}^{\tau},\mathbf{\Phi}^{\tau}) \tag{19.3}$$

for internal precategories. After that we take an object P in $\mathbf{C}^{\tau}$, and construct a further induced adjunction in the same way as the adjunction (13.2) was constructed out of an object B in **C**; we display it as

$$(I^{P}, H^{P},\eta^{P},\varepsilon^{P}) : \mathbf{F}^{\tau}(P) \to \mathbf{\Phi}^{\tau}(IP). \tag{19.4}$$

where we write IP instead of $I^{\tau}(P)$, since $I^{\tau}(P)$ is nothing but the composite of $P : \tau \to \mathbf{C}$ with $I : \mathbf{C} \to \mathbf{X}$.

From Remark 14.3(b) we obtain:

Lemma 19.1. If (Q,κ) is a discrete opfibration over P, in which κ_0 is a trivial covering, then κ_1 and κ_2 also are trivial coverings, and $I^{P}(Q,\kappa)$ is a discrete opfibration over $I^{\tau}(P)$. □

Corollary 19.2. The adjunction (19.4) induces an equivalence between:

(a) the full subcategory in $\mathbf{F}^{\tau}(P)$ with objects all (Q,κ) that are discrete opfibrations with κ_0 being a trivial covering, and

(b) the full subcategory in $\mathbf{\Phi}^{\tau}(I^{\tau}(P))$ with objects all objects in it that are discrete opfibrations. □

Identifying now discrete opfibrations with actions (see Theorem 11.11), we obtain

Theorem 19.3. The adjunction (19.4) induces an equivalence between:

(a) the full subcategory $\mathrm{Triv}(\mathbf{C}^{P})$ in $\mathbf{C}^{P}$ with objects all $A = (A_0,\pi,\xi)$ in $\mathbf{C}^{P}$, in which π is a trivial covering;

(b) the full subcategory $\mathbf{X}^{IP}\cap\mathbf{\Phi}$ in $\mathbf{X}^{IP}$ with objects all $X = (X_0,\pi,\xi)$ in $\mathbf{X}^{P}$, in which π is in $\mathbf{\Phi}$. □

– after which we are ready to prove:

Theorem 19.4. ("The fundamental theorem of Galois theory") Let $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ be a fixed abstract Galois structure satisfying Convention 5.4 as above, and let (E,p) be a monadic extension of an object B in $\mathbf{C}$. Then sending an object (A,α) in $\mathrm{Spl}(E,p)$ to the triple $(I(E\times_B A),I(\mathrm{proj}_1),I(\mathrm{proj}_1)\times I(\mathrm{proj}_2))$, determines a category equivalence

$$\mathrm{Spl}(E,p) \to \mathbf{X}^{I(\mathrm{Eq}(p))}\cap\mathbf{\Phi} \qquad (19.5)$$

(denoting by proj_i ($i = 1, 2$) suitable projections, in particular using proj_1 for both $E\times_B A \to E$ and $E\times_B E \to E$, and using the notation of Theorem 19.3 for $P = \mathrm{Eq}(p)$).

Proof. All we need is to consider the diagram

$$\begin{array}{ccc} \mathbf{F}(B) & \longrightarrow & \mathbf{F}(E)^T \approx \mathbf{C}^{\mathrm{Eq}(p)}\cap\mathbf{F} \\ \uparrow & & \uparrow \\ \mathrm{Spl}(E,p) & \dashrightarrow & \mathrm{Triv}(\mathbf{C}^{\mathrm{Eq}(p)}) \sim \mathbf{X}^{I(\mathrm{Eq}(p))}\cap\mathbf{\Phi}, \end{array} \qquad (19.6)$$

in which:

(i) T is the monad determined by the monadic functor $p^* : \mathbf{F}(B) \to \mathbf{F}(E)$, and $\mathbf{F}(B) \to \mathbf{F}(E)^T$ is the comparison functor, which is a category equivalence since p^* is monadic.

(ii) $\mathbf{F}(E)^T \approx \mathbf{C}^{\mathrm{Eq}(p)}\cap\mathbf{F}$ is the isomorphism established in the same way as the isomorphism (12.7) in Theorem 2.1. It therefore sends a T-algebra (D,δ,ζ) to the triple $(D,\delta,\zeta\bar{\delta})$, where $\delta = \langle\mathrm{proj}_1,\mathrm{proj}_3\rangle : (E\times_{(p,p)}E)\times_{(\mathrm{proj}_2,\delta)}D \to E\times_{(p,p\delta)}D$ as in Theorem 12.1.

(iii) Calculating the composite $\mathbf{F}(B) \to \mathbf{F}(E)^T \approx \mathbf{C}^{\mathrm{Eq}(p)}\cap\mathbf{F}$ we easily conclude that it sends an object (A,α) to the triple $(E\times_B A,\mathrm{proj}_1,\mathrm{proj}_1\times\mathrm{proj}_2)$, where proj_i ($i = 1, 2$) are the same as in the formulation of the theorem.

(iv) The vertical arrows are the inclusion functors.

(v) (A,α) belongs to $\mathrm{Spl}(E,p)$ exactly when $(E\times_B A,\mathrm{proj}_1)$ is a trivial covering. Therefore (iii) tells us that the composite $\mathbf{F}(B) \to \mathbf{F}(E)^T \approx \mathbf{C}^{\mathrm{Eq}(p)}\cap\mathbf{F}$ determines the dotted arrow in (11.4), and that that arrow is an equivalence of categories.

(vi) $\mathrm{Triv}(\mathbf{C}^{\mathrm{Eq}(p)}) \sim \mathbf{X}^{I(\mathrm{Eq}(p))}\cap\mathbf{\Phi}$ is the equivalence described in Theorem 11.3 (for $P = \mathrm{Eq}(p)$).

(vii) The desired equivalence is the composite of the equivalences $\mathrm{Spl}(E,p) \sim \mathrm{Triv}(\mathbf{C}^{\mathrm{Eq}(p)})$ and $\mathrm{Triv}(\mathbf{C}^{\mathrm{Eq}(p)}) \sim \mathbf{X}^{I(\mathrm{Eq}(p))}\cap\mathbf{\Phi}$. □

Remark 19.5. (a) According to this theorem it is good to write

$$\mathrm{Gal}(E,p) = I(\mathrm{Eq}(p)), \qquad (19.7)$$

and call this internal precategory the Galois pregroupoid of the monadic extension (E,p). Here "pregroupoid" (rather than "precategory") refers to a certain extra structure, that makes $I(\mathrm{Eq}(p))$ a groupoid whenever it is a category. And in fact it is a groupoid whenever (E,p) is *normal*, which means that (E,p) belongs to $\mathrm{Spl}(E,p)$. Other reasonable synonyms of "normal" are *Galois covering* and *regular covering*. Furthermore, for a normal (E,p), $I(\mathrm{Eq}(p))$ is a group if and only if E is *connected*, i.e. $I(E)$ is a terminal object in $\mathbf{X}$.

(b) There is also a reasonable way to define *fundamental groupoids* as "the largest" Galois groupoids. □

20. Back to the classical cases

In this section we consider the simplest applications the fundamental theorem of categorical Galois theory.

The classical form of the fundamental theorem of Galois theory is usually formulated as follows:

Theorem 20.1. Let $B \subseteq E$ be a finite Galois field extension, and $\mathrm{Aut}_B(E)$ its Galois group. Then:

(a) The correspondences

$$F \mapsto \mathrm{Aut}_F(E) \text{ and } H \mapsto E^H = \{x \in E \mid g \in H \Rightarrow g(x) = x\} \tag{20.1}$$

determine inverse to each other and inclusion reversing bijections between the lattice $\mathrm{Sub}(E/B)$ of field subextensions of $B \subseteq E$, and the lattice $\mathrm{Sub}(\mathrm{Aut}_B(E))$ of subgroups in $\mathrm{Aut}_B(E)$.

(b) If $B \subseteq F$ is a field subextension of $B \subseteq E$, then every B-algebra homomorphism from F to E extends to a B-algebra automorphism of E.

(c) A field subextension $B \subseteq F$ of $B \subseteq E$ is a Galois extension if and only if its corresponding subgroup $\mathrm{Aut}_F(E)$ is a normal subgroup in $\mathrm{Aut}_B(E)$. In this case every B-algebra automorphism of E restricts to a B-algebra automorphism of F, yielding a short exact sequence

$$0 \to \mathrm{Aut}_F(E) \to \mathrm{Aut}_B(E) \to \mathrm{Aut}_B(F) \to 0 \tag{20.2}$$

of groups. □

How does this theorem follow from Theorem 19.4?

Answering this question requires a number of simple observations:

(i) Every statement of Theorem 20.1 is a statement about purely-categorical properties of the category $\underline{\mathrm{Sub}}(E/B)$ of subextensions of the field extension $B \subseteq E$. The only thing that needs an explanation here, is that E itself can be defined categorically as a special object in $\underline{\mathrm{Sub}}(E/B)$. For, just observe that it is the only weak terminal object (i.e. the only object that admits morphisms from all other objects into it).

(ii) Moreover, it turns out that the category $\underline{\mathrm{Sub}}(E/B)^{\mathrm{op}}$ is equivalent to the category of transitive (=one-orbit) $\mathrm{Aut}_B(E)$-sets – *which is known as Grothendieck's form of the fundamental theorem of Galois theory* – and every statement of Theorem 20.1 follows from this fact.

(iii) Furthermore, it is sufficient to know that $\underline{\mathrm{Sub}}(E/B)^{\mathrm{op}}$ is equivalent to the category of transitive G-sets for some monoid G, because this fact itself implies that G is isomorphic to $\mathrm{Aut}_B(E)$. Indeed:

• We know that $\underline{\mathrm{Sub}}(E/B)$ has a unique weakly terminal object, namely E, and that the endomorphism monoid of this object is $\mathrm{Aut}_B(E)$.
• On the other hand G acts on itself via its multiplication, and this object is weakly initial in the category of transitive G-sets; and its endomorphism monoid is isomorphic to G.
• Therefore the equivalence of $\underline{\mathrm{Sub}}(E/B)^{\mathrm{op}}$ to the category of transitive G-sets implies that G is isomorphic to $\mathrm{Aut}_B(E)$.

(iv) Let us now apply Theorem 19.4 to the situation considered in Section 16. As follows from the equivalence (a)⇔(c) in Theorem 16.3, which assumes that $B \subseteq E$ is a Galois extension and $p : E \to B$ is the inclusion map $B \to E$, in that situation we have

$$\mathrm{Spl}(E,p) \sim \mathrm{Fam}_{\mathbf{X}}(\underline{\mathrm{Sub}}(E/B)^{\mathrm{op}}). \tag{20.3}$$

At the same time Theorem 19.4 tells us that the category $\mathrm{Spl}(E,p)$ is equivalent to the category of finite G-sets for some finite monoid G – namely for $G = L(I(\mathrm{Eq}(p)))$, where L is the functor (11.9), and $L(I(\mathrm{Eq}(p)))$ is indeed a monoid since $I(E)$ has only one element.

(v) As follows from (iv), $\underline{\mathrm{Sub}}(E/B)^{\mathrm{op}}$ must be equivalent to the category of transitive G-sets, as desired. Therefore Theorem 20.1 indeed follows from Theorem 19.4.

The situation with covering spaces is very similar: many standard text books in algebraic topology show how the connected covering spaces of a “good” space B are “classified” via subgroups of the fundamental group of B by proving a theorem similar to Theorem 12.1, usually not showing the categorical result behind, which is:

Theorem 20.2. Let B be a connected locally connected topological space, admitting a universal covering space (E,p) over it. Then the category of covering spaces over B is equivalent to the category of $\mathrm{Aut}(E,p)$-sets.

– and this theorem can easily be obtained as a corollary of Theorem 19.4, using the results of Section 17. Recall, however, that what is called a universal covering space of B is in fact a weakly initial object in the category of non-empty covering spaces over B, and that “weakness” can be avoided by using pointed spaces.

Applying Theorem 19.4 to the situation considered in Section 18, we obtain, in particular, a description of the category $\mathrm{Centr}(B)$ of central extension of an arbitrary group B. The full explanation would involve some homological algebra and internal category theory (in “nice” categories), which would take us too far. Therefore let us just mention that it becomes especially simple when B is perfect, i.e. when $[B,B] = B$: in this case

$$\mathrm{Centr}(B) \sim ((\text{Abelian groups}){\downarrow}H_2(A,\boldsymbol{Z})), \tag{20.4}$$

which presents the second homology group $H_2(A,\boldsymbol{Z})$ as a certain “Galois group”, and implies the well-known result saying that every perfect group has a universal central extension.

Finally, let us mention one less familiar examples of Galois theories very briefly; being less familiar it was, however, the original motivating example for categorical Galois theory:

Example 20.3. The system $(\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\boldsymbol{\Phi})$ described below is an admissible Galois structure in which ε is an isomorphism:

• **C** is the opposite category of commutative unitary rings;
• **X** is the opposite category of (unitary) Boolean rings, or, equivalently, the opposite category of Boolean algebras; up to a category equivalence we can identify **X** with the category of Stone spaces (=profinite topological spaces = compact totally disconnected Hausdorff spaces = compact 0-dimensional Hausdorff spaces = compact topological spaces in which every two points can be separated by a closed-and-open subset);
• $I : \mathbf{C} \to \mathbf{X}$ is sending rings to the Boolean rings of their idempotents, or, considering **X** as the category of Stone spaces, I is defined by

$I(A)$ = Boolean spectrum of A = Stone space of the Boolean algebra of idempotents in A = the space of connected components of the Zariski spectrum of A; (20.5)

• $H : \mathbf{X} \to \mathbf{C}$ is defined by

$$H(X) = \mathrm{hom}(X,\boldsymbol{Z}), \tag{20.6}$$

where X is any object in **X** considered as a topological space, $\boldsymbol{Z}$ is the ring of integers equipped with the discrete topology, and $\mathrm{hom}(X,\boldsymbol{Z})$ is set of continuous maps $X \to \boldsymbol{Z}$ with the ring structure induced by the ring structure of $\boldsymbol{Z}$;

• η and ε are defined accordingly, and **F** and **Φ** are the classes of all morphisms in **C** and **X** respectively.

The covering morphisms with respect to this Galois structure are the same as what A. R. Magid calls *componentially locally strongy separable* algebras; they are defined as follows:

(a) a commutative (unitary) algebra S over a commutative (unitary) ring R is said to be *separable* if it is *projective* as an $S\otimes_R S$-module;

(b) a commutative separable R-algebra S is said to be *strongly separable* if it is *projective* as an R-module;

(c) an R-algebra S is said to be *locally strongly separable* if every finite subset in it is contained in a strongly separable R-subalgebra;

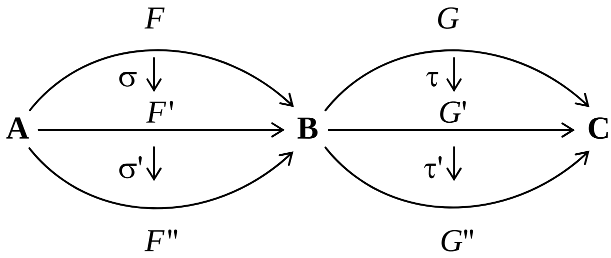
(d) a commutative R-algebra S is said to be *componentially locally strongy separable* if its all Boolean localizations S_x are locally strongly separable R_x-algebras; here, for a maximal ideal x of the Boolean ring of idempotents in R, the Boolean localizations S_x is defined as the quotient algebra S/Sx.

And these componentially locally strongy separable algebras were the most general algebras involved in Magid's *separable Galois theory of commutative rings*. For a field extension $B \subseteq E$ we have:

E is a separable B-algebra $\Leftrightarrow$ E is a strongly separable B-algebra
$\Leftrightarrow$ E is a finite separable extension of B. □ (20.7)

A1. Remarks on functors and natural transformations

A1.1. Natural transformations can be composed vercally and horizontally, and these operations agree via the middle interchange law. For, consider the diagram



(A1.1)

of categories, functors, and natural transformations. While the vertical composite

$$\sigma'\sigma : F \to F'' \text{ is defined by } (\sigma'\sigma)_A = \sigma'_A\sigma_A : F(A) \to F''(A), \tag{A1.2}$$

the vertical composite

$$\tau\sigma : GF \to G'F' \text{ is defined by } (\tau\sigma)_A = G'(\sigma_A)\tau_{F(A)} = \tau_{F'(A)}G(\sigma_A) : GF(A) \to G'F'(A) \tag{A1.3}$$

(in both cases for all objects A in **A**); here the equality $G'(\sigma_A)\tau_{F(A)} = \tau_{F'(A)}G(\sigma_A)$ is simply the commutativity of the naturality square

$$\begin{array}{ccc} GF(A) & \xrightarrow{G(\sigma_A)} & GF'(A) \\ \tau_{F(A)}\downarrow & & \downarrow\tau_{F'(A)} \\ G'F(A) & \xrightarrow[G'(\sigma_A)]{} & G'F'(A) \end{array} \tag{A1.4}$$

Furthermore, the rows and the columns of (A1.4) are in fact components of the natural transformations

$$G\sigma : GF \to GF' \text{ defined by } (G\sigma)_A = G(\sigma_A) \tag{A1.5}$$

and

$$\tau F : GF \to G'F \text{ defined by } (\tau F)_A = \tau_{F(A)} \tag{A1.6}$$

respectively. Using these natural transformations, the commutativity of (A1.4) for all A in **A** can be expressed as the commutativity of

$$\begin{array}{ccc} GF & \xrightarrow{G\sigma} & GF' \\ \tau F\downarrow & & \downarrow\tau F' \\ G'F & \xrightarrow[G'\sigma]{} & G'F' \end{array} \tag{A1.7}$$

We also have

$$G\sigma = 1_G\sigma \text{ and } \tau F = \tau 1_F, \tag{A1.8}$$

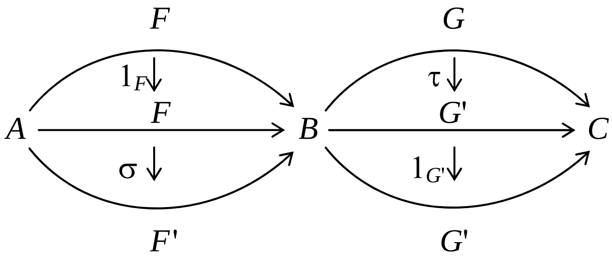
and the commutativity of (A1.7), written as the equality

$$(1_{G'}\sigma)(\tau 1_F) = (\tau 1_{F'})(1_G\sigma), \qquad \text{(A1.9)}$$

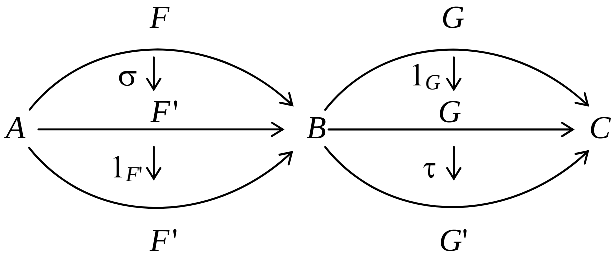
can be deduced from the *middle interchange law*

$$(\tau'\tau)(\sigma'\sigma) = (\tau'\sigma')(\tau\sigma), \qquad \text{(A1.10)}$$

written here for the situation (A1.1). Indeed, applying (A1.10) to



and



we obtain

$$(1_{G'}\tau)(\sigma 1_F) = (1_{G'}\sigma)(\tau 1_F) \text{ and } (\tau 1_G)(1_{F'}\sigma) = (\tau 1_{F'})(1_G\sigma) \qquad \text{(A1.11)}$$

respectively, which gives

$$(1_{G'}\sigma)(\tau 1_F) = (1_{G'}\tau)(\sigma 1_F) = \tau\sigma = (\tau 1_G)(1_{F'}\sigma) = (\tau 1_{F'})(1_G\sigma) \qquad \text{(A1.12)}$$

On the other hand the middle interchange law (A1.10) can itself be obtained using the appropriate commutative diagrams of the form (A1.7), which is easy to show using the diagram

$$\begin{array}{ccccc} GF & \xrightarrow{G\sigma} & GF' & \xrightarrow{G\sigma'} & GF'' \\ \tau F \downarrow & & \downarrow \tau F' & & \downarrow \tau F'' \\ G'F & \xrightarrow[G'\sigma]{} & G'F' & \xrightarrow[G'\sigma']{} & G'F'' \\ \tau' F \downarrow & & \downarrow \tau' F' & & \downarrow \tau' F'' \\ G''F & \xrightarrow[G''\sigma]{} & G''F' & \xrightarrow[G''\sigma']{} & G''F'' \end{array} \qquad \text{(A1.13)}$$

whose four small squares are of the form (A1.7) (for various functors involved): one way of doing it is to write

$(\tau'\tau)(\sigma'\sigma) = ((\tau'\tau)F'')(G(\sigma'\sigma))$ (by the definition of the horizontal composite of $\tau'\tau$ with $\sigma'\sigma$)
$= (\tau'F'')(\tau F'')(G\sigma')(G\sigma)$ (by obvious properties of the "usual" composition)
$= (\tau'F'')(G'\sigma')(\tau F')(G\sigma)$ (by commutativity of the right-hand top square in (A1.13)
$= (\tau'\sigma')(\tau\sigma)$ (by the definition of the horizontal composites of τ' with σ' and of τ with σ).

Note, however, that good understanding of all these calculations requires seeing horizontal composition as functors

$$\mathbf{Cat}(\mathbf{B},\mathbf{C}) \times \mathbf{Cat}(\mathbf{A},\mathbf{B}) \to \mathbf{Cat}(\mathbf{A},\mathbf{C}), \qquad \text{(A1.14)}$$

where $\mathbf{Cat}(\mathbf{A},\mathbf{B})$ denotes the category of all functors $\mathbf{A} \to \mathbf{B}$, etc.

A1.2. The hom functors. For a fixed object A in a category $\mathbf{X}$ one can form the *covariant hom functor*

$\hom_{\mathbf{X}}(A,-) : \mathbf{X} \to \mathbf{Sets}$, sending a morphism $f : X \to Y$ of $\mathbf{X}$ to the map
$\hom_{\mathbf{X}}(A,f) : \hom_{\mathbf{X}}(A,X) \to \hom_{\mathbf{X}}(A,Y)$ defined by $\alpha \mapsto f\alpha$ (A1.15)

and the *contravariant hom functor*

$\hom_{\mathbf{X}}(-,A) : \mathbf{X}^{op} \to \mathbf{Sets}$, sending a morphism $f : X \to Y$ of $\mathbf{X}$ to the map
$\hom_{\mathbf{X}}(f,A) : \hom_{\mathbf{X}}(Y,A) \to \hom_{\mathbf{X}}(X,A)$ defined by $\alpha \mapsto \alpha f$. (A1.16)

Moreover, these two constructions agree in the sense that one can also form the functor

$\hom : \mathbf{X}^{op}\times\mathbf{X} \to \mathbf{Sets}$, sending a morphism $(f,f') : (X,X') \to (Y,Y')$ of $\mathbf{X}^{op}\times\mathbf{X}$
to the map $\hom_{\mathbf{X}}(f,f') : \hom_{\mathbf{X}}(X,X') \to \hom_{\mathbf{X}}(Y,Y')$ defined by $\varphi \mapsto f'\varphi f$, (A1.17)

and we have

$$\hom_{\mathbf{X}}(A,f) = \hom_{\mathbf{X}}(1_A,f) \text{ and } \hom_{\mathbf{X}}(f,A) = \hom_{\mathbf{X}}(f,1_A) \qquad \text{(A1.18)}$$

in the situations (A1.15) and (A1.16), and

$$\hom_{\mathbf{X}}(f,f') = \hom_{\mathbf{X}}(Y,f')\hom_{\mathbf{X}}(f,X') = \hom_{\mathbf{X}}(f,Y')\hom_{\mathbf{X}}(X,f') \qquad \text{(A1.19)}$$

in the situation (A1.17).

Note that we use "covariant hom functor" and "contravariant hom functor" only as convenient expressions, not as instances of "covariant/contravariant functors" – assuming the convention that there are only functors that are always covariant, and a "contravariant functor", say, from $\mathbf{A}$ to $\mathbf{B}$, should either be seen as a functor $\mathbf{A}^{op} \to \mathbf{B}$ or as functor $\mathbf{A} \to \mathbf{B}^{op}$ (and these two functors are dual to each other). For instance it is important that the contravariant hom functor $\hom_{\mathbf{X}}(-,A)$ is defined as a functor $\mathbf{X}^{op} \to \mathbf{Sets}$, and not as a functor $\mathbf{X} \to \mathbf{Sets}^{op}$.

A2. Limits and colimits.

A2.1. General case. For a graph G =

$$G_1 \overset{d}{\underset{c}{\rightrightarrows}} G_0 \qquad \text{(A2.1)}$$

we will write, as usually, $f : x \to y$ when f is in G_1 and $d(f) = x$ and $c(f) = y$. For a category **C** and a diagram $D : G \to \mathbf{C}$ a *cone* over D is a system $(C,\varphi) = (C,(\varphi_x : C \to D(x))_{x\in G_0})$, in which C is an object in **C**, and $\varphi_x : C \to D(x)$ $(x \in G_0)$ morphisms in **C**, making the diagram

$$\begin{array}{ccc} & C & \\ {\scriptstyle\varphi_x}\swarrow & & \searrow{\scriptstyle\varphi_y} \\ D(x) & \xrightarrow{\quad D(f) \quad} & D(y) \end{array} \qquad \text{(A2.2)}$$

commute for every $f : x \to y$ in G. A morphism $\gamma : (C,\varphi) \to (C',\varphi')$ of cones over D is a morphism $\gamma : C \to C'$ in **C**, making the diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad \gamma \quad} & C' \\ {\scriptstyle\varphi_x}\searrow & & \swarrow{\scriptstyle\varphi_y} \\ & D(x) & \end{array} \qquad \text{(A2.3)}$$

commute for every x in G. The category of cones over D will be denoted by Con(D), and its terminal object

$$\lim D = (\lim D,\pi) \qquad \text{(A2.4)}$$

(provided it exists) is called the limit of D. The morphisms π_x are then called the limit projections. There are many important special cases, some of which are listed below.

A2.2. Products. In the notation above, when G_1 is empty, and therefore the graph G can be identified with the set G_0, we write

$$\lim D = \textstyle\prod_{x\in G} D(x) = (\prod_{x\in G} D(x),\pi) \qquad \text{(A2.5)}$$

and call this limit the product of the family $(D(x))_{x\in G}$. In particular, it is easy to that:

- When G is empty, $\prod_{x\in G} D(x)$ is nothing but the terminal object in **C**.
- When $G = \{x\}$ is a one-element set, $\prod_{x\in G} D(x) = D(x)$.
- When G has (exactly) two elements, whose images under D are A and B, we have $\prod_{x\in G} D(x) = A\times B$.

And more generally, when G has n elements, whose images under D are $A_1, \ldots, A_n$, it is convenient to write $\prod_{x \in G} D(x) = A_1 \times \ldots \times A_n$.

A2.3. Infima. If **C** is an ordered set considered as a category, then for every $D : G \to \mathbf{C}$ we have

$$\lim D = \prod_{x \in G_0} D(x) = \bigwedge_{x \in G_0} D(x) = \inf\{D(x) \mid x \in G_0\}, \tag{A2.6}$$

i.e. $\lim D$ is the infimum of the set $\{D(x) \mid x \in G_0\}$ in **C**.

A2.4. Equalizers. Let G be a graph that has two objects x and y, and two morphisms from x to y, and let D be the diagram sending those two morphisms to

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \tag{A2.7}$$

Then to give a cone over D is to give a morphism $h : X \to A$ with $fh = gh$. Therefore the limit of D can be identified with a pair (E,e), in which $e : E \to A$ is a morphism in **C** such that:

(a) $fe = ge$;

(b) if $fh = gh$ as above, then there exists a unique morphism $u : X \to E$ with $eu = h$.

Such a pair (E,e) is called the equalizer of the pair (f,g).

A2.5. Pullbacks. Let G be a graph that has three objects x, y, and z, one morphism from x to z, and one morphism from y to z, and let D be the diagram sending those two morphisms to

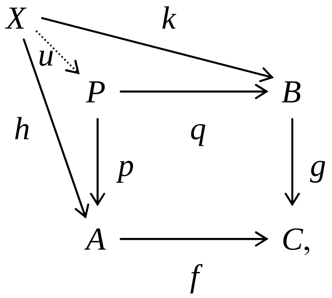
$$\begin{array}{ccc} & & B \\ & & \downarrow g \\ A & \xrightarrow[f]{} & C \end{array} \tag{A2.8}$$

Then to give a cone over D is to give a morphisms $h : X \to A$ and $k : X \to B$ with $fh = gk$. Therefore the limit of D can be identified with a triple (P,p,q), in which $p : P \to A$ and $q : Q \to B$ are morphisms in **C** such that:

(a) $fp = gq$;

(b) if $fh = gk$ as above, then there exists a unique morphism $u : X \to P$ with $pu = h$ and $qu = k$.

As suggested by the display



(A2.9)

the limit of D is called the pullback of f and g. One also says that:

- the square formed by f, g, p, q is a pullback square, or a cartesian square;
- p is a pullback of g along f, and q is a pullback of f along g;
- P is a fibred product of (A,f) and (B,g) (since indeed, $(P,fp) = (P,gq)$ is the product of (A,f) and (B,g) in the category $(\mathbf{C}\downarrow C)$; another good reason is that, say, for $\mathbf{C} = \mathbf{Sets}$, it turnes out that the fibres of $fp = gq$ are the products of the corresponding fibres of f and g). One also writes $P = A\times_{(f,g)}B = A\times_C B$.

A2.6. Examples of limits. In many concrete categories, including **Sets**, all varieties of universal algebras, and the category of topological spaces, limits can be constructed as follows: the products are the same as the usual cartesian products, and then

$$\lim D = \{(a_x)_{x\in G_0} \in \textstyle\prod_{x\in G_0}D(x) \mid D(f)(a_x) = a_y \text{ for each } f : x \to y \text{ in } G\}, \qquad \text{(A2.10)}$$

in the notation above, with $\pi_x : \lim D \to D(x)$ being induced by the corresponding usual product projection for each x in G_0. In particular the equalizer of a pair (A2.7) of parallel morphisms in $\mathbf{C}$ can be identified with

$$\{a \in A \mid f(a) = g(a)\}, \qquad \text{(A2.11)}$$

and for the pullback in (A2.9) we can write

$$A\times_{(f,g)}B = \{(a,b) \in A\times B \mid f(a) = g(b)\} \qquad \text{(A2.12)}$$

A2.7. Colimits. The colimit of a diagram $D : G \to \mathbf{C}$ is the same as the limit of the dual diagram $D^{op} : G^{op} \to \mathbf{C}^{op}$. That is, the notion of colimit is simply dual to the notion of limit. And all special limits above have their dual versions: coproducts are dual to products, coequalizers to equalizers, and pushouts to pullbacks. The standard notation is:

- colim D – for the colimit of a diagram D;
- $\sum_{x\in G}D(x)$, or $\coprod_{x\in G}D(x)$ – for the coproduct of the family $(D(x))_{x\in G}$;
- $A+B = A\amalg B$ for the coproduct of A and B, and accordingly for pushouts.

However the constructions of colimits in familiar categories are usually more complicated than those of limits. When we say that limits in varieties of universal algebras and in the category of topological spaces are "constructed in the same way as in the category of sets", it first of all means that the forgetful functors from all these categories to sets *preserve limits* (in the obvious sense). This, however, is usually not the case for colimits. Say, for a variety $\mathbf{C}$ of universal algebras, the colimit of a diagram $D : G \to \mathbf{C}$ can be constructed in several steps as follows:

• we take A to the free algebra on the disjoint union of all $D(x)$ ($x \in G_0$);
• define the congruence ~ on A as the smallest congruence E for which the composite of the canonical maps $D(x) \to A$ and $A \to A/E$ is a homomorphism of algebras;
• then one can show that $A/\sim$ becomes the colimit of D.

A3. Galois connections

A *Galois connection* between ordered sets L and M is a pair of maps

$$L \rightleftarrows M, \text{ both written as } x \mapsto x^*, \tag{A3.1}$$

and satisfying the following conditions:

$$x \leq y \Rightarrow y^* \leq x^* \text{ for all } x \text{ and } y \text{ in } L \text{ and for all } x \text{ and } y \text{ in } M; \tag{A3.2}$$
$$x \leq x^{**} \text{ for all } x \text{ in } L \text{ and for all } x \text{ in } M. \tag{A3.3}$$

That is, a Galois connection between L and M is nothing but an adjunction $L \to M^{\mathrm{op}}$, or, equivalently, an adjunction $M \to L^{\mathrm{op}}$. And just as any adjunction $\mathbf{X} \to \mathbf{A}$ determines a monad on $\mathbf{X}$, any Galois connection above determines *closure operators* on L and on M, both given by

$$c(x) = x^{**}. \tag{A3.4}$$

Let us recall here that in general a closure operator on ordered sets is unary operation c satisfying the following conditions:

$$\begin{aligned} &x \leq y \Rightarrow c(x) \leq c(y);\\ &x \leq c(x);\\ &cc(x) = c(x). \end{aligned} \tag{A3.5}$$

And if c is defined via a Galois connection as above, then the conditions (A3.5) easily follow from (A3.2) and (A3.3) of course; the crucial observation is the equality

$$x^{***} = x^*, \tag{A3.6}$$

in which $x^* \leq x^{***}$ by (A3.3) applied to x^*, and $x^{***} \leq x^*$ by (A3.2) applied to (A3.3).

As usually, an element x is called closed (under a given closure operator c) if $c(x) = x$. From the equality (A3.6) we easily conclude:

Theorem A3.1. Any Galois connection (A3.1) induces inverse to each other bijections between the set of closed elements in L and the set of closed elements in M. □

When L and M are power sets ordered by inclusion, the Galois connections between L and M are nothing but binary relations between the ground sets. More precisely, we have:

Theorem A3.2. Let X and Y be arbitrary sets and $\mathrm{P}(X)$ and $\mathrm{P}(Y)$ their power sets. Then:

(a) For any Galois connection between $\mathrm{P}(X)$ and $\mathrm{P}(Y)$, and x in X and y in Y, we have:

$$x \in \{y\}^* \Leftrightarrow y \in \{x\}^* \qquad \text{(A3.7)}$$

(b) Associating to a Galois connection between P(X) and P(Y) the binary relation $\alpha \subseteq X\times Y$ defined by

$$\alpha = \{(x,y) \mid x \in \{y\}^*\} = \{(x,y) \mid y \in \{x\}^*\} \qquad \text{(A3.8)}$$

determined a bijection from the set of all Galois connections between P(X) and P(Y) and power set P($X\times Y$). The inverse bijection sends $\alpha \subseteq X\times Y$ to the Galois connection between P(X) and P(Y) defined by

$$A^* = \{y \in Y \mid a \in A \Rightarrow (a,y) \in \alpha\} \text{ for } A \subseteq X, \qquad \text{(A3.9)}$$
$$B^* = \{x \in X \mid b \in B \Rightarrow (x,b) \in \alpha\} \text{ for } B \subseteq Y. \qquad \text{(A3.10)}$$

Proof. (a): We have

$x \in \{y\}^* \Leftrightarrow \{x\} \subseteq \{y\}^* \Rightarrow \{y\}^{**} \subseteq \{x\}^*$ (by (A3.2))
$\Rightarrow \{y\} \subseteq \{x\}^*$ (by (A3.3))
$\Leftrightarrow y \in \{x\}^*$.

Therefore $x \in \{y\}^* \Rightarrow y \in \{x\}^*$. Similarly (and "symmetrically") the converse implication also holds.

(b): It is easy to see that (A3.9) and (A3.10) indeed define a Galois connection. That is, we have maps

$$\boxed{\begin{array}{l}\text{Galois connections}\\ \text{between}\\ \text{P}(X)\text{ and P}(Y)\end{array}} \; \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \; \text{P}(X\times Y) \qquad \text{(A3.11)}$$

sending Galois connections to the corresponding binary relations defined by (A3.8) and sending binary relations to the corresponding Galois connections defined by (A3.9) and (A3.10), and we have to show that $\psi\varphi$ and $\varphi\psi$ are the identity maps.

To show that $\psi\varphi$ is the identity map is to show that, for every Galois connection between P(X) and P(Y), we have

$$A^* = \{y \in Y \mid a \in A \Rightarrow y \in \{a\}^*\} \text{ for } A \subseteq X,$$
$$B^* = \{x \in X \mid b \in B \Rightarrow x \in \{b\}^*\} \text{ for } B \subseteq Y.$$

or, equivalently, to show that

$$A^* = \bigcap_{a\in A}\{a\}^* \text{ for } A \subseteq X, \qquad \text{(A3.12)}$$

$$B^* = \bigcap_{b\in B}\{b\}^* \text{ for } B \subseteq Y. \qquad \text{(A3.13)}$$

We have:

$$y \in A^* \Leftrightarrow \forall_{a\in A}((a,y) \in \alpha) \Leftrightarrow \forall_{a\in A}(y \in \{a\}^*) \Leftrightarrow y \in \bigcap_{a\in A}\{a\}^*,$$

which proves (A3.12), and (A3.13) can be proved similarly.

To show that φψ is the identity map is (according to (A3.8)) to show that, for every binary relation $\alpha \subseteq X\times Y$, we have

$$\alpha = \{(x,y) \mid x \in \{y\}^*\}, \text{ where } \{y\}^* = \{x \in X \mid (x,y) \in \alpha\},$$

i.e.

$$\alpha = \{(x,y) \mid (x,y) \in \alpha\},$$

which is trivial. □

Remark A3.3. To construct a closure operator out of a Galois connection via (A3.4) is a special case of constructing a monad out of an adjunction. But are there also general theorems about adjoint functors that would give Theorems A3.1 and A3.2 as special cases? Yes, but they are far more sophisticated and we shall not need them here. □