ON THE LOCALLY CONVEX SPACE OF RAPIDLY DECREASING DISTRIBUTIONS

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ABSTRACT. It is proved that in the set $\mathcal{O}'_C$ of rapidly decreasing distributions on $\mathbb{R}^n$ the two topologies yield the same bounded sets and coincide on bounded sets:
(i) the projective topology corresponding to the inductive topology in the predual space,
(ii) the operator topology related to the action of $\mathcal{O}'_C$ on $\mathcal{S}$ by convolution.

1. THE SPACE OF RAPIDLY DECREASING DISTRIBUTIONS ON $\mathbb{R}^n$ AND ITS PREDUAL SPACE

1.1. **The J. Horváth space $\mathcal{O}_C$ of slowly increasing C^∞-functions on $\mathbb{R}^n$.** Let

$$C_b^\infty = \Big\{\varphi \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} |\partial^\alpha \varphi(x)| < \infty \text{ for every } \alpha \in \mathbb{N}_0^n\Big\},$$
$$C_0^\infty = \Big\{\varphi \in C^\infty(\mathbb{R}^n) : \lim_{|x|\to\infty} |\partial^\alpha \varphi(x)| = 0 \text{ for every } \alpha \in \mathbb{N}_0^n\Big\},$$
$$\mathcal{D}_{L^p} = \{\varphi \in C^\infty(\mathbb{R}^n) : \partial^\alpha\varphi \in L^p(\mathbb{R}^n) \text{ for every } \alpha \in \mathbb{N}_0^n\} \quad \text{if } p \in [1,\infty[.$$

Moreover, for $\mu \in \mathbb{R}$ let

$$\tilde{S}_\mu = (1+|\cdot|^2)^{\mu/2} C_b^\infty, \quad S_\mu = (1+|\cdot|^2)^{\mu/2} C_0^\infty, \quad S_\mu^p = (1+|\cdot|^2)^{\mu/2} \mathcal{D}_{L^p}.$$

All the spaces defined above are contained in $C^\infty(\mathbb{R}^n)$ and are topological Fréchet spaces.

Lemma. *Whenever $\mu \in \mathbb{R}$, the function $f_\mu : \mathbb{R}^n \ni x \mapsto (1+|x|^2)^{\mu/2} \in \mathbb{R}^+$ belongs to $\tilde{S}_\mu$. Moreover, whenever $\lambda, \mu \in \mathbb{R}$, multiplication by f_μ yields isomorphisms of the Fréchet spaces $\tilde{S}_\lambda$ onto $\tilde{S}_{\lambda+\mu}$, S_λ onto $S_{\lambda+\mu}$, and S_λ^p onto $S_{\lambda+\mu}^p$, and multiplication by $f_{-\mu}$ yields the inverse isomorphisms.*

The above lemma is a consequence of the equality

$$\partial^\alpha (1+|x|^2)^{\mu/2} = (1+|x|^2)^{\mu/2-|\alpha|} P_\alpha(x) \tag{1.1}$$

in which P_α is a polynomial on $\mathbb{R}^n$ of degree no greater than $|\alpha|$. The equality (1.1) appears in [H, Sect. 2.5, Example 8] and can be proved by induction on the length $|\alpha|$ of the multiindex α.

Corollary. *Whenever $\mu \in \mathbb{R}$ and $\lambda \in \,]n/p, \infty[$, there are continuous imbeddings*

$$S_\mu^p \subset S_\mu \subset \tilde{S}_\mu \subset S_{\mu+\lambda}^p. \tag{1.2}$$

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From the Remark in [B, Sect. II.2.4] and from the imbeddings (1.2) it follows that all the inductive limits

$$\lim_{\mu\to\infty}\operatorname{ind} \tilde{S}_\mu, \quad \lim_{\mu\to\infty}\operatorname{ind} S_\mu \quad \text{and} \quad \lim_{\mu\to\infty}\operatorname{ind} S^p_\mu, \quad p \in [1,\infty[,$$

define the same locally convex space contained in $C^\infty(\mathbb{R}^n)$. This space, denoted by $\mathcal{O}_C$, was discovered by J. Horváth (before 1966 when the first edition of his book was published by Addison-Wesley). J. Horváth represented $\mathcal{O}_C$ as $\lim\operatorname{ind}_{\mu\to\infty} S_\mu$. See [H, Sect. 2.12, Example 9].

1.2. **The strong projective topology in** $(\mathcal{O}_C)'$**.** The *strong projective topology* in the space $(\mathcal{O}_C)'$ dual to $\mathcal{O}_C$ is defined as the weakest locally convex topology τ in $(\mathcal{O}_C)'$ such that for every $\mu \in [0,\infty[$ the locally convex space $((\mathcal{O}_C)', \tau)$ is continuously imbedded in the strong dual space $(\tilde{S}_\mu)'$. The same holds when $(\tilde{S}_\mu)'$ is replaced by $(S^p_\mu)'$ or $(S_\mu)'$. In this connection let us stress that $(S^p_\mu)'$ and $(S_\mu)'$ have transparent form: $(S^p_\mu)' = (1+|\cdot|^2)^{-\mu/2}(\mathcal{D}_{L^1})'$ and $(S_\mu)' = (1+|\cdot|^2)^{-\mu/2}(C^\infty_0)'$ where $(\mathcal{D}_{L^1})'$ is the space of bounded distributions on $\mathbb{R}^n$, and $(C^\infty_0)'$ is the space of integrable distributions on $\mathbb{R}^n$. See [S, Sect. VI.8] and [H, Sect. 4.11, Corollary to Proposition 6].

For every $\mu \in [0,\infty[$ let $\tilde{B}_\mu$, B_μ and B^p_μ denote the families of all bounded subsets of the spaces $\tilde{S}_\mu$, S_μ and S^p_μ. The imbeddings (1.2) imply that

$$\bigcup_{\mu\in[0,\infty[} \tilde{B}_\mu = \bigcup_{\mu\in[0,\infty[} B_\mu = \bigcup_{\mu\in[0,\infty[} B^p_\mu$$

Let U denote the common value of these unions. From [R-R, Sect. V.4, Proposition 15] it follows that the strong projective topology in $(\mathcal{O}_C)'$ is equal to the topology of uniform convergence on subsets of $\mathcal{O}_C$ belonging to U. Since U is a covering of $\mathcal{O}_C$ by bounded subsets of $\mathcal{O}_C$, it follows that the strong projective topology in $(\mathcal{O}_C)'$ is an $\mathfrak{S}$-topology. See [B, Sect. III.3.1].

1.3. **The L. Schwartz space** $\mathcal{O}'_C$ **of rapidly decreasing distributions.** Whenever $\mu \in \mathbb{R}$, the L. Schwartz space $\mathcal{S}$ of rapidly decreasing C^∞-functions on $\mathbb{R}^n$ is sequentially dense in S^1_μ, and so $\mathcal{S}$ is sequentially dense in $\mathcal{O}_C = \lim\operatorname{ind}_{\mu\to\infty} S^1_\mu$ in the inductive topology. It follows that every $T \in \mathcal{S}'$ has at most one extension to a continuous linear functional on $\mathcal{O}_C$. The *L. Schwartz space* $\mathcal{O}'_C$ *of rapidly decreasing distributions on* $\mathbb{R}^n$ is defined as follows. As a set,

$$\mathcal{O}'_C = \{T \in \mathcal{S}' : T \text{ extends uniquely to a continuous linear functional on } \mathcal{O}_C\}. \tag{1.3}$$

The topology in $\mathcal{O}'_C$ is defined as the one induced from $(\mathcal{O}_C)'$ equipped with the strong projective topology.

1.4. **L. Schwartz's original definition of** $\mathcal{O}'_C$**.** In the original definition of $\mathcal{O}'_C$ formulated by L. Schwartz [S, Sect. VII.5] the space $\mathcal{O}_C$ is not used. Since $\mathcal{S}$ is dense in S^1_μ, it follows that for every $T \in \mathcal{S}'$ and $\mu \in [0,\infty[$ there is at most one $T_\mu \in (S^1_\mu)'$ extending T. Schwartz's definition of $\mathcal{O}'_C$ is as follows. As a set,

$$\mathcal{O}'_C = \{T \in \mathcal{S}' : \text{whenever } \mu \in [0,\infty[,\ (1+|\cdot|^2)^{\mu/2}T \text{ is continuous on } \mathcal{S} \text{ in the topology induced from } \mathcal{D}_{L^1}\}.$$

Thus,

$$\mathcal{O}'_C = \{T \in \mathcal{S}' : \text{whenever } \mu \in [0, \infty[, T \text{ extends uniquely to a} \\ \text{continuous linear functional } T_\mu \text{ on } S^1_\mu\},$$

whence (1.3) follows by [B, Sect. II.2.2, Corollary to Proposition 1] or by [R-R, Sect. V.2, Proposition 5]. Moreover L. Schwartz defined the topology in $\mathcal{O}'_C$ by distinguishing the class of convergent nets in $\mathcal{O}'_C$. Namely a net $(T_\iota)_{\iota \in J} \subset \mathcal{O}'_C$ is convergent if and only if for every $\mu \in [0, \infty[$ and $B \in B^1_\mu$ the net of non-negative numbers $(\sup_{\phi \in B} |T_{\iota,\mu}(\phi)|)_{\iota \in J}$ is convergent. The last means that the L. Schwartz topology in $\mathcal{O}'_C$ coincides with the one induced from $(\mathcal{O}_C)'$ equipped with the strong projective topology.

2. Rapidly decreasing distributions acting by convolution in $\mathcal{S}$

The present section is devoted mainly to characterization of rapidly decreasing distributions as those slowly increasing distributions, convolution with which maps continuously $\mathcal{S}$ into $\mathcal{S}$. Whenever $\phi \in C^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we denote by ϕ_x the translate of ϕ by x, i.e. the function $\phi_x : \mathbb{R}^n \ni y \mapsto \phi(x+y) \in \mathbb{R}^n$. Following [K, Vol. 2, Sect. CC.III.3°], by a *periodic partition of unity* on $\mathbb{R}^n$ we mean a partition of unity $\{\varphi_z : z \in \mathbb{Z}^n\}$ consisting of the translates of a function $\varphi \in C^\infty_c(\mathbb{R}^n)$.

Proposition 1. *For every set $\{T_\iota : \iota \in J\} \subset \mathcal{S}'$ the following three conditions are equivalent:*

(a) *$\{T_\iota : \iota \in J\} \subset \mathcal{S}'$ is a set of distributions equicontinuous with respect to the topology in $\mathcal{S}$ induced from $\mathcal{O}_C$,*
(b) *$\{[T_\iota *]|_S : \iota \in J\}$ is an equicontinuous set of operators belonging to $L(\mathcal{S}, \mathcal{S})$,*
(c) *whenever $\{\varphi_z : z \in \mathbb{Z}^n\}$ is a periodic partition of unity on $\mathbb{R}^n$, then*

$$\sum_{z \in \mathbb{Z}^n} \sup_{\iota \in J} |T_\iota(\varphi_z \phi)| < \infty \qquad \textit{for every } \phi \in \mathcal{O}_C.$$

For the main result of the present section it is superfluous to consider in Proposition 1 the sets of distributions instead of single distributions. However, in subsequent sections we shall use (a) and (b) for sets of distributions. We shall prove (a)⇒(b)⇒(c)⇒(a). To this end, for C^∞-functions φ on $\mathbb{R}^n$, we shall use the seminorms

$$\rho_{\mu,\alpha}(\varphi) = \sup_{x \in \mathbb{R}^n} (1 + |x|)^{-\mu} |\partial^\alpha \varphi(x)|.$$

For instance, the family of seminorms $\{\rho_{-\mu,\alpha} : \mu \in \mathbb{N},\ \alpha \in \mathbb{N}^n_0\}$ determines the topology of the L. Schwartz space $\mathcal{S}$ of rapidly decreasing C^∞-functions on $\mathbb{R}^n$, and for every fixed $\mu \in \mathbb{R}$ the family of seminorms $\{\rho_{\mu,\alpha} : \alpha \in \mathbb{N}^n_0\}$ determines the topology of the space $\tilde{S}_\mu$.

Proof of (a)⇒(b). If (a) holds, then for every $\mu \in [0, \infty[$ the set of distributions $\{T_\iota : \iota \in J\} \subset \mathcal{S}'$ is equicontinuous with respect to the topology induced in $\mathcal{S}$ from $\tilde{S}_\mu$, i.e. for every $\mu \in [0, \infty[$ there are $\nu_\mu \in \mathbb{N}_0$ and $C_\mu \in [0, \infty[$ such that

$$|T_\iota(\varphi)| \le C_\mu \sup_{|\alpha| \le \nu_\mu} \rho_{\mu,\alpha}(\varphi) \qquad \text{for every } \varphi \in \mathcal{S} \text{ and } \iota \in J. \tag{2.1}$$

It follows that whenever $\varphi \in \mathcal{S}$, $\iota \in J$ and $x \in \mathbb{R}^n$, then for every $\mu \in [0, \infty[$ one has

$$\begin{aligned}|(T_\iota * \varphi)(x) = |T_\iota((\varphi_x)^\vee)| &\le C_\mu \sup_{y\in\mathbb{R}^n,\, |\alpha|\le\nu_\mu} (1+|y|)^{-\mu}|\partial^\alpha \varphi(x-y)| \\ &\le C_\mu \sup_{|\alpha|\le\nu_\mu} \rho_{-\mu,\alpha}(\varphi) \sup_{y\in\mathbb{R}^n} (1+|y|)^{-\mu}(1+|x-y|)^{-\mu} \\ &\le C_\mu \sup_{|\alpha|\le\nu_\mu} \rho_{-\mu,\alpha}(\varphi)(1+|x|)^{-\mu}\end{aligned}$$

where the last inequality follows from the fact that $1 + |x| \le (1 + |y|)(1 + |x - y|)$. Consequently, for every $\mu \in [0, \infty[$ there are $\nu_\mu \in \mathbb{N}_0$ and $C_\mu \in [0, \infty[$ such that

$$\rho_{-\mu,0}(T_\iota * \varphi) \le C_\mu \sup_{|\alpha|\le\nu_\mu} \rho_{-\mu,\alpha}(\varphi) \quad \text{for every } \varphi \in \mathcal{S} \text{ and } \iota \in J. \tag{2.2}$$

Applying (2.2) to $\partial^\beta\varphi$ in place of φ, we infer that for every $\mu \in [0, \infty[$ there are $\nu_\mu \in \mathbb{N}_0$ and $C_\mu \in [0, \infty[$ such that

$$\rho_{-\mu,\beta}(T_\iota * \varphi) \le C_\mu \sup_{|\alpha|\le\nu_\mu} \rho_{-\mu,\alpha+\beta}(\varphi) \quad \text{for every } \varphi \in \mathcal{S},\ \iota \in J \text{ and } \beta \in \mathbb{N}_0^n,$$

which implies (b). □

Proof of (b)⇒(c). Whenever $\iota \in J$, $\psi \in \mathcal{S}$ and $x \in \mathbb{R}^n$, then

$$T_\iota(\psi) = T_\iota((((\psi_x)^\vee)_x)^\vee) = [T_\iota * (\psi_x)^\vee](x),$$

so that for our periodic partition of unity consisting of shifts of a function $\varphi \in C^\infty(\mathbb{R}^n)$ we have

$$T_\iota(\varphi_z\phi) = [T_\iota * (\varphi\phi_{-z})^\vee](-z).$$

Consequently, whenever $\iota \in J$, $\kappa \in [0, \infty[$ and $\phi \in C^\infty(\mathbb{R}^n)$, then

$$|T_\iota(\varphi_z\phi)| \le \rho_{-\kappa,0}(T_\iota * (\varphi\phi_{-z})^\vee) \cdot (1+|z|)^{-\kappa} \quad \text{for every } z \in \mathbb{Z}^n. \tag{2.3}$$

Assume now that (b) holds. Then (2.3) implies that for every $\kappa \in [0, \infty[$ there are $C_\kappa \in [0, \infty[$, $\lambda_\kappa \in [0, \infty[$ and $\nu_\kappa \in \mathbb{N}$ such that, for every $\phi \in C^\infty(\mathbb{R}^n)$ and $z \in \mathbb{Z}^n$,

$$\begin{aligned}\sup_{\iota\in J} |T_\iota(\varphi_z\phi)| &\le C_\kappa \cdot \sup_{|\alpha|\le\nu_\kappa} \rho_{-\lambda_\kappa,\alpha}((\varphi\phi_{-z})^\vee) \cdot (1+|z|)^{-\kappa} \\ &\le C_\kappa \cdot \sup_{x\in\mathbb{R}^n,\, |\alpha|\le\nu_\kappa} (1+|x|)^{\lambda_\kappa}|\partial^\alpha(\varphi(x)\phi(x-z))| \cdot (1+|z|)^{-\kappa} \\ &\le C_\kappa(1+r)^{\lambda_\kappa} \cdot \sup_{x\in\mathbb{R}^n,\, |\alpha|\le\nu_\kappa} |\partial^\alpha(\varphi(x)\phi(x-z))| \cdot (1+|z|)^{-\kappa}\end{aligned}$$

where

$$r = \sup\{|x| : x \in \operatorname{supp}\varphi\}.$$

From this, by the Leibniz formula for partial derivatives of a product of functions, it follows that for every $\kappa \in [0, \infty[$, $\phi \in C^\infty(\mathbb{R}^n)$ and $z \in \mathbb{Z}^n$ one has

$$\sup_{\iota\in J} |T_\iota(\varphi_z\phi)| \le D_\kappa \cdot \sup_{x\in z+\operatorname{supp}\varphi,\, |\alpha|\le\nu_\kappa} |\partial^\alpha\phi(x)| \cdot (1+|z|)^{-\kappa} \tag{2.4}$$

where

$$D_\kappa = LC_\kappa(1+r)^{\lambda_\kappa} \sup_{x\in\mathbb{R}^n,\, |\alpha|\le\nu_\kappa} |\partial^\alpha\varphi(x)|,$$

L being an absolute constant equal to the maximum of the coefficients in the Leibniz formula.

Till now we assumed that (b) holds and ϕ is an arbitrary function belonging to $C^\infty(\mathbb{R}^n)$. Henceforth we shall assume that (b) holds and $\phi \in \mathcal{O}_C$. As a set, $\mathcal{O}_C$ is equal to $\bigcup_{\mu\in[0,\infty[} \tilde{S}_\mu$. If $\mu \in [0,\infty[$ and $\phi \in \tilde{S}_\mu$ then

$$\begin{aligned}\sup_{x\in z+\operatorname{supp}\varphi,\,|\alpha|\le\nu_\kappa} |\partial^\alpha\phi(x)| &\le \sup_{|\alpha|\le\nu_\kappa} \rho_{\mu,\alpha}(\phi)\cdot(1+r+|z|)^\mu \\ &\le \sup_{|\alpha|\le\nu_\kappa} \rho_{\mu,\alpha}(\phi)\cdot(1+r)^\mu(1+|z|)^\mu,\end{aligned}$$

so that, by (2.4), for every $\kappa \in [0,\infty[$ and $z \in \mathbb{Z}^n$ one has

$$\sup_{\iota\in J} |T_\iota(\varphi_z\phi)| \le D_\kappa \sup_{|\alpha|\le\nu_\kappa} \rho_{\mu,\alpha}(\phi)(1+r)^\mu(1+|z|)^{\mu-\kappa}. \tag{2.5}$$

Fix now $\alpha \in]n,\infty[$ and take a $\phi \in \tilde{S}_\mu$ where $\mu \in [0,\infty[$. Let $\kappa = a+\mu$. Then, by (2.5),

$$\sup_{\iota\in J} |T_\iota(\varphi_z\phi)| \le M(\phi)\cdot(1+|z|)^{-a} \qquad \text{for every } z\in\mathbb{Z}^n,$$

where

$$M(\phi) = D_{a+\mu} \sup_{|\alpha|\le\nu_{a+\mu}} \rho_{\mu,\alpha}(\phi)(1+r)^\mu \in [0,\infty[.$$

It follows that

$$\sum_{z\in\mathbb{Z}^n} \sup_{\iota\in J} |T_\iota(\varphi_z\phi)| \le M(\phi) \sum_{z\in\mathbb{Z}^n} (1+|z|)^{-a},$$

so that (c) will be proved if we show that the multiple series $\sum_{z\in\mathbb{Z}^n}(1+|z|)^{-a}$ is convergent.

To prove that convergence, fix $\rho \in [n^{1/2},\infty[$ and for every $z \in \mathbb{Z}^n$ define $B_z := \{x \in \mathbb{R}^n : |x-z| \le \rho\}$. Then $\{B_z : z \in \mathbb{Z}^n\}$ is a covering of $\mathbb{R}^n$. If $x \in B_z$, then $1+|x| \le 1+|z|+\rho \le (1+|z|)(1+\rho)$, so that $(1+|z|)^{-a} \le (1+\rho)^a(1+|x|)^{-a}$. Hence

$$(1+|z|)^{-a} \le V^{-1}(1+\rho)^a \int_{B_z} (1+|x|)^{-a}\,dx \qquad \text{for every } z\in\mathbb{Z}^n,$$

where V is the volume of B_z, independent of z. It follows that

$$\sum_{z\in\mathbb{Z}^n} (1+|z|)^{-a} \le KV^{-1}(1+\rho)^a \int_{\mathbb{R}^n} (1+|x|)^{-a}\,dx < \infty,$$

where K denotes the order of the covering $\{B_z : z \in \mathbb{Z}^n\}$ of $\mathbb{R}^n$. □

Proof of (c)⇒(a). Suppose that (c) holds. We shall construct the extensions $\tilde{T}_\iota$ of the distributions T_ι by the series expansions

$$\tilde{T}_\iota(\phi) = \sum_{z\in\mathbb{Z}^n} T_\iota(\varphi_z\phi), \qquad \phi\in\mathcal{O}_C, \tag{2.6}$$

where $\{\varphi_z : z \in \mathbb{Z}^n\}$ is a periodic partition of unity on $\mathbb{R}^n$ consisting of translates of a function $\varphi \in C_c^\infty(\mathbb{R}^n)$. From (c) it follows that for every $\phi \in \mathcal{O}_C$ the numerical multiple series in (2.6) is absolutely convergent. For every $\iota \in J$, $k \in \mathbb{N}$ and $\phi \in \mathcal{O}_C$ let

$$\tilde{T}_{\iota,k}(\phi) := \sum_{|z|\le k} T_\iota(\varphi_z\phi).$$

Then

$$\text{each } \tilde{T}_{\iota,k} \text{ is a continuous linear functional on } \mathcal{O}_C, \tag{2.7}$$

because for every $z \in \mathbb{Z}^n$ the mapping $\mathcal{O}_C \ni \phi \mapsto \varphi_z \phi \in C_c^\infty(\mathbb{R}^n)$ is continuous. Whenever $\iota \in J$ and $\phi \in \mathcal{O}_C$ are fixed, by (2.4) the sequence $(\tilde{T}_{\iota,k}(\phi))_{k\in\mathbb{N}}$ of complex numbers is convergent and

$$\lim_{k\to\infty} \tilde{T}_{\iota,k}(\phi) = \tilde{T}_\iota(\phi). \tag{2.8}$$

Since, being the inductive limit of Fréchet spaces, $\mathcal{O}_C$ is a barrelled space, from (2.7) and (2.8), by the Banach–Steinhaus theorem, it follows that

whenever $\iota \in J$, then $\tilde{T}_\iota$ is a continuous linear functional on $\mathcal{O}_C$.

Furthermore, from (c) and (2.8) it follows that whenever $\phi \in \mathcal{O}_C$, then $\{\tilde{T}_{\iota,k}(\phi) : \iota \in J, k \in \mathbb{N}\}$ and hence also $\{\tilde{T}_\iota(\phi) : \iota \in J\}$ are bounded subsets of $\mathbb{C}$. Since $\mathcal{O}_C$ is barrelled, from boundedness of the sets $\{\tilde{T}_\iota(\phi) : \iota \in J\}$, $\phi \in \mathcal{O}_C$, and from [B, Sect. III.3.6, Theorem 2] it follows that $\{\tilde{T}_\iota(\phi) : \iota \in J\}$ is an equicontinuous set of linear functionals on $\mathcal{O}_C$.

It remains to prove that $\tilde{T}_\iota|_{\mathcal{S}} = T_\iota$ for every $\iota \in J$. To this end, notice that if $\phi \in C_c^\infty(\mathbb{R}^n)$ then $\tilde{T}_\iota(\phi) = T_\iota(\phi)$ because $\tilde{T}_{\iota,k}(\phi) = T_\iota(\phi)$ for every k so large that $\operatorname{supp}\varphi_z \cap \operatorname{supp}\phi = \emptyset$ whenever $|z| > k$. The equality $\tilde{T}_\iota(\phi) = T_\iota(\phi)$ for $\phi \in \mathcal{S}$ is a consequence of the analogous equality for $\phi \in C_c^\infty(\mathbb{R}^n)$ and dense continuous imbeddings $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S} \subset \mathcal{O}_C$. □

Theorem 1. *Whenever $T \in \mathcal{S}'$, then $T \in \mathcal{O}_C'$ if and only if $[T\,*]|_{\mathcal{S}} \in L(\mathcal{S}, \mathcal{S})$. Moreover, if $T \in \mathcal{S}'$ and $T * \mathcal{S} \subset \mathcal{S}$, then $[T\,*]|_{\mathcal{S}} \in L(\mathcal{S}, \mathcal{S})$.*

Proof. The first assertion of the theorem follows from [E, Theorem 2] and is also a consequence of the equivalence (a)⇔(b) in Proposition 1. The proof of the second assertion is as follows. If $T \in \mathcal{S}'$ and $T*\mathcal{S} \subset \mathcal{S}$, then, by [H, Sect. 4.11, Proposition 7], $[T\,*]|_{\mathcal{S}} \subset L(\mathcal{S}, \mathcal{O}_C)$, so that $[T\,*]|_{\mathcal{S}}$ is a closed operator from $\mathcal{S}$ into $\mathcal{S}$. Since $\mathcal{S}$ is an F-space, by the closed graph theorem, closedness of $[T\,*]|_{\mathcal{S}}$ implies its continuity. □

Since $[T\,*]|_{\mathcal{S}} \in L(\mathcal{S}, \mathcal{S})$ for every $T \in \mathcal{O}_C'$, we can equip $\mathcal{O}_C'$ with the topology induced from $L_b(\mathcal{S}, \mathcal{S})$ via the mapping $\mathcal{O}_C' \ni T \mapsto [T\,*]|_{\mathcal{S}} \in L(\mathcal{S}, \mathcal{S})$. This topology will be called the *strong operator topology.*

Theorem 2. *The strong projective topology in $\mathcal{O}_C'$ is no weaker than the strong operator topology.*

Proof. The topology in $L_b(\mathcal{S}, \mathcal{S})$ is determined by the system of seminorms

$$p_{\mu,\alpha,A}(L) = \sup_{\varphi\in A,\, x\in\mathbb{R}^n} (1+|x|)^\mu |\partial^\alpha (L\varphi)(x)| \tag{2.9}$$

where L denotes an operator belonging to $L_b(\mathcal{S}, \mathcal{S})$, and the entities labelling the system are: bounded subsets A of $\mathcal{S}$, non-negative numbers μ, and multiindices $\alpha \in \mathbb{N}_0^n$. Since convolution commutes with differentiations, for operators of the form $L = T\,*$ the system of seminorms (2.9) is equivalent to the system

$$\{q_{\mu,A} : \mu \in [0,\infty),\ A \text{ a bounded subset of } \mathcal{S}\} \tag{2.10}$$

where

$$q_{\mu,A}(T) = \sup_{\varphi\in A,\, x\in\mathbb{R}^n} (1+|x|)^\mu |(T*\varphi)(x)| = \sup_{\varphi\in A,\, x\in\mathbb{R}^n} |\tilde{T}((1+|\cdot|)^\mu * \varphi^\vee)| \tag{2.11}$$

for $T \in \mathcal{O}'_C$. Theorem 3 follows once it is proved that whenever $\mu \in [0, \infty[$ and A is a bounded subset of $\mathcal{S}$, then

$$B = \{(1 + |\cdot|)^\mu * \varphi^\vee : \varphi \in A\} \tag{2.12}$$

is a bounded subset of $\tilde{S}_\mu$, i.e. $\sup_{\phi \in B} \rho_{\mu,\alpha}(\phi) < \infty$ for every $\alpha \in \mathbb{N}_0^n$. But if B has the form (2.12) and $\lambda \in]\mu + n, \infty[$, then

$$\begin{aligned}
\sup_{\phi \in B} \rho_{\mu,\alpha}(\phi) &= \sup_{\varphi \in A,\, x \in \mathbb{R}^n} (1 + |x|)^{-\mu} \left| \int_{\mathbb{R}^n} (1 + |x - y|)^\mu \partial^\alpha (\varphi^\vee)(y)\, dy \right| \\
&\leq \sup_{\varphi \in A,\, x \in \mathbb{R}^n} \rho_{-\lambda,\alpha}(\varphi) \int_{\mathbb{R}^n} (1 + |x|)^{-\mu} (1 + |x - y|)^\mu (1 + |y|)^{-\lambda}\, dy \\
&\leq \sup_{\varphi \in A} \rho_{-\lambda,\alpha}(\varphi) \int_{\mathbb{R}^n} (1 + |y|)^{\mu - \lambda}\, dy
\end{aligned}$$

where the last term is finite because A is a bounded subset of $\mathcal{S}$, and $\mu - \lambda < -n$. □

3. Equicontinuity and boundedness

Theorem 3. *For every set $\{T_\iota : \iota \in J\} \subset \mathcal{S}'$ of distributions the following four conditions are equivalent:*

- (a) *for every $\iota \in J$ the distribution T_ι can be (uniquely) extended to a linear functional $\tilde{T}_\iota$ continuous on $\mathcal{O}_C$, and $\{\tilde{T}_\iota : \iota \in J\}$ is an equicontinuous set of linear functionals on $\mathcal{O}_C$,*
- (a)′ *for every $\iota \in J$ the distribution T_ι can be (uniquely) extended to a linear functional $\tilde{T}_\iota$ continuous on $\mathcal{O}_C$, and $\{\tilde{T}_\iota : \iota \in J\}$ is a bounded subset of $(\mathcal{O}_C)'$ in the strong projective topology,*
- (b) *$\{[T_\iota *]|_{\mathcal{S}} : \iota \in J\}$ is an equicontinuous subset of $L(\mathcal{S}, \mathcal{S})$,*
- (b)′ *$\{[T_\iota *]|_{\mathcal{S}} : \iota \in J\}$ is a bounded subset of $L_b(\mathcal{S}, \mathcal{S})$.*

Proof. From Theorem 2 we know that (a)⇔(b). Moreover, the space $\mathcal{S}$ is barrelled as a Fréchet space, and $\mathcal{O}_C$ is barrelled as the inductive limit of Fréchet (and hence barrelled) spaces. From barrelledness of $\mathcal{O}_C$ and $\mathcal{S}$ the equivalences (a)⇔(a)′ and (b)⇔(b)′ follow in view of [O, Sect. 4.2, Theorem 4.16] or [B, Sect. III.3.4, Theorem 1, and Sect. III.3.6, Proposition 7 and Theorem 2]. □

4. Coincidence of the strong operator topology and the strong projective topology on bounded subsets of $\mathcal{O}'_C$

By Theorems 2 and 3 the strong projective topology in $\mathcal{O}'_C$ is no weaker than the strong operator topology in $\mathcal{O}'_C$, and the bounded subsets of $\mathcal{O}'_C$ are the same for both the topologies. The following theorem implies that both topologies coincide on bounded subsets of $\mathcal{O}'_C$.

Theorem 4. *Let $(T_\iota)_{\iota \in J}$ be a net in $\mathcal{O}'_C$. If the net $([T_\iota *]|_{\mathcal{S}})_{\iota \in J}$ of convolution operators is bounded and is convergent in $L_b(\mathcal{S}, \mathcal{S})$, then the net $(T_\iota)_{\iota \in J}$ is convergent in the projective topology of $\mathcal{O}'_C$.*

Proof. Suppose that the net $([T_\iota *]|_{\mathcal{S}})_{\iota \in J}$ is bounded and converges to zero in $L_b(\mathcal{S}, \mathcal{S})$. Let ψ and η be non-negative C^∞-functions on $\mathbb{R}^n$ such that $\operatorname{supp} \psi \subset \{x \in \mathbb{R}^n : |x| \leq 2\}$, $\operatorname{supp} \eta \subset \{x \in \mathbb{R}^n : |x| \geq 1\}$, and $\psi(x) + \eta(x) = 1$ for every $x \in \mathbb{R}^n$. For every $r > 0$ and $x \in \mathbb{R}^n$ let $\psi_r(x) = \psi(r^{-1}x)$, $\eta_r(x) = \eta(r^{-1}x)$. If $r > 0$

is fixed, then ψ_r and η_r are non-negative C^∞-functions on $\mathbb{R}^n$, $\operatorname{supp}\psi_r \subset \{x \in \mathbb{R}^n : |x| \le 2r\}$, $\operatorname{supp}\eta_r \subset \{x \in \mathbb{R}^n : |x| \ge r\}$, $\psi_r(x) + \eta_r(x) = 1$ for every $x \in \mathbb{R}^n$, and

$$\text{for every } r_0 > 0 \text{ the set } \{\eta_r : r \ge r_0\} \text{ is a bounded subset of } C_b^\infty. \tag{4.1}$$

Whenever $\mu \in [0,\infty[$ and a bounded subset $\mathbf{B}$ of $\tilde{S}_\mu$ are fixed, then $\sup_{\phi\in\mathbf{B}} |\tilde{T}_\iota(\phi)| \le A_{r,\iota} + B_r$ for every $r > 0$ and $\iota \in J$ where $A_{r,\iota} = \sup_{\phi\in\mathbf{B}} |\tilde{T}_\iota(\psi_r\phi)|$, $B_r = \sup_{\iota\in J,\,\phi\in\mathbf{B}} |\tilde{T}_\iota(\eta_r\phi)|$. Hence, in view of the definition of the strong projective topology in $\mathcal{O}'_C$, Theorem 5 follows once it is proved that

$$\lim_\iota A_{r,\iota} = 0 \quad \text{for every fixed } r > 0, \tag{4.2}$$

$$\lim_{r\to\infty} B_r = 0. \tag{4.3}$$

If $r > 0$ is fixed and $\mathbf{B}$ is a bounded subset of $\tilde{S}_\mu$, then $C = \{\phi_r\phi : \phi \in \mathbf{B}\}$ is a bounded subset of $C_c^\infty(\mathbb{R}^n)$ and, a fortiori, a bounded subset of $\mathcal{S}$. It follows that

$$\lim_\iota A_{r,\iota} = \lim_\iota \sup_{\varphi\in C} |T_\iota(\varphi)| = \lim_\iota \sup_{\varphi\in C} |[T_\iota * \varphi^\vee](0)| = 0,$$

so that condition (4.2) is satisfied.

Since the net of operators $([T_\iota *]|_{\mathcal{S}})_{\iota\in J} \subset L(\mathcal{S},\mathcal{S})$ is bounded in the topology of $L_b(\mathcal{S},\mathcal{S})$, from the equivalence $(\mathrm{b})'\Leftrightarrow(\mathrm{a})'$ of Theorem 3 it follows that the net $(\tilde{T}_\iota)_{\iota\in J} \subset (\mathcal{O}_C)'$ is bounded in the strong projective topology of $(\mathcal{O}_C)'$. The proof of (4.3) will be based on the inequality (1.1) and the corresponding lemma from Section 1. Suppose that $r_0 > 0$, $\mu \ge 0$ and $\lambda > 0$ are fixed and $\mathbf{B}$ is a bounded subset of $\tilde{S}_\mu$. Then $\mathbf{C} := \{(1 + |\cdot|^2)^{\lambda/2}\phi : \phi \in \mathbf{B}\}$ is a bounded subset of $\tilde{S}_{\mu+\lambda}$. Whenever $r \ge r_0 \ge 1$, then

$$B_r = \sup_{\iota\in J,\,\phi\in\mathbf{C}} |\tilde{T}_\iota((1 + |\cdot|^2)^{-\lambda/2}\eta_r\phi)| \tag{4.4}$$

$$\le (1+r^2)^{-\lambda/2} \sup_{\iota\in J,\,\phi\in\mathbf{C}} \left|\tilde{T}_\iota\left(\left(\frac{1+|\cdot|^2}{1+r^2}\right)^{-\lambda/2}\eta_r\phi\right)\right| \le (1+r_0)^{-\lambda/2} \sup_{\iota\in J,\,\psi\in\mathbf{D}} |\tilde{T}_\iota(\psi)|$$

where

$$\mathbf{D} = \left\{\left(\frac{1+|\cdot|^2}{1+r^2}\right)^{-\lambda/2}\eta_r\phi : r \ge 1,\ \phi \in \mathbf{C}\right\}.$$

Since $\operatorname{supp}(\eta,\phi) \subset \{x \in \mathbb{R}^n : |x| \ge r\}$, from (4.1), (1.1), and the Leibniz formula one infers that, together with $\mathbf{C}$, also $\mathbf{D}$ is a bounded subset of $\tilde{S}_{\mu+\lambda}$. Since the strong projective topology in $(\mathcal{O}_C)'$ coincides with the topology of uniform convergence on subsets of $\mathcal{O}_C$ belonging to $\bigcup_{\mu\in[0,\infty[} \tilde{B}_\mu$, it follows that

$$\sup_{\iota\in J,\,\psi\in\mathbf{D}} |\tilde{T}_\iota(\psi)| < \infty. \tag{4.5}$$

It remains to observe that (4.4) and (4.5) imply (4.3). □

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