ON THE EXCHANGE BETWEEN CONVOLUTION AND MULTIPLICATION VIA THE FOURIER TRANSFORMATION

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Abstract. The paper contains a complete proof of the theorem of L. Schwartz on Fourier exchange between convolution and multiplication. The main tool is the L. Schwartz one-to-one correspondence between distributions on $\mathbb{R}^n$ and translation invariant operators from the test space into $C^\infty(\mathbb{R}^n)$.

1. Content of the paper

We consider the L. Schwartz space $\mathcal{S}'$ of slowly increasing distributions on $\mathbb{R}^n$, and the L. Schwartz space $\mathcal{O}'_C$ of rapidly decreasing distributions on $\mathbb{R}^n$. $\mathcal{O}_C$ denotes the predual space of $\mathcal{O}'_C$, determined by J. Horváth, such that $\mathcal{O}'_C = \{T \in \mathcal{S}' :$ T extends uniquely to a continuous linear functional $\tilde{T}$ on $\mathcal{O}_C\}$. We use the one-to-one correspondence between distributions $U \in \mathcal{S}'$ and the operators $A_U : \mathcal{S} \ni \phi \mapsto U * \phi \in C^\infty$. If $T \in \mathcal{O}'_C$ and $U \in \mathcal{S}'$, then the convolutions $U * T$ and $T * U$ are defined by the equalities $A_{U*T} = A_U \circ A_T$ and $A_{T*U} = A_{\tilde{T}} \circ A_U$. It turns out that the distributions $U * T \in \mathcal{S}'$ and $T * U \in \mathcal{S}'$ so defined are equal, and that $(U * T)|_{C_c^\infty} = (T * U)|_{C_c^\infty}$ is equal to the commutative convolution of T with U defined by the general method involving tensor products.

By approximating the operators A_U, $U \in \mathcal{S}'$, by operators A_ϕ, $\phi \in C_c^\infty$, we prove the L. Schwartz theorem on Fourier exchange between convolution and multiplication for convolution of $T \in \mathcal{O}'_C$ and $U \in \mathcal{S}'$. Let us stress that the original proof of this theorem [S2, Sect. VII.8, Theorem XV] was incomplete (see L. Schwartz's own remark in [S2, pp. 269–270]). Notice also that the proof of Schwartz's theorem was announced (but not published) by J. Horváth [H1, Sect. 4.11, Theorem 3].

2. Convolution and Fourier transformation in $\mathcal{S}$

The present section contains a concise presentation of the fundamental properties of convolution and Fourier transformation in the L. Schwartz space $\mathcal{S}$ of rapidly decreasing functions on $\mathbb{R}^n$. The subsequent sections are devoted to extension of these properties onto the dual space of $\mathcal{S}$, i.e. the space $\mathcal{S}'$ of slowly increasing distributions on $\mathbb{R}^n$.

Convolution on $\mathcal{S} \times \mathcal{S}$ is a continuous bilinear symmetric mapping $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ denoted by $*$ and defined by

$$(2.1) \qquad (\varphi * \psi)(x) = \int_{\mathbb{R}^n} \varphi(y)\psi(x-y)\,dy \quad \text{for } \varphi, \psi \in \mathcal{S} \text{ and } x \in \mathbb{R}^n.$$

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The Fourier transformation is a continuous linear mapping $\mathcal{S} \to \mathcal{S}$ denoted by $\mathcal{F}$ and defined by

$$(\mathcal{F}\varphi)(x) = \int_{\mathbb{R}^n} e^{-i\langle x,y\rangle}\varphi(y)\,dy \quad \text{ for } \varphi \in \mathcal{S} \text{ and } x \in \mathbb{R}^n.$$

It follows from Fubini's theorem that whenever $\varphi, \psi \in \mathcal{S}$, then

$$\mathcal{F}(\varphi * \psi) = (\mathcal{F}\varphi)\cdot(\mathcal{F}\psi) \quad (\textit{the Fourier exchange equality}) \tag{2.2}$$

and

$$\langle \mathcal{F}\varphi, \psi\rangle = \langle \varphi, \mathcal{F}\psi\rangle \quad (\textit{the Parseval equality}) \tag{2.3}$$

where $\langle \varphi, \psi\rangle = \int_{\mathbb{R}^n} \varphi(x)\psi(x)\,dx$.

Now we are going to prove that the *Fourier transformation* $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ *is an automorphism of* $\mathcal{S}$ *such that* $\mathcal{F}^{-1} = (2\pi)^{-n}\mathcal{F}^{\vee}$. To this end for every $t \in]0,\infty[$ we consider the gaussian function G_t on $\mathbb{R}^n$, and the normalized gaussian function N_t on $\mathbb{R}^n$, defined by

$$G_t(x) = \exp\left(-\frac{t|x|^2}{2}\right) \quad \text{and} \quad N_t(x) = (2\pi t)^{-n/2}\exp\left(-\frac{|x|^2}{2t}\right)$$

for $x \in \mathbb{R}^n$. Starting from the one-dimensional equality

$$(2\pi)^{-1/2}\int_{\mathbb{R}} e^{-ixy}\exp\left(-\frac{y^2}{2}\right)dy = \exp\left(-\frac{x^2}{2}\right)$$

whose proof can be found in the Remark at the end of [Y, Sect. VI.1], one concludes that

$$N_t = (2\pi)^{-n}\mathcal{F}(G_t).$$

From now on, we follow [M, Sect. III.2]. Let $\varphi \in \mathcal{S}$, $t \in]0,\infty[$ and $x \in \mathbb{R}^n$. By the Fubini–Tonelli theorem the integrability on $\mathbb{R}^{2n}$ of the function $\mathbb{R}^{2n} \ni (y,z) \mapsto |\varphi(y)|\exp(-t|z|^2/2) \in [0,\infty[$ implies that

$$\begin{aligned}
[N_t * \varphi](x) &= (2\pi)^{-n}\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} e^{-i\langle x-y,z\rangle}\exp\left(-\frac{t|z|^2}{2}\right)dz\right]\varphi(y)\,dy \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i\langle x,z\rangle}\exp\left(-\frac{t|z|^2}{2}\right)\left[\int_{\mathbb{R}^n} e^{-i\langle y,z\rangle}\varphi(-y)\,dy\right]dz \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i\langle x,z\rangle}\exp\left(-\frac{t|z|^2}{2}\right)[\mathcal{F}(\varphi^{\vee})](z)\,dz.
\end{aligned} \tag{2.4}$$

Now we shall pass to the limit as $t \downarrow 0$ in both sides of (2.4). Since $\mathcal{F}(\varphi^{\vee}) \in \mathcal{S}$ and $\exp(-t|z|^2/2) \le 1$ for $t \ge 0$, and since $\lim_{t\downarrow 0}\exp(-t|z|^2/2) = 1$ almost uniformly with respect to $z \in \mathbb{R}^n$, it follows by the Lebesgue dominated convergence theorem that when $t \downarrow 0$, the right hand side of (2.4) converges to

$$(2\pi)^{-n}\int_{\mathbb{R}^n} e^{-i\langle x,z\rangle}[\mathcal{F}(\varphi^{\vee})](z)\,dz = (2\pi)^{-n}[[\mathcal{F}\circ\mathcal{F}](\varphi^{\vee})](x).$$

On the other hand,

$$\int_{\mathbb{R}^n} N_t(x)\,dx = \int_{\mathbb{R}^n} N_1(x)\,dx = (2\pi)^{-n/2}\int_{\mathbb{R}^n} G_1(x)\,dx = 1,$$

and $\lim_{t\downarrow 0}\sup_{|x|\ge r} N_t(x) = 0$ for every $r \in]0,\infty[$, whence $\lim_{t\downarrow 0}[N_t * \varphi](x) = \varphi(x)$. Consequently, by (2.4),

$$\varphi = (2\pi)^{-n}[\mathcal{F}\circ\mathcal{F}](\varphi^{\vee}) \quad \text{ for every } \varphi \in \mathcal{S}. \tag{2.5}$$

From (2.5) it follows that the mapping $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is invertible and maps $\mathcal{S}$ onto $\mathcal{S}$, and

$$\mathcal{F}^{-1} = (2\pi)^{-n}\mathcal{F}^{\vee}. \tag{2.6}$$

Notice that the operator semigroup $([N_t\, *]|_{\mathcal{S}})_{t>0} \subset L(\mathcal{S},\mathcal{S})$ is the well-known heat (or diffusion) one-parameter convolution semigroup. By (2.4) and (2.6) we have

$$[N_t\, *]|_{\mathcal{S}} = [\mathcal{F}^{-1} \circ (G_t\, \cdot) \circ \mathcal{F}]|_{\mathcal{S}},$$

so that $([G_t\, \cdot]|_{\mathcal{S}})_{t>0} \subset L(\mathcal{S},\mathcal{S})$ is the corresponding one-parameter semigroup of Fourier multipliers. This correspondence is an example of Fourier exchange between convolution and multiplication.

3. Locally convex spaces associated with slowly increasing distributions

In the space $\mathcal{D}'$ of all distributions on $\mathbb{R}^n$ L. Schwartz distinguished the linear subset $\mathcal{S}'$ of *slowly increasing distributions on* $\mathbb{R}^n$ in which Fourier transformation acts transparently. There are two equivalent definitions of $\mathcal{S}'$:

(a) *definition by distributional differentiation of continuous functions of polynomial growth*: for every $\mu \in \mathbb{R}^n$ let $B_\mu \subset C(\mathbb{R}^n)$ be the Banach space with the norm $\|f\|_\mu = \sup_{x\in\mathbb{R}^n}(1+|x|)^{-\mu}|f(x)|$. Then

$$\mathcal{S}' = \{U \in \mathcal{D}'(\mathbb{R}^n) : U = \partial^\alpha f \text{ for some } \alpha \in \mathbb{N}_0^n \text{ and } f \in B_\mu \text{ where } \mu \in [0,\infty[\},$$

(b) *definition by duality*: $\mathcal{S}'$ is the locally convex space of continuous linear functionals on the space $\mathcal{S}$ of rapidly decreasing C^∞-functions on $\mathbb{R}^n$.

The equivalence of these two definitions is a consequence of [S2, Sect. VII.4, Theorem VI.1^0, p. 239].

In the theory of slowly increasing distributions there appear locally convex spaces of C^∞-functions ϕ on $\mathbb{R}^n$ whose topology can be determined by seminorms of the form

$$\rho_{\mu,\alpha}(\phi) = \sup_{x\in\mathbb{R}^n}(1+|x|)^{-\mu}|\partial^\alpha\phi(x)|,$$

where $\mu \in \mathbb{R}$ and $\alpha \in \mathbb{N}_0^n$.

1. For every fixed $\mu \in \mathbb{R}$ let

$$\tilde{S}_\mu = \{\phi \in C^\infty(\mathbb{R}^n) : \rho_{\mu,\alpha}(\phi) < \infty \text{ for every } \alpha \in \mathbb{N}_0^n\}.$$

Endowed with the topology determined by the family of seminorms $\{\rho_{\mu,\alpha} : \alpha \in \mathbb{N}_0^n\}$, $\tilde{S}_\mu$ is a Fréchet space. Examples of functions $\phi \in \tilde{S}_\mu$ are $\phi = f * \psi$ where $f \in B_\mu$ and $\psi \in C_c^\infty(\mathbb{R}^n)$. By [H1, Sect. 2.5, Example 8] for every $\mu \in \mathbb{R}$ the function $\phi(x) = (1+|x|^2)^{\mu/2}$ belongs to $\tilde{S}_\mu$.

2. $\mathcal{S} := \bigcap_{\mu\in\mathbb{R}} \tilde{S}_\mu$ is the L. Schwartz space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$. Endowed with the topology determined by the family of seminorms $\{\rho_{-\mu,\alpha} : \mu \in \mathbb{N},\ \alpha \in \mathbb{N}_0^n\}$, $\mathcal{S}$ is a Fréchet space.

3. $\mathcal{O}_C := \bigcup_{\mu\in\mathbb{R}} \tilde{S}_\mu = \{\phi \in C^\infty(\mathbb{R}^n)$: there is $\mu \in [0,\infty[$ such that $\rho_{\mu,\alpha}(\phi) < \infty$ for every $\alpha \in \mathbb{N}_0^n\}$ is the space of some (not all) infinitely differentiable functions on $\mathbb{R}^n$ whose partial derivatives of all orders have at most polynomial growth at infinity (for every $\phi \in \mathcal{O}_C$ the same growth for all partial derivatives). The natural topology in $\mathcal{O}_C$ is the inductive topology determined by the family $\{\tilde{S}_\mu : \mu \in \mathbb{R}\}$

of subspaces of $\mathcal{O}_C$. In this sense we write $\mathcal{O}_C = \liminf_{\mu\to\infty} \tilde{S}_\mu$. The space $\mathcal{O}_C$ endowed with an inductive topology (determined by spaces S_μ similar to $\tilde{S}_\mu$) was introduced by J. Horváth in [H1, Sect. 2.12, Example 9]. The inductive topology in $\mathcal{O}_C$ is no weaker than the topology induced in $\mathcal{O}_C$ from $\mathcal{S}'$ equipped with the strong dual topology.

It is remarkable that whenever $\mu \in [0,\infty[$, then C_0^∞ is not dense in $\tilde{S}_\mu$, but C_c^∞ is dense in $\mathcal{O}_C = \liminf_{\mu\to\infty} \tilde{S}_\mu$. Indeed, let $(\chi_k)_{k\in\mathbb{N}} \subset C_c^\infty$ be a sequence such that $\chi_k(x) = 1$ whenever $|x| \le k$, and $\sup_{k\in\mathbb{N},\, x\in\mathbb{R}^n} |\partial^\alpha \chi_k(x)| < \infty$ for every $\alpha \in \mathbb{N}_0^n$. If $\phi \in \mathcal{O}_C$, then $\phi \in \tilde{S}_\mu$ for some $\mu \in [0,\infty[$, $\chi_k\phi \in C_c^\infty$ for every $k \in \mathbb{N}$, and whenever $\varepsilon > 0$, then $\lim_{k\to\infty} \chi_k\phi = \phi$ in the topology of the Fréchet space $\tilde{S}_{\mu+\varepsilon}$, so that a fortiori $\lim_{k\to\infty} \chi_k\phi = \phi$ in the topology of $\mathcal{O}_C$.

Our spaces $\tilde{S}_\mu$ differ from the spaces $S_\mu = (1+|x|)^{\mu/2} C_0^\infty$ used by J. Horváth. But $\tilde{S}_{\mu-\varepsilon} \subset S_\mu \subset \tilde{S}_\mu$ for every $\mu \in \mathbb{R}$ and $\varepsilon \in]0,\infty[$, so that $\mathcal{O}_C$ is the same (see the Remark in [B, Sect. II.2.4]).

4. An important non-Fréchet space of C^∞-functions on $\mathbb{R}^n$ is the space $\mathcal{O}_M$ of all functions whose partial derivatives have at most polynomial growth, that is,

$$\mathcal{O}_M = \{\phi \in C^\infty : \text{for every } \alpha \in \mathbb{N}_0^n \text{ there is } \mu \in [0,\infty[\text{ such that } \rho_{\mu,\alpha}(\phi) < \infty\}.$$

The space $\mathcal{O}_M$ coincides with the algebra of multipliers of $\mathcal{S}$. An example of $\phi \in \mathcal{O}_M \setminus \mathcal{O}_C$ is $\phi(x) = \exp(i|x|^2)$.

Apart from the above spaces of C^∞-functions and the space $\mathcal{S}'$ of slowly increasing distributions we shall use the L. Schwartz limes-space $\mathcal{O}'_C$ of *rapidly decreasing distributions* on $\mathbb{R}^n$. According to J. Horváth [H1, Sect. 4.11], as a set,

$$\mathcal{O}'_C = \{T \in \mathcal{S}' : T \text{ extends to a continuous linear functional } \tilde{T} \text{ on } \mathcal{O}_C\}.$$

The extension is unique, because $\mathcal{S}$ is dense in $\mathcal{O}_C$. By [E1, Theorem 2], as a set,

$$\mathcal{O}'_C = \{T \in \mathcal{S}' : [T\,*]|_{\mathcal{S}} \in L(\mathcal{S},\mathcal{S})\}, \tag{3.1}$$

so that it is reasonable to equip $\mathcal{O}'_C$ with the *operator topology* induced from $L_b(\mathcal{S},\mathcal{S})$ via the mapping $\mathcal{O}'_C \ni T \mapsto [T\,*]|_{\mathcal{S}} \in L(\mathcal{S},\mathcal{S})$. In accordance with [S2, Sect. VII.5, Theorem IX.1^0] one has

(3.2) $\mathcal{O}'_C = \{T \in \mathcal{S}'$: for every $\mu \in]0,\infty[$ the distribution T can be represented as a finite sum $T = \sum_{|\gamma|\le G} \partial^\gamma f_\gamma$ where $f_\gamma \in B_{-\mu}$ for every $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \le G\}$.

Here

$$B_{-\mu} = \Big\{ f \in C(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} |x|^\mu |f(x)| < \infty \Big\} \qquad \text{for } \mu \in [0,\infty[.$$

For instance, if $T \in \mathcal{S}'$ is represented by the function $\phi(x) = \exp(i|x|^2)$, $x \in \mathbb{R}^n$, then $T \in \mathcal{O}'_C$. The proof based on (3.2) (and on Hermite polynomials) is given in [S2, Sect. VII.5, Example on p. 245].

4. Convolutions of distributions with C^∞-functions

For every function f on $\mathbb{R}^n$ and every $x \in \mathbb{R}^n$ we define the translate of f by x to be the function $\tau_x f$ on $\mathbb{R}^n$ given by $(\tau_x f)(y) = f(x+y)$ for $y \in \mathbb{R}^n$. In this definition τ_x is the operator of translation by x whose domain consists of all functions defined on $\mathbb{R}^n$. If $U \in \mathcal{S}'$ and $\phi \in \mathcal{S}$ then the convolution $U * \phi$ is defined as the C^∞-function on $\mathbb{R}^n$ given by $[U*\phi](x) = U((\tau_x\phi)^\vee)$ for $x \in \mathbb{R}^n$. Whenever $\tilde{T} \in (\mathcal{O}_C)'$, then for every $\psi \in \mathcal{O}_C$ the convolution $\tilde{T} * \psi$ is defined as the function on $\mathbb{R}^n$ given by $[\tilde{T} * \psi](x) = \tilde{T}((\tau_x\psi)^\vee)$ for $x \in \mathbb{R}^n$. In order to be sure that the above definitions are correct we have to know that the spaces $\mathcal{S}$ and $\mathcal{O}_C$ are invariant with respect to translations and reflections. For $\mathcal{S}$ this is well known, and for $\mathcal{O}_C$ it follows from the estimation (4.1) below.

Theorem 4.1.

(i) *If* $U \in \mathcal{S}'$, *then* $[U\,*]|_{\mathcal{S}} \in L(\mathcal{S}, \mathcal{O}_C)$.
(ii) $T \in \mathcal{O}_C'$ *if and only if* $[T\,*]|_{\mathcal{S}} \in L(\mathcal{S}, \mathcal{S})$.
(iii) *If* $T \in \mathcal{O}_C'$, *then* $[\tilde{T}\,*]|_{\mathcal{O}_C} \in L(\mathcal{O}_C, \mathcal{O}_C)$.

Denote by $L_b(\mathcal{S}, \mathcal{O}_C)$ the set $L(\mathcal{S}, \mathcal{O}_C)$ of continuous linear operators equipped with the topology of uniform convergence on bounded subsets of $\mathcal{S}$. Theorem 4.1(i) implies that $\mathcal{S}'$ can be endowed with the topology induced from $L_b(\mathcal{S}, \mathcal{O}_C)$ via the mapping $\mathcal{S}' \ni U \mapsto [U\,*]|_{\mathcal{S}} \in L(\mathcal{S}, \mathcal{O}_C)$. This topology in $\mathcal{S}'$ will be called the *operator topology*. It is no weaker than the strong dual topology in $\mathcal{S}'$ because if $(U_\iota)_{\iota \in J}$ is a net in $\mathcal{S}'$ such that $[U_\iota\,*]|_{\mathcal{S}} \to 0$ in $L_b(\mathcal{S}, \mathcal{O}_C)$, then $U_\iota(\phi) = [[U_\iota\,*](\phi^\vee)](0) \to 0$ in $\mathbb{C}$ uniformly with respect to ϕ ranging over any bounded subset of $\mathcal{S}$, i.e. $U_\iota \to 0$ in the strong dual topology of $\mathcal{S}'$.

Proof of Theorem 4.1. (i) This assertion goes back to J. Horváth [H1, Sect. 4.11, Proposition 7]. For our spaces $\tilde{S}_\mu$ we have only to care about the seminorms $\rho_{\mu,\alpha}$ and not about more subtle asymptotic behaviour as in the case of the J. Horváth spaces S_μ. Therefore our proof is shorter. Let $U \in \mathcal{S}'$. According to the definition of $\mathcal{S}'$ by distributional differentiation of continuous functions of at most polynomial growth, there are $\mu \in [0, \infty[$, $\alpha \in \mathbb{N}_0^n$ and $f \in B_\mu$ such that $U = \partial^\alpha f$. It follows that whenever $\varphi \in \mathcal{S}$, $\beta \in \mathbb{N}_0^n$ and $\lambda \in]n, \infty[$, then

$$\begin{aligned}\rho_{\mu,\beta}(U * \varphi) = \rho_{\mu,\alpha+\beta}(f * \varphi) &\le \sup_{x \in \mathbb{R}^n} (1 + |x|)^{-\mu} \int_{\mathbb{R}^n} |f(x - y)|\, |\partial^{\alpha+\beta}\varphi(y)|\, dy \\ &\le \sup_{x \in \mathbb{R}^n} (1 + |x|)^{-\mu} \|f\|_{B_\mu} \rho_{-\mu-\lambda,\alpha+\beta}(\varphi) \int_{\mathbb{R}^n} (1 + |x - y|)^{\mu} (1 + |y|)^{-\mu-\lambda}\, dy \\ &\le \|f\|_{B_\mu} \int_{\mathbb{R}^n} (1 + |y|)^{-\lambda}\, dy\, \rho_{-\mu-\lambda,\alpha+\beta}(\varphi),\end{aligned}$$

because $(1+|x|)^{-\mu}(1+|x-y|)^{\mu}(1+|y|)^{-\mu} \le 1$. This shows that $[U\,*]|_{\mathcal{S}} \in L(\mathcal{S}, \tilde{S}_\mu)$. Since $\tilde{S}_\mu$ is continuously imbedded in $\mathcal{O}_C$, it follows that $[U\,*]|_{\mathcal{S}} \in L(\mathcal{S}, \mathcal{O}_C)$.

(ii) This assertion is nothing but (3.1).

(iii) Since $\mathcal{O}_C = \lim\operatorname{ind}_{\mu\to\infty} \tilde{S}_\mu$, by [R-R, Sect. V.2, Proposition 5] it will follow that $[\tilde{T}\,*]|_{\mathcal{O}_C} \in L(\mathcal{O}_C, \mathcal{O}_C)$ once it is proved that $[\tilde{T}\,*]|_{\tilde{S}_\mu} \in L(\tilde{S}_\mu, \mathcal{O}_C)$ for every $\mu \in [0, \infty[$. Since $\tilde{S}_\mu$ is continuously imbedded in $\mathcal{O}_C$, it suffices to prove that $[\tilde{T}\,*]|_{\tilde{S}_\mu} \in L(\tilde{S}_\mu, \tilde{S}_\mu)$ for every $\mu \in [0, \infty[$.

Notice first that if $\Psi \in \tilde{S}_\mu$, then $\tau_x \Psi \in \tilde{S}_\mu$ for every $x \in \mathbb{R}^n$: this follows from the estimation

$$\begin{aligned}\rho_{\mu,\beta}(\tau_x \Psi) &= \sup_{y \in \mathbb{R}^n} (1 + |y|)^{-\mu} |\partial^\beta \Psi(x + y)| \\ &\le \sup_{y \in \mathbb{R}^n} (1 + |y|)^{-\mu} [(1 + |x + y|)^{\mu} \rho_{\mu,\beta}(\Psi)] \le (1 + |x|)^{\mu} \rho_{\mu,\beta}(\Psi),\end{aligned} \tag{4.1}$$

which is a consequence of $1+|x+y| \le (1+|x|)(1+|y|)$. Furthermore,

$$\begin{aligned}\rho_{\mu,\alpha}(\tilde{T}*\Psi) &= \sup_{x\in\mathbb{R}^n}(1+|x|)^{-\mu}|\partial^\alpha(\tilde{T}*\Psi)(x) \\ &= \sup_{x\in\mathbb{R}^n}(1+|x|)^{-\mu}|\tilde{T}((\tau_x\partial^\alpha\Psi)^\vee)|.\end{aligned} \tag{4.2}$$

Since $\tilde{T}|_{\tilde{S}_\mu}$ is a continuous linear functional on the Fréchet space $\tilde{S}_\mu$, there are $C \in]0,\infty[$ and $k \in \mathbb{N}$, depending only on $\tilde{T}|_{\tilde{S}_\mu}$, such that $|\tilde{T}|_{\tilde{S}_\mu}(\Phi)| \le C\sum_{|\beta|\le k}\rho_{\mu,\beta}(\Phi)$ for every $\Phi \in \tilde{S}_\mu$. Hence from (4.2) and (4.1) it follows that whenever $T \in \mathcal{O}'_C$, $\mu \in [0,\infty[$, $\alpha \in \mathbb{N}_0^n$ and $\Psi \in \tilde{S}_\mu$, then

$$\rho_{\mu,\alpha}(\tilde{T}*\Psi) \le \sup_{x\in\mathbb{R}^n}(1+|x|)^{-\mu}C\sum_{|\beta|\le k}\rho_{\mu,\beta}(\tau_x\partial^\alpha\Psi) \le C\sum_{|\beta|\le k}\rho_{\mu,\alpha+\beta}(\Psi),$$

proving that $[\tilde{T}*]|_{\tilde{S}_\mu} \in L(\tilde{S}_\mu,\tilde{S}_\mu)$. □

Lemma 4.2. *If $U \in \mathcal{S}'$ and $\varphi,\psi \in \mathcal{S}$, then*

$$U*(\varphi*\psi) = (U*\varphi)*\psi.$$

Proof. There are $\alpha \in \mathbb{N}_0^n$ and $\mu \in [0,\infty[$ such that $U = \partial^\alpha f$ for some $f \in B_\mu$. Therefore $U*(\varphi*\psi) = f*\partial^\alpha(\varphi*\psi) = (f*\partial^\alpha\varphi)*\psi = (U*\varphi)*\psi$ by Fubini's theorem, because convolutions involving f are absolutely convergent integrals, in view of elementary inequalities similar to that used in the proof of Theorem 4.1. □

5. Sequential denseness of C_c^∞ in $\mathcal{S}'$ and of $\mathcal{E}'$ in $\mathcal{O}'_C$ in operator topologies

Theorem 5.1. *C_c^∞ is sequentially dense in $\mathcal{S}'$ in the operator topology.*

Proof. Let $U \in \mathcal{S}'$. Then there are $\mu_0 \in [0,\infty[$, $\alpha \in \mathbb{N}_0^n$ and $f \in B_{\mu_0}$ such that $U = \partial^\alpha f$. Take a continuous $[0,1]$-valued function χ on $\mathbb{R}^n$ such that $\chi(x) = 1$ if $|x| \le 1$ and $\chi(x) = 0$ if $|x| \ge 2$. For every $k \in \mathbb{N}$ define $\chi_k \in C(\mathbb{R}^n)$ by

$$\chi_k(x) = \chi(k^{-1}x) \tag{5.1}$$

and choose $\psi_k \in C_c^\infty(\mathbb{R}^n)$ such that

$$\|\psi_k - \chi_k f\|_{B_{\mu_0}} \le k^{-1}. \tag{5.2}$$

Such a ψ_k exists because $\chi_k f$ is continuous on $\mathbb{R}^n$ and has bounded support. Let $\mu \in]\mu_0,\infty[$. Then $\|\psi_k - \chi_k f\|_{B_\mu} \le \|\psi_k - \chi_k f\|_{B_{\mu_0}} \le k^{-1}$ and

$$\begin{aligned}\|\chi_k f - f\|_{B_\mu} &= \sup_{|x|\ge k}(1+|x|)^{-\mu}|(\chi_k(x)-1)f(x)| \\ &\le (1+k)^{-(\mu-\mu_0)}\sup_{|x|\in\mathbb{R}^n}(1+|x|)^{-\mu_0}|(\chi_k(x)-1)f(x)| \\ &\le (1+k)^{-(\mu-\mu_0)}\|f\|_{B_{\mu_0}}.\end{aligned}$$

Hence

$$\text{whenever } \mu \in]\mu_0,\infty[, \text{ then } \|\psi_k - f\|_{B_{\mu_0}} \le k^{-1} + \|f\|_{B_{\mu_0}}(1+k)^{-(\mu-\mu_0)}. \tag{5.3}$$

Now define

$$\varphi_k = \partial^\alpha\psi_k. \tag{5.4}$$

We shall verify that $(\varphi_k)_{k\in\mathbb{N}} \subset C_0^\infty$ is a good approximating sequence for U.

We begin by proving that

(5.5) for every $\mu \in]\mu_0, \infty[$ the sequence $([(\varphi_k - U) *]|_{\mathcal{S}})_{k\in\mathbb{N}} \subset L(\mathcal{S}, \tilde{S}_\mu)$ of convolution operators converges to zero pointwise on $\mathcal{S}$.

To this end we take any $\varphi \in \mathcal{S}$ and for every $\beta \in \mathbb{N}_0^n$ we estimate $\rho_{\mu,\beta}((\varphi_k - U)*\varphi)$. Since $(\varphi_k - U)*\varphi = (\psi_k - f)*\partial^\alpha\varphi$, we have $\rho_{\mu,\beta}((\varphi_k - U)*\varphi) = \rho_{\mu,\alpha+\beta}((\psi_k - f)*\varphi)$. Moreover, whenever $\lambda \in]n, \infty]$ and $x \in \mathbb{R}^n$, then

$$
\begin{aligned}
|[\partial^{\alpha+\beta}((\psi_k - f) * \varphi)](x) &= |[(\psi_k - f) * \partial^{\alpha+\beta}\varphi](x)| \\
&\le \|\psi_k - f\|_{B_\mu} \cdot \rho_{-\mu-\lambda,\alpha+\beta}(\varphi) \cdot \int_{\mathbb{R}^n} (1 + |x - y|)^\mu (1 + |y|)^{-\mu-\lambda}\, dy \\
&\le \|\psi_k - f\|_{B_\mu} \cdot \rho_{-\mu-\lambda,\alpha+\beta}(\varphi) \cdot \int_{\mathbb{R}^n} (1 + |y|)^{-\lambda}\, dy \cdot (1 + |x|)^\mu,
\end{aligned}
$$

because $(1 + |x - y|)^\mu (1 + |y|)^{-\mu} (1 + |x|)^{-\mu} \le 1$. Hence, by (5.3),

$$
\begin{aligned}
\rho_{\mu,\beta}((\varphi_k - U) * \varphi) &= \rho_{\mu,\alpha+\beta}((\psi_k - f) * \varphi) \\
&= \sup_{x\in\mathbb{R}^n} (1 + |x|)^{-\mu} |[\partial^{\alpha+\beta}((\psi_k - f) * \varphi)](x)| \\
&\le (k^{-1} + \|f\|_{B_{\mu_0}} (1 + k)^{-(\mu-\mu_0)}) \cdot \rho_{\mu-\lambda,\alpha+\beta}(\varphi) \int_{\mathbb{R}^n} (1 + |y|)^{-\lambda}\, dy
\end{aligned}
$$

for every $\mu \in]\mu_0, \infty[$, $\beta \in \mathbb{N}_0^n$, $\varphi \in \mathcal{S}$ and $k \in \mathbb{N}$, so that indeed (5.5) holds.

Since the space $\mathcal{S}$ is Fréchet and Montel, by the Banach–Steinhaus theorem, (5.5) implies

(5.6) whenever $\mu \in]\mu_0, \infty[$, then $\lim_{k\to\infty}[(\varphi_k - U) *]|_{\mathcal{S}} = 0$ in the topology of $L_b(\mathcal{S}, \tilde{S}_\mu)$.

Since $L_b(\mathcal{S}, \tilde{S}_\mu)$ is continuously imbedded in $L_b(\mathcal{S}, \mathcal{O}_C)$, by Theorem 4.1(i) from (5.6) it follows that $\lim_{k\to\infty}[\varphi_k *]|_{\mathcal{S}} = [U *]|_{\mathcal{S}}$ in the topology of $L_b(\mathcal{S}, \mathcal{O}_C)$. This means that $\lim_{k\to\infty} \varphi_k = U$ in the operator topology of $\mathcal{S}'$. □

Theorem 5.2. *Let $\alpha \in C_c^\infty$ be such that $\alpha(x) = 1$ if $|x| \le 1$, and $\alpha(x) = 0$ if $|x| \ge 2$. Define the sequence $(\alpha_k)_{k\in\mathbb{N}} \subset C_c^\infty$ by $\alpha_k(x) = \alpha(k^{-1}x)$. Let $T \in \mathcal{O}'_C$. Then $\alpha_k T \in \mathcal{O}'_C$ for every $k \in \mathbb{N}$, and $\lim_{k\to\infty} \alpha_k T = T$ in the operator topology of $\mathcal{O}'_C$.*

Proof. Since $\alpha_k \in C_c^\infty$, it follows that $\alpha_k T \in \mathcal{E}' \subset \mathcal{O}'_C$ where the last inclusion is a consequence of (3.1). Thus the assertion is equivalent to

(5.7) whenever B is a bounded subset of $\mathcal{S}(\mathbb{R}^n)$, then $\lim_{k\to\infty}[(\alpha_k - 1)T] * \varphi = 0$ in the topology of $\mathcal{S}$, uniformly with respect to φ ranging over B.

Since $\mathcal{S}$ is a complete metrizable space, and moreover $\mathcal{S}$ is a Montel space, the Banach–Steinhaus theorem implies that (5.7) is equivalent to

$$
\lim_{k\to\infty} [(1 - \alpha_k)T] * \varphi = 0 \quad \text{in } \mathcal{S}, \text{ for every } \varphi \in \mathcal{S}. \tag{5.8}
$$

In terms of seminorms determining the topology of $\mathcal{S}$, (5.8) takes the form

$$
\text{whenever } \mu \in [0, \infty[\text{ and } \beta \in \mathbb{N}_0^n, \text{ then } \lim_{k\to\infty} \rho_{\mu,\beta}([(1 - \alpha_k)T] * \varphi) = 0 \tag{5.9}
$$

where

$$
\rho_{\mu,\beta}([(1 - \alpha_k)T] * \varphi) = \sup_{x\in\mathbb{R}^n} (1 + |x|)^\mu \big| [(1 - \alpha_k)T] * \partial^\beta\varphi \big](x) \big|.
$$

By (3.2) for any fixed $\mu \in [0,\infty[$ and $\lambda \in]n,\infty[$ the distribution $T \in \mathcal{O}'_C$ can be represented as a finite sum

$$T = \sum_{|\gamma|\le G} \partial^\gamma f_\gamma$$

where $\gamma \in \mathbb{N}_0^n$ and $f_\gamma \in B_{-\mu-\lambda}$. One has

$$(5.10)\quad \big[[(1-\alpha_k)T] * \partial^\beta\varphi\big](x) = \sum_{|\gamma|\le G} \int_{\mathbb{R}^n} f_\gamma(y)(-1)^{|\gamma|}\partial^\gamma\big[[1-\alpha_k](y)[\partial^\beta\varphi](x-y)\big]\,dy = \sum_{|\gamma|\le G}\sum_{0\le\delta\le\gamma}\binom{\gamma}{\delta}(-1)^{|\gamma|+|\gamma-\delta|}\int_{\mathbb{R}^n} f_\gamma(y)[\partial^\delta(1-\alpha_k)](y)[\partial^{\beta+\gamma-\delta}\varphi](x-y)\,dy.$$

Since $[1-\alpha_k](y)$ vanishes when $|y|\le k$, the integrals in (5.10) can be restricted to $\{y\in\mathbb{R}^n : |y|\ge k\}$, so that

$$(5.11)\quad \big|\big[[(1-\alpha_k)T] * \partial^\beta\varphi\big](x)\big| \le \sum_{|y|\le G}\sum_{0\le\delta\le\gamma}\binom{\gamma}{\delta}\|f_\gamma\|_{B_{-\mu-\lambda}}\cdot\sup_{y\in\mathbb{R}^n}|[\partial^\delta(1-\alpha_k)](y)|\cdot\rho_{\mu,\beta+\gamma-\delta}(\varphi) \cdot\int_{|\gamma|\ge k}(1+|y|)^{-\mu-\lambda}(1+|x-y|)^{-\mu}\,dy.$$

Since $[1-\alpha_k](y) = [1-\alpha](k^{-1}y)$, one has $\sup_{y\in\mathbb{R}^n}|[\partial^\delta(1-\alpha_k)](y)| = k^{-|\delta|}C_\delta$ where $C_\delta = \sup_{y\in\mathbb{R}^n}|[\partial^\delta(1-\alpha)](y)|$. Hence from (5.11) one obtains the inequality

$$\big|\big[[(1-\alpha_k)T] * \partial^\beta\varphi\big](x)\big| \le K_{\varphi,\mu,\lambda,\beta}\int_{|y|\ge k}(1+|y|)^{-\mu-\lambda}(1+|x-y|)^{-\mu}\,dy$$

where

$$K_{\varphi,\mu,\lambda,\beta} = \sum_{|y|\le G}\sum_{0\le\delta\le\gamma}\binom{\gamma}{\delta}\|f_\gamma\|_{B_{-\mu-\lambda}}C_\delta\rho_{\mu,\beta+\gamma-\delta}(\varphi)$$

is finite and independent of k. It follows that

$$(5.12)\quad \rho_{\mu,\beta}([(1-\alpha_k)T]*\varphi) \le K_{\varphi,\mu,\lambda,\beta}\sup_{x\in\mathbb{R}^n}\int_{|y|\ge k}(1+|y|)^{-\lambda}[(1+|x|)^\mu(1+|y|)^{-\mu}(1+|x-y|)^{-\mu}]\,dy.$$

Since $(1+|x|)^\mu \le (1+|y|+|x-y|)^\mu \le (1+|y|)^\mu(1+|x-y|)^\mu$, in (5.12) the expression in square brackets in the integrand is never greater than one, so that for any fixed $\varphi\in\mathcal{S}$, $\mu\in[0,\infty[$ and $\beta\in\mathbb{N}_0^n$ one has

$$\rho_{\mu,\beta}([(1-\alpha_k)T]*\varphi) \le K_{\varphi,\mu,\lambda,\beta}\int_{|y|\ge k}(1+|y|)^{-\lambda}\,dy,$$

proving (5.9). □

6. Convolutions of slowly increasing distributions

6.1. **The L. Schwartz theorem on representation of distributions by translation invariant operators on infinitely differentiable functions.** Let $\mathcal{D}$ be a locally convex space contained in $C^\infty(\mathbb{R}^n)$. Assume that $\mathcal{D}$ is invariant with respect to translations and reflections. Let $\mathcal{D}'$ be the dual space of $\mathcal{D}$, and let $\Lambda(\mathcal{D}, C^\infty)$ be the space of continuous linear operators from $\mathcal{D}$ into $C^\infty(\mathbb{R}^n)$ commuting with translations.

For every $U \in \mathcal{D}'$ and $\phi \in \mathcal{D}$ one can define the convolution $U * \phi$ to be the $C^\infty(\mathbb{R}^n)$ function given by $[U * \phi](x) = U((\tau_x\phi)^\vee)$ for $x \in \mathbb{R}^n$. This yields an operator $A_U : \mathcal{D} \ni \phi \mapsto U * \phi \in C^\infty(\mathbb{R}^n)$.

When $\mathcal{D} = C_c^\infty(\mathbb{R}^n)$, the following theorem reduces to the famous theorem of L. Schwartz quoted e.g. in [H, Sect. 4.2, Theorem 4.2.1] and [Y, Sect. VI.3, Theorem 2].

Theorem 6.1. *The assignment $U \mapsto A_U$ maps $\mathcal{D}'$ onto $\Lambda(\mathcal{D}, C^\infty)$ in one-to-one manner.*

Proof. To prove surjectivity, take any $A \in \Lambda(\mathcal{D}, C^\infty)$ and let $U \in \mathcal{D}'$ be defined by $U(\phi^\vee) = [A(\phi)](0)$ for $\phi \in \mathcal{D}$. Then $[U * \phi](x) = U((\tau_x\phi)^\vee) = [A(\tau_x\phi)](0) = [A(\phi)](x)$, so that $A = A_U$.

To prove injectivity, suppose $U \in \mathcal{D}'$ and $[U *]|_{\mathcal{D}} = 0$. Then $U(\phi) = [U * \phi^\vee](0) = 0$ for every $\phi \in \mathcal{D}$, so that $U = 0$. □

6.2. **Extensions of slowly increasing distributions.** A locally convex space $\mathbf{D}$ of C^∞-functions on $\mathbb{R}^n$ will be called *admissible* if it satisfies the following two conditions:

(6.1) $\mathcal{S}$ is densely and continuously imbedded in $\mathbf{D}$, and $\mathbf{D}$ is continuously imbedded in C^∞,

(6.2) $\mathbf{D}$ is invariant with respect to translations and reflections.

Let $U \in \mathcal{S}'$ and let $\mathbf{D}$ be an admissible locally convex space of C^∞-functions on $\mathbb{R}^n$. Then there is at most one element $\tilde{U}$ of the space $\mathbf{D}'$ dual to $\mathbf{D}$ such that $\tilde{U}|_{\mathcal{S}} = U$. The extendability of $U \in \mathcal{S}'$ to $\tilde{U} \in \mathbf{D}'$ is equivalent to continuity of U with respect to the topology induced on $\mathcal{S}$ from $\mathbf{D}$.

All the above is generalized in [H1, Sect. 4.2] and formulated there in terms of *normal spaces of distributions.* The only NSD used in the present paper are the admissible locally convex spaces of C^∞-functions on $\mathbb{R}^n$.

Theorem 6.2. *Let $T \in \mathcal{O}_C'$ and let $\tilde{T} \in (\mathcal{O}_C)'$ be the unique extension of T to a continuous linear functional on $\mathcal{O}_C$. Whenever $\varphi \in \mathcal{S}$ and $f \in \mathcal{O}_C$, then*

$$\tilde{T} * (f * \varphi) = f * (T * \varphi). \tag{6.3}$$

*P*roof. In (6.3) there appear convolutions of $f \in \mathcal{O}_C$ with φ and with $T * \varphi$, both belonging to $\mathcal{S}$. These convolutions make sense because if $f \in \mathcal{O}_C$, then f belongs to the Horváth space S_μ for some $\mu \in [0, \infty[$, and whenever $f \in S_\mu$, $\psi \in \mathcal{S}$ and $\lambda \in]n, \infty[$, then

$$\begin{aligned}\rho_{\mu,\alpha}(f*\psi) &= \sup_{x\in\mathbb{R}^n}(1+|x|)^{-\mu}|(f*\partial^\alpha\psi)(x)|\\ &\le \sup_{x\in\mathbb{R}^n}(1+|x|)^{-\mu}\rho_{\mu,0}(f)\rho_{-\mu-\lambda,\alpha}(\psi)\int_{\mathbb{R}^n}(1+|x-y|)^{\mu}(1+|y|)^{-\mu-\lambda}\,dy\\ &\le \rho_{\mu,0}(f)\rho_{-\mu-\lambda,\alpha}(\psi)\int_{\mathbb{R}^n}(1+|y|)^{-\lambda}\,dy<\infty\end{aligned}$$

by the inequality $1+|x-y|\le(1+|x|)(1+|y|)$.

By (3.2) for every $\mu\in[0,\infty[$ and $\lambda\in]n,\infty[$ the distribution $T\in\mathcal{O}'_C$ can be represented as a finite sum $\sum_{|p|\le P}\partial^p f_p$ with each f_p in $B_{-\mu-\lambda}$. The sum $\Sigma := \sum_{|p|\le P}\partial^p f_p$ is a distribution uniquely extendable to a continuous linear functional on the admissible Horváth space S_μ, and $\Sigma*\phi\in S_\mu$ for every $\phi\in S_\mu$: indeed,

$$\begin{aligned}\rho_{\mu,\alpha}(\Sigma*\phi) &= \sup_{x\in\mathbb{R}^n}(1+|x|)^{-\mu}\Big|\Big(\Big(\sum_{|p|\le P}f_p\Big)*\partial^{\alpha+p}\phi\Big)(x)\Big|\\ &\le \sup_{x\in\mathbb{R}^n}(1+|x|)^{-\mu}\sum_{|p|\le P}\|f_p\|_{B_{-\mu-\lambda}}\rho_{\mu,\alpha+p}(\phi)\\ &\qquad\qquad\cdot\int_{\mathbb{R}^n}(1+|y|)^{-\mu-\lambda}(1+|x-y|)^{\mu}\,dy\\ &\le \sum_{|p|\le P}\|f_p\|_{B_{-\mu-\lambda}}\rho_{\mu,\alpha+p}(\phi)\int_{\mathbb{R}^n}(1+|y|)^{-\lambda}\,dy.\end{aligned}$$

Since the spaces $\mathcal{O}_C$ and S_μ are admissible, it follows that $\tilde{T}|_{S_\mu}$ is the unique extension of T to a continuous linear functional on S_μ. Consequently,

$$\tilde{T}|_{S_\mu}=\Sigma.$$

Since $\mathcal{O}_C=\bigcup_{\mu\in[0,\infty[}S_\mu$, it follows that whenever $f\in\mathcal{O}_C$ and $\varphi\in\mathcal{S}$, then $f\in S_\mu$ for some $\mu\in[0,\infty[$, and so

$$\tilde{T}*(f*\varphi)=\Sigma*(f*\varphi)=\sum_{|p|\le P}f_p*(f*\partial^p\varphi)$$

where in the rightmost expression all convolutions are classical convolutions of functions on $\mathbb{R}^n$. Similarly,

$$f*(T*\varphi)=f*(\Sigma*\varphi)=\sum_{|p|\le P}f*(f_p*\partial^p\varphi).$$

In order to complete the proof of Theorem 6.2 it remains to prove that whenever $|p|\le P$, and $\varphi\in\mathcal{S}$ and $f\in S_\mu$ for some $\mu\in[0,\infty[$, then $f*(f_p*\varphi)=f_p*(f*\varphi)$. To this end notice that

$$(f*(f_p*\varphi))(x)=\int_{\mathbb{R}^n}f(x-y)\Big[\int_{\mathbb{R}^n}f_p(z)\varphi(y-z)\,dz\Big]\,dy$$

where the double iterated integral on the right side is absolutely convergent. Indeed,

$$\int_{\mathbb{R}^n} |f(x-y)| \left[\int_{\mathbb{R}^n} |f_p(z)|\, |\varphi(y-z)|\, dz \right] dy \le \rho_{\mu,0}(f) \|f_p\|_{B_{-\mu-\lambda}} \rho_{-\mu-\lambda,0}(\varphi)$$
$$\cdot \int_{\mathbb{R}^n} (1+|x-y|)^{\mu} \left[\int_{\mathbb{R}^n} (1+|z|)^{-\mu-\lambda} (1+|y-z|)^{-\mu-\lambda}\, dz \right] dy$$
$$\le \rho_{\mu,0}(f) \|f_p\|_{B_{-\mu-\lambda}} \rho_{-\mu-\lambda,0}(\varphi) \cdot \Big(\sup_{y,z\in\mathbb{R}^n} (1+|x-y|)^{\mu} (1+|z|)^{-\mu} (1+|y-z|)^{-\mu} \Big)$$
$$\cdot \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} (1+|z|)^{-\lambda} (1+|y-z|)^{-\lambda}\, dz \right] dy$$
$$\le \rho_{\mu,0}(f) \|f_p\|_{B_{-\mu-\lambda,0}}(\varphi) (1+|x|)^{\mu} \int_{\mathbb{R}^n} (1+|y|)^{-\lambda}\, dy \int_{\mathbb{R}^n} (1+|z|)^{-\lambda}\, dz < \infty,$$

because

$$\begin{aligned}(1+|x-y|)^{\mu} &\le (1+|x|+|y|)^{\mu} \le (1+|x|)^{\mu}(1+|y|)^{\mu} \\ &\le (1+|x|)^{\mu}(1+|z|+|y-z|)^{\mu} \le (1+|x|)^{\mu}(1+|z|)^{\mu}(1+|y-z|)^{\mu}.\end{aligned}$$

Hence, by the Fubini–Tonelli theorem (see [El, Sect. V.2, Theorem 2.4, p. 179]),

$$\begin{aligned}(f*(f_p*\varphi))(x) &= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} f(x-y)\varphi(y-z)\, dy \right] f_p(z)\, dz \\ &= \int_{\mathbb{R}^n} [(f*\varphi)(x-z)] f_p(z)\, dz = (f_p*(f*\varphi))(x). \qquad \square\end{aligned}$$

6.3. **Convolution of an ordered pair of distributions**

Proposition 6.3.1. *Let $U, V \in \mathcal{S}'$. Suppose that $\mathbf{D}$ is an admissible locally convex space of C^∞-functions on $\mathbb{R}^n$ such that*

$$A_V \in \Lambda(\mathcal{S}, \mathbf{D}), \tag{6.4}$$

and let $\tilde{U} \in \mathbf{D}'$ be the the extension of U. Then there is a unique distribution $W \in \mathcal{S}'$ such that

$$W(\varphi) = \tilde{U}((V*\varphi^\vee)^\vee) \quad \text{for every } \varphi \in \mathcal{S}. \tag{6.5}$$

The distribution $W \in \mathcal{S}'$ defined by (6.5) will be denoted by $\mathbf{D}$-$(U*V)$ and called the $\mathbf{D}$-*convolution of the ordered pair* (U, V). The equality (6.5) is equivalent to

$$A_W = A_{\tilde{U}} \circ A_V, \tag{6.6}$$

so that the convolution $W = \mathbf{D}$-$(U*V)$ is determined by the superposition $A_{\tilde{U}} \circ A_V$ of the operator $A_{\tilde{U}}$, which belongs to $\Lambda(\mathbf{D}, C^\infty)$, with the operator $A_V \in \Lambda(\mathcal{S}, \mathbf{D})$. Existence of the superposition is guaranteed by the condition (6.4). The formula (6.5) can be written equivalently as $W(\varphi) = \tilde{U}(V^\vee * \varphi)$.

6.4. **The commutative convolution of two distributions.** Commutative convolution of distributions T and U on $\mathbb{R}^n$ is discussed in [S2, Sect. VI.2]. The requirements are that for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ both the expressions

$$T_x \cdot (U_y \cdot \varphi(x+y)) \quad \text{and} \quad U_x \cdot (T_y \cdot \varphi(x+y))$$

have to make sense and be equal to $[\varphi(x+y)(T\otimes U)](\mathbb{1}_{\mathbb{R}^{2n}})$. The last expression is well defined if $\varphi(x+y)(T\otimes U)$ is an integrable distribution on $\mathbb{R}^n$.

In the following we limit ourselves to slowly increasing distributions and we assume that $\varphi \in \mathcal{S}$. Since $U_y \cdot \varphi(x+y) = [U * \varphi^\vee](x)$ and generally the range of $[U *]|_\mathcal{S}$ need not be contained in $\mathcal{S}$, it is visible that the first of our expressions makes sense only if T is suitably extended. A certain method of extending U was presented in Section 6.3. Let $\mathbf{D}$ be an admissible locally convex space of C^∞-functions on $\mathbb{R}^n$ such that $A_U \in \Lambda(\mathcal{S}, \mathbf{D})$ and let $\tilde{T}$ be the extension of T to an element of the space $\mathbf{D}'$ dual to $\mathbf{D}$. Then for every $\varphi \in \mathcal{S}$, instead of $T_x(U_x(\varphi(x+y)))$ we try to use $\tilde{T}_x(U_x(\varphi(x+y))) = \tilde{T}((U * \varphi^\vee)^\vee)$.

Theorem 6.4.1. *Let $T \in \mathcal{O}'_C$ and $U \in \mathcal{S}'$. By* (3.1) *the convolution $\mathcal{S}$-$(U*T)$ of the ordered pair (U, T) exists, and by Theorem* 4.1(i)&(iii) *the convolution $\mathcal{O}_C$-$(T*U)$ of the ordered pair (T, U) exists. We have*

$$\mathcal{S}\text{-}(U * T) = \mathcal{O}_C\text{-}(T * U) \tag{6.7}$$

where both sides belong to $\mathcal{S}'$. The common value of both sides will be denoted by $T \between U$ and called the commutative convolution of the distributions T and U. Moreover, whenever $\varphi \in C_c^\infty(\mathbb{R}^n)$, then

$$\varphi(\hat{x}+\hat{y})(T \otimes U) \text{ is an integrable distribution on } \mathbb{R}^n \tag{6.8}$$

and

$$[T \between U](\varphi) = [\varphi(\hat{x}+\hat{y})(T \otimes U)](\mathbb{1}_{\mathbb{R}^{2n}}). \tag{6.9}$$

Proof. The existence of $V := \mathcal{S}$-$(U*T) \in \mathcal{S}'$ is a consequence of (3.1). The existence of $W := \mathcal{O}_C$-$(T \otimes U) \in \mathcal{S}'$ is a consequence of Theorem 4.1(i)&(iii). The equality $V = W$ can be proved by the method of C. Chevalley [C, p. 112]. Namely, since $\mathcal{S} * \mathcal{S}$ is dense in $\mathcal{S}$, it is sufficient to prove that whenever $\varphi, \psi \in \mathcal{S}$, then

$$V * (\varphi * \psi) = (U * \varphi) * (T * \psi)$$

and

$$W * (\varphi * \psi) = (U * \varphi) * (T * \psi).$$

To this end notice that

$$A_T \in L(\mathcal{S}, \mathcal{S}), \quad A_U \in L(\mathcal{S}, \mathcal{O}_C), \quad A_{\tilde{T}} \in L(\mathcal{O}_C, \mathcal{O}_C),$$
$$A_V = A_U \circ A_T \quad \text{and} \quad A_W = A_{\tilde{T}} \circ A_U.$$

It follows that

$$\begin{aligned} V * (\varphi * \psi) = A_V(\varphi * \psi) &= [A_U \circ A_T](\varphi * \psi) \\ &= A_U(T * (\varphi * \psi)) \\ &= A_U((T * \varphi) * \psi) \quad \text{(by Lemma 4.2)} \\ &= U * ((T * \psi) * \varphi) \\ &= (U * \varphi) * (T * \psi) \quad \text{(by Lemma 4.2).} \end{aligned}$$

Similarly

$$\begin{aligned} W * (\varphi * \psi) = A_W(\varphi * \psi) &= [A_{\tilde{T}} \circ A_U](\varphi * \psi) \\ &= A_{\tilde{T}}(U * (\varphi * \psi)) \\ &= A_{\tilde{T}}((U * \varphi) * \psi) \quad \text{(by Lemma 4.2)} \\ &= \tilde{T} * ((U * \varphi) * \psi) \\ &= (U * \varphi) * (T * \psi) \quad \text{(by Theorem 6.2).} \end{aligned}$$

Now we come to relations (6.8) and (6.9) which involve the tensor product $\otimes : \mathcal{D}'(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^{2n})$. The relation (6.8) was proved by J. Horváth in [H2, Example 6, pp. 186–187]. An equality of type (6.9), with (6.8) as a condition sufficient and necessary for the existence of convolution, was proposed by L. Schwartz [S3] as a general definition of the convolution of distributions. Such a definition was also used in [H2]. In Theorem 6.4.1 the definition of convolution of distributions is different, and (6.9) is an equality proving the equivalence of the two definitions.

In order to prove (6.9) we first have to show that the right side of (6.9) makes sense. By (6.8), whenever $T \in \mathcal{O}_C$, $U \in \mathcal{S}'$ and $\varphi \in C_c^\infty$, then $\varphi(\hat{x}+\hat{y})(T * U)$ is an integrable distribution on $\mathbb{R}^{2n}$, and hence it is equal to a finite sum $\sum_p \partial^{\alpha_p}\mu_p$ of distributional partial derivatives of regular Borel complex measures μ_p with finite variation on $\mathbb{R}^{2n}$ (see [K-R, Sects. 3.2–3.5, pp. 48–50] and [Ru, Chap. 6, Theorem 6.19]). It follows that every bounded measurable function on $\mathbb{R}^n$ is absolutely integrable with respect to each μ_p, and therefore the distribution $\varphi(\hat{x}+\hat{y})(T \otimes U)$, integrable on $\mathbb{R}^{2n}$, is a continuous linear functional on $C_b^\infty(\mathbb{R}^{2n})$. The right side of (6.9) is the value of this functional on the sample function $\mathbb{1}_{\mathbb{R}^{2n}}$ which belongs to $C_b^\infty(\mathbb{R}^{2n})$.

To prove (6.9) we shall show that if $\varphi \in C_c^\infty(\mathbb{R}^n)$, then

$$[\varphi(\hat{x}+\hat{y})(T \otimes U)](\mathbb{1}_{\mathbb{R}^{2n}}) = U((T * \varphi^\vee)^\vee) = [\mathcal{S}\text{-}(U * T)](\varphi). \tag{6.10}$$

To this end we shall rely on Theorem 5.2 and on the definition of the tensor product of distributions in [Y, Sect. I.14, Theorem 2, p. 66][1]. Let $(\alpha_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ be the sequence described in Theorem 5.2, and let $T \in \mathcal{O}'(\mathbb{R}^n)$, $U \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then $\alpha_k T \in \mathcal{O}'(\mathbb{R}^n)$ for every $k \in \mathbb{N}$, so that, by [H2, Example 6, pp. 186–187], $\varphi(\hat{x}+\hat{y})((\alpha_k T) \otimes U)$ is an integrable distribution on $\mathbb{R}^n$, and

$$[\varphi(\hat{x}+\hat{y})((\alpha_k T) \otimes U)](\mathbb{1}_{\mathbb{R}^{2n}}) = [T \otimes U](\alpha_k(\hat{x})\varphi(\hat{x}+\hat{y})).^2$$

Since $\alpha_k(\hat{x})\varphi(\hat{x}+\hat{y}) \in C_c^\infty(\mathbb{R}^{2n})$, by [Y, Sect. I.14, Theorem 2], the expression on the right side of the last equality is equal to $U_y((\alpha_k T)_x(\varphi(x+y))) = U(((\alpha_k T) * \varphi^\vee)^\vee)$. By Theorem 5.2, whenever $\varphi \in C_c^\infty(\mathbb{R}^n)$ is fixed, then

$$\lim_{k\to\infty} ((\alpha_k T) * \varphi^\vee)^\vee = (T * \varphi^\vee)^\vee \quad \text{in the topology of } \mathcal{S},$$

whence

$$\lim_{k\to\infty} U(((\alpha_k T) * \varphi^\vee)^\vee) = U((T * \varphi^\vee)^\vee) \quad \text{in } \mathbb{C},$$

and so

$$\lim_{k\to\infty} [\varphi(\hat{x}+\hat{y})((\alpha_k T) \otimes U)](\mathbb{1}_{\mathbb{R}^{2n}}) = U((T * \varphi^\vee)^\vee). \tag{6.11}$$

---

[1] The same definition is used in [V, Sect. II.7.1]. See also [H1, Sect. 4.8, Proposition 2].

[2] This equality is a consequence of the property of the tensor product od distribution stated explicitly in [Y, Sect. II.7.3, equality (12)], and can be proved by replacing $\mathbb{1}_{\mathbb{R}^{2n}}$ by $\eta(\hat{x}/l, \hat{y}/l)$ and $\alpha_k(\hat{x})\varphi(\hat{x}+\hat{y})$ by $\alpha_k(\hat{x})\varphi(\hat{x}+\hat{y})\eta(\hat{x}/l, \hat{y}/l)$, and passing to the limit as $l \to \infty$. Here $\eta(\hat{x}, \hat{y}) \in C_c^\infty(\mathbb{R}^{2n})$ takes values in $[0,1]$, $\eta(x,y) = 1$ whenever $|x| + |y| \le 1$, $\eta(x,y) = 0$ whenever $|x| + |y| \ge 2$. The sequence $(\eta(\hat{x}/l, \hat{y}/l))_{l\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^{2n})$ converges on $\mathbb{R}^{2n}$ to $\mathbb{1}_{\mathbb{R}^{2n}}$ in the sense of majorized pointwise convergence, and all its partial derivatives converge to zero uniformly on $\mathbb{R}^{2n}$. Similar sequences appear in Vladimirov's definition of convolution of distributions [V, Sect. II.7.4] equivalent to the definition of Schwartz–Horváth. See [O, Sect. 3].

Finally

$$\lim_{k\to\infty}[\varphi(\hat{x}+\hat{y})((\alpha_k T)\otimes U)](\mathbb{1}_{\mathbb{R}^{2n}}) = \lim_{k\to\infty}[\varphi(\hat{x}+\hat{y})(T\otimes U)](\alpha_k(\hat{x})(\mathbb{1}_{\mathbb{R}^{2n}})(\hat{x}+\hat{y})) = [\varphi(\hat{x}+\hat{y})(T\otimes U)](\mathbb{1}_{\mathbb{R}^{2n}})), \tag{6.12}$$

because the distribution $[\varphi(\hat{x}+\hat{y})(T\otimes U)]$ is integrable on $\mathbb{R}^{2n}$ and $\lim_{k\to\infty}\alpha_k(\hat{x})\mathbb{1}_{\mathbb{R}^{2n}}(\hat{x},\hat{y}) = \mathbb{1}_{\mathbb{R}^{2n}}$ in the sense of majorized pointwise convergence in $C_b^\infty(\mathbb{R}^{2n})$. See [K-R, Sect. 3.4, p. 49] for a more exact definition of that convergence, related to the Lebesgue dominated convergence theorem. The equalities (6.11) and (6.12) imply (6.10). □

7. Extension of Fourier transformation from $\mathcal{S}$ onto $\mathcal{S}'$

Let $\mathcal{S}$ be the space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$ equipped with its usual topology of a Fréchet space, and let $\mathcal{S}'$ be its dual space, i.e. the space of slowly increasing distributions on $\mathbb{R}^n$, equipped with the strong dual topology. The Fourier transformation $\mathcal{F}:\mathcal{S}\to\mathcal{S}$ is a topological linear automorphism of $\mathcal{S}$.

By [Y, Sect. VI.2, Corollary 2], $\mathcal{F}'$, the mapping dual to $\mathcal{F}$, is a linear topological automorphism of $\mathcal{S}'$ equipped with the $*$-weak topology, so that, by [B, Sect. IV.4.2, Proposition 6], $\mathcal{F}'$ is also a linear topological automorphism of $\mathcal{S}'$ equipped with the strong dual topology. Moreover, by Theorem 5.1 and the Parseval equality (2.3), $\mathcal{F}':\mathcal{S}'\to\mathcal{S}'$ is an extension of $\mathcal{F}:\mathcal{S}\to\mathcal{S}$ by continuity. In what follows we shall write $\mathcal{F}$ instead of $\mathcal{F}'$.

8. Fourier exchange on $\mathcal{S}'\times\mathcal{S}$

Lemma 8.1. *If $\Psi\in\mathcal{O}_M$, then the mapping $\mathcal{S}'\ni U\mapsto\Psi\cdot U\in\mathcal{S}'$ is continuous in the strong dual topology of $\mathcal{S}'$.*

The above lemma is part of the hypocontinuity theorem for multiplication [S2, Sect. VII.5, Theorem XI]. Only this part is needed below.

Proof of Lemma 8.1. Let $\Psi\in\mathcal{O}_M$, and let $(U_\iota)_{\iota\in J}\subset\mathcal{S}'$ be a net converging to $U\in\mathcal{S}'$ in the strong dual topology of $\mathcal{S}'$. From [K3, Sect. 2.3, Proposition 2] it follows that whenever B is a bounded subset of $\mathcal{S}$, then $\Psi\cdot B$ is also a bounded subset of $\mathcal{S}$. Consequently, $\lim_\iota(\Psi\cdot U_\iota)(\varphi) = U(\Psi\cdot U)(\varphi) = (\Psi\cdot U)(\varphi)$ uniformly with respect to $\varphi\in B$. □

Theorem 8.2. *Whenever $U\in\mathcal{S}'$ and $\varphi\in\mathcal{S}$, then*

$$\mathcal{F}(U*\varphi) = \mathcal{F}(\varphi)\cdot\mathcal{F}(U). \tag{8.1}$$

Proof. If $U\in\mathcal{S}'$, then by Theorem 5.1 there is a sequence $(\varphi_k)_{k\in\mathbb{N}}\subset C_c^\infty$ converging to U in the operator topology of $\mathcal{S}'$. This means that $\lim_k\varphi_k*\varphi = U*\varphi$ in the topology of $\mathcal{O}_C$ uniformly with respect to φ ranging over any bounded subset of $\mathcal{S}$. It follows that whenever $\varphi\in\mathcal{S}$ is fixed, then $\lim_{k\to\infty}\varphi_k*\varphi = U*\varphi$ in the strong dual topology of $\mathcal{S}'$. Furthermore, by (2.2), one has

$$\mathcal{F}(\varphi_k*\varphi) = \mathcal{F}(\varphi)\cdot\mathcal{F}(\varphi_k) \quad \text{for every } \varphi\in\mathcal{S}. \tag{8.2}$$

Since $\mathcal{F}(\varphi)\in\mathcal{S}\subset\mathcal{O}_M$, from Lemma 8.1 it follows that $\lim_{k\to\infty}\mathcal{F}(\varphi)\cdot\mathcal{F}(\varphi_k) = \mathcal{F}(\varphi)\cdot\mathcal{F}(U)$ in the strong dual topology of $\mathcal{S}'$. Moreover the continuity of $\mathcal{F}$:

$\mathcal{S}' \to \mathcal{S}'$ implies that $\lim_{k\to\infty} \mathcal{F}(\varphi_k * \varphi) = \mathcal{F}(U * \varphi)$ in the strong dual topology of $\mathcal{S}'$. Therefore (8.1) follows from (8.2) by passing to the limit in the strong dual topology of $\mathcal{S}'$. □

9. Fourier isomorphism of $\mathcal{O}'_C$ onto $\mathcal{O}_M$

Proposition 9.1. *Let $T \in \mathcal{O}'_C$ and let $\mathcal{F}$ denote the Fourier transformation understood as an automorphism of $\mathcal{S}'$. Then $\mathcal{F}(T) \in \mathcal{S}'$ is a distribution represented by a function belonging to $\mathcal{O}_M$.*

Proof. By Theorem 8.2 the distribution $e^{-|x|^2/2}\mathcal{F}(T) \in \mathcal{S}'$ is equal to the distribution $(2\pi)^{n/2}\mathcal{F}(T * e^{-|x|^2/2})$. Since $\mathcal{F}|_{\mathcal{S}}$ is an automorphism of $\mathcal{S}$ and $[T\,*]|_{\mathcal{S}} \in L(\mathcal{S},\mathcal{S})$, the distribution $(2\pi)^{n/2}\mathcal{F}(T * e^{-|x|^2/2})$ is represented by the function $\psi = (2\pi)^{n/2}(\mathcal{F}|_{\mathcal{S}})(T * e^{-|x|^2/2})$ belonging to $\mathcal{S}$. Consequently, the distribution $\mathcal{F}(T) \in \mathcal{S}'$ is represented by the function $\eta := e^{|x|^2/2}\psi$, which is infinitely differentiable on $\mathbb{R}^n$.

It remains to prove that η is a multiplier of $\mathcal{S}$, i.e. $\varphi{\cdot}\eta \in \mathcal{S}$ whenever $\varphi \in \mathcal{S}$. To this end take any $\varphi \in \mathcal{S}$ and notice that, by Theorem 8.2, $\varphi{\cdot}\mathcal{F}(T) = \mathcal{F}(T*\mathcal{F}^{-1}(\varphi)) \in \mathcal{S}'$. Both sides of this equality are represented by the function $\varphi \cdot \eta \in C^\infty$, and also by the function $(\mathcal{F}|_{\mathcal{S}})(T*\mathcal{F}^{-1}(\varphi))$, which belongs to $\mathcal{S}$. The two infinitely differentiable functions must be equal, because they represent the same distribution. It follows that $\varphi \cdot \eta \in \mathcal{S}$, as claimed. □

Theorem 9.2. *The Fourier transformation $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ maps $\mathcal{O}_C$ in one-to-one manner onto $(\mathcal{O}_M)$, the subset of $\mathcal{S}'$ consisting of the distributions represented by functions belonging to $\mathcal{O}_M$.*

Proof. Denote by $(\mathcal{S})$ the set of distributions which are represented by functions belonging to $\mathcal{S}$. The fact that the Fourier transformation (understood as a linear topological automorphism of $\mathcal{S}'$) maps $\mathcal{O}'_C$ in one-to-one manner onto $(\mathcal{O}_M)$ follows from the inclusions $\mathcal{F}(\mathcal{O}'_C) \subset (\mathcal{O}_M)$ and $\mathcal{F}((\mathcal{O}_M)) \subset \mathcal{O}'_C$. The former inclusion is a consequence of Proposition 9.1. In order to prove the latter, take any distribution $U \in \mathcal{F}((\mathcal{O}_M))$. Then $\mathcal{F}(U) \in (\mathcal{F}\circ\mathcal{F})((\mathcal{O}_M)) = (2\pi)^n(\mathcal{O}_M)^\vee = (\mathcal{O}_M)$, so that $\mathcal{F}(\varphi)\cdot\mathcal{F}(U) \in (\mathcal{S})$ for every $\varphi \in \mathcal{S}$, because $\mathcal{O}_M$ is the space of multipliers of $\mathcal{S}$. Furthermore,

$$\begin{aligned}
&\mathcal{F}(\varphi)\cdot\mathcal{F}(U) \in (\mathcal{S}) \text{ for every } \varphi \in \mathcal{S}\\
&\qquad\qquad \Leftrightarrow \mathcal{F}(U*\varphi) \in (\mathcal{S}) \text{ for every } \varphi \in \mathcal{S} \text{ (by Theorem 8.2)}\\
&\qquad\qquad \Leftrightarrow U*\varphi \in (\mathcal{S}) \text{ for every } \varphi \in \mathcal{S}\\
&\qquad\quad \text{(since } \mathcal{F} \text{ is an automorphism of } \mathcal{S}' \text{ and } \mathcal{F}|_{\mathcal{S}} \text{ is an automorphism of } \mathcal{S}).
\end{aligned}$$

Whenever $\varphi \in \mathcal{S}$, then $U*\varphi \in C^\infty$. The conjunction of $U*\varphi \in (\mathcal{S})$ and $U*\varphi \in C^\infty$ is equivalent to $U*\varphi \in \mathcal{S}$. Thus $[U\,*]$ maps $\mathcal{S}$ into $\mathcal{S}$. By Theorem 4.1(i), $[U\,*]$ is a closed operator from $\mathcal{S}$ into $\mathcal{S}$. Hence, by the closed graph theorem, $[U\,*]|_{\mathcal{S}} \in L(\mathcal{S},\mathcal{S})$, so that $U \in \mathcal{O}'_C$, by (3.1). □

Theorem 9.3. *The Fourier transformation determines a continuous linear isomorphism between the space $\mathcal{O}'_C$ equipped with the operator topology and the space $\mathcal{O}_M$ equipped with the strong topology.*

Proof. The strong topology in $\mathcal{O}_M$ is defined in [S2, Sect. VII.5], and is discused in [K3, Sect. 2.2]. A net of distributions $(T_\iota)_{\iota\in J} \subset \mathcal{O}'_C$ converges to the distribution $T \in \mathcal{O}'_C$ in the operator topology in $\mathcal{O}'_C$ if and only if

(9.1) whenever B is a bounded subset of $\mathcal{S}$, then $\lim_\iota (T_\iota * \varphi) = T * \varphi$ in the topology of $\mathcal{S}$, uniformly with respect to φ ranging over B.

According to [K3, Sect. 2.2], a net of C^∞-functions $(\Phi_\iota)_{\iota\in J} \subset \mathcal{O}_M$ is convergent to a C^∞-function $\Phi \in \mathcal{O}_M$ in the strong topology of $\mathcal{O}_M$ if and only if

(9.2) whenever B is a bounded subset of $\mathcal{S}$, then $\lim_\iota (\Phi_\iota \cdot \varphi) = \Phi \cdot \varphi$ in the topology of $\mathcal{S}$, uniformly with respect to φ ranging over B.

Suppose that (9.1) holds and $\Phi_\iota = \widetilde{\mathcal{F}T_\iota}$, $\Phi = \widetilde{\mathcal{F}T}$ are functions belonging to $\mathcal{O}_M$, representing the distributions $\mathcal{F}T_\iota$, $\mathcal{F}T$. Such functions Φ_ι and Φ exist in view of Theorem 9.2. By (3.1) and Theorem 8.2, for every $\varphi \in \mathcal{S}$ one has

$$(9.3) \qquad [\mathcal{F}|_{\mathcal{S}}](T_\iota * \varphi) = \mathcal{F}(T_\iota * \varphi) = \Phi_\iota \cdot \hat{\varphi}, \quad [\mathcal{F}|_{\mathcal{S}}](T * \varphi) = \mathcal{F}(T * \varphi) = \Phi \cdot \hat{\varphi}.$$

Since $\mathcal{F}|_{\mathcal{S}}$, the restriction of $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ to $\mathcal{S}$, is a continuous linear automorphism of $\mathcal{S}$, so that $\mathcal{F}|_{\mathcal{S}}$ maps bounded subsets of $\mathcal{S}$ onto bounded subsets of $\mathcal{S}$, it follows that $\lim_\iota (T_\iota * \varphi) = T * \varphi$ in the topology of $\mathcal{S}$, uniformly with respect to φ ranging over any bounded subset of $\mathcal{S}$, if and only if the same holds for $\lim_\iota [\mathcal{F}|_{\mathcal{S}}](T_\iota * \varphi) = [\mathcal{F}|_{\mathcal{S}}](T * \varphi)$. Moreover, if B is a subset of $\mathcal{S}$, then $\hat{B} = \{\hat{\varphi} \in \mathcal{S} : \varphi \in B\}$ is bounded if and only if B is. Therefore, by (9.3), (9.1) implies (9.2) for C^∞-functions Φ_ι and Φ defined by the equalities $\Phi_\iota = \widetilde{\mathcal{F}T_\iota}$, $\Phi = \widetilde{\mathcal{F}T}$.

The proof of the implication (9.2)⇒(9.1) for $\Phi_\iota = \widetilde{\mathcal{F}T\iota}$, $\Phi = \widetilde{\mathcal{F}T}$ is analogous. □

Corollary 9.4. *All gaussian functions belong to $\mathcal{O}_M$ and determine distributions belonging to $\mathcal{O}'_C$.*

Proof. A gaussian function in the sense of [H, Sect. 7.6] is a function of the form

$$\phi(x) = \exp\bigl(-\tfrac{1}{2}\langle Ax, x\rangle\bigr), \qquad x \in \mathbb{R}^n,$$

where A is a symmetric complex invertible $n\times n$-matrix such that the quadratic form $\langle(\operatorname{Re} A)x, x\rangle$ is non-negative semi-definite. All gaussian functions belong to $\mathcal{O}_M$: this follows by repeated application of the chain rule for differentiation, and the Leibniz rule. As a result, whenever $\alpha \in \mathbb{N}_0^n$, then $\partial^\alpha \phi = P_\alpha \cdot \phi$ where P_α is a polynomial on $\mathbb{R}^n$ with complex coefficients.

By [H, Sect. 7.6, Theorem 7.6.1] the Fourier transform of the gaussian function $\phi(x) = \exp\bigl(-\frac{1}{2}\langle Ax, x\rangle\bigr)$ is

$$\hat{\phi}(x) = c \exp\bigl(-\tfrac{1}{2}\langle A^{-1}x, x\rangle\bigr),$$

i.e. a gaussian function multiplied by a complex constant $c \neq 0$. Hence the Fourier transformation preserves the class of gaussian functions up to multiplicative constants and, by Theorem 9.3, gaussian functions determine distributions belonging to $\mathcal{O}'_C$. □

10. Fourier exchange theorem of L. Schwartz

Theorem 10.1 ([S2, Sect. VII.8, Theorem XV], [H1, Sect. 4.11, Theorem 3])**.** *If $T \in \mathcal{O}'_C$ and $U \in \mathcal{S}'$, then the commutative convolution $T \mathbin{\text{☼}} U \in \mathcal{S}'$ in the sense of Theorem* 6.4.1 *exists, and*

$$\mathcal{F}(T \mathbin{\text{☼}} U) = \widetilde{\mathcal{F}(T)} \cdot \mathcal{F}(U).$$

Here $\mathcal{F}$ denotes the Fourier transformation understood as a linear topological automorphism of $\mathcal{S}'$, and $\cdot$ stands for multiplication of the distribution $\mathcal{F}(U) \in \mathcal{S}'$ by the function $\widetilde{\mathcal{F}(T)}$ belonging to $\mathcal{O}_M$, representing the distribution $\mathcal{F}(T)$.

Proof. Let $T \in \mathcal{O}'_C$ and $U \in \mathcal{S}'$. Theorem 6.4.1 implies that the commutative convolution $T \mathbin{☼} U \in \mathcal{S}'$ exists. By Theorem 5.1 there exists a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset C_c^\infty$ such that $\lim_{k\to\infty} \varphi_k = U$ in the operator topology of $\mathcal{S}'$, i.e. $\lim_{k\to\infty} A_{\varphi_k} = A_U$ in $L_b(\mathcal{S}, \mathcal{O}_C)$. Since, by (3.1), $A_T \in L(\mathcal{S}, \mathcal{S})$, it follows that $\lim_{k\to\infty}(A_{\varphi_k} \circ A_T) = (\lim_{k\to\infty} A_{\varphi_k}) \circ A_T = A_U \circ A_T = \mathcal{S}\text{-}(U * T)$ in $L_b(\mathcal{S}, \mathcal{O}_C)$. Moreover, by Lemma 4.2, whenever $\varphi \in \mathcal{S}$, then $[A_{\varphi_k} \circ A_T](\varphi) = \varphi_k * (T * \varphi) = (T * \varphi_k) * \varphi$. It follows that $\lim_{k\to\infty} T * \varphi_k = \mathcal{S}\text{-}(U * T) = U \mathbin{☼} T$ in the operator topology of $\mathcal{S}'$, whence a fortiori

$$\lim_{k\to\infty} T * \varphi_k = T \mathbin{☼} U \quad \text{in the strong dual topology of } \mathcal{S}'. \tag{10.1}$$

Since $\mathcal{F}$ is a linear topological automorphism of $\mathcal{S}'$, Theorem 9.2 and Lemma 8.1 imply that $\mathcal{F}(T) \subset (\mathcal{O}_M)$ and

$$\lim_{k\to\infty} \widetilde{\mathcal{F}(T)} \cdot \mathcal{F}(\varphi_k) = \widetilde{\mathcal{F}(T)} \cdot \mathcal{F}(U) \quad \text{in the strong dual topology of } \mathcal{S}'. \tag{10.2}$$

Finally, by Theorem 8.2,

$$\widetilde{\mathcal{F}(T)} \cdot \mathcal{F}(\varphi_k) = \mathcal{F}(T * \varphi_k) \quad \text{for every } k \in \mathbb{N}. \tag{10.3}$$

The equalities (10.1)–(10.3) imply that $\widetilde{\mathcal{F}(T)} \cdot \mathcal{F}(U) = \mathcal{F}(T \mathbin{☼} U)$. □

Remark. The equality (3.1) is involved in several reasonings which finally lead to Theorem 10.1. Conversely, (3.1) can be deduced from Theorem 6.4.1 (a part of Theorem 10.1), Theorem 8.2 (a particular case of Theorem 10.1), Proposition 9.1 and Theorem 10.1. Namely, from the above mentioned four theorems one can infer that

(10.4) if $T \in \mathcal{O}'_C$ and $\varphi \in \mathcal{S}$, then $T * \varphi \in \mathcal{S}$ and the distribution $T \mathbin{☼} (\varphi)$ is represented by the function $T * \varphi$,

where (φ) denotes the distribution represented by φ.

Indeed, let $T \in \mathcal{O}'_C$ and $\varphi \in \mathcal{S}$. By Theorem 8.2 the distribution $\mathcal{F}(T*\varphi)$ is equal to $\mathcal{F}(T)\cdot\mathcal{F}(\varphi)$ where $\mathcal{F}$ is understood as an automorphism of $\mathcal{S}'$. By Proposition 9.1, the distribution $\mathcal{F}(T)$ is represented by a function $\widetilde{\mathcal{F}(T)}$ belonging to $\mathcal{O}_M$. It follows that

$$(\mathcal{F}|_{\mathcal{S}})(T * \varphi) = \widetilde{\mathcal{F}(T)} \cdot (\mathcal{F}|_{\mathcal{S}})(\varphi) \tag{10.5}$$

where both sides are functions. Since $\mathcal{O}_M$ is the space of multipliers of $\mathcal{S}$ and $\mathcal{F}|_{\mathcal{S}}$ is an automorphism of $\mathcal{S}$, it follows that

$$T * \varphi \in \mathcal{S}. \tag{10.6}$$

By Theorem 10.1 one has

$$\mathcal{F}(T \mathbin{☼} (\varphi)) = \widetilde{\mathcal{F}(T)} \cdot \mathcal{F}(\varphi) \tag{10.7}$$

where $\mathcal{F}$ is understood as an automorphism of $\mathcal{S}'$. From (10.7) and (10.5) it follows that the distribution $T \mathbin{☼} (\varphi)$ is represented by the function $T * \varphi$. Together with (10.6), this completes the proof of (10.4).

11. H. Bremermann's approach to convolution

H. Bremermann [Br, Sects. 8.27–29 and 14.10–12] defines the convolution $U * V$ of two distributions U and V by the formula $U * V = \mathcal{F}^{-1}(\mathcal{F}(U) \cdot \mathcal{F}(V))$. He treats the Fourier transformation $\mathcal{F}$ as an isomorphism of the space $\mathcal{D}'(\mathbb{R}^n)$ of all distributions on $\mathbb{R}^n$ onto the space $\mathcal{Z}(\mathbb{C}^n)$ dual to the function algebra $\mathcal{Z}(\mathbb{C}^n)$ of those entire functions on $\mathbb{C}^n$ which are classical Fourier transforms of test functions belonging to $C_c^\infty(\mathbb{R}^n)$. The Fourier transforms of distributions on $\mathbb{R}^n$ with compact support turn out to be multipliers of $\mathcal{Z}(\mathbb{C}^n)$. H. Bremermann applies the above method to the convolution of an arbitrary distribution on $\mathbb{R}^n$ with a distribution on $\mathbb{R}^n$ with compact support.

12. Some consequences of the Fourier exchange theorem

First, the Fourier exchange theorem constitutes a base for L. Schwartz extension [S1] of I. G. Petrovskiĭ's method [P] of spatial Fourier transformation in partial differential equations.

Moreover, by the equality $\mathcal{F}\mathcal{O}'_C = (\mathcal{O}_M)$ which is part of the Fourier exchange theorem and [K3, Sect. 3.1, Theorem 3], *the matricial distribution* $T \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ *is the infinitesimal generator of a one-parameter infinitely differentiable convolution semigroup* $(T_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ *if and only if*

$$\max\big\{\{0\} \cup \operatorname{Re}\sigma(\hat{T}(x))\big\} = O(\log|x|) \quad \textit{as } x \in \mathbb{R}^n \textit{ and } |x| \to \infty. \tag{12.1}$$

For $T \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ the condition (12.1) is equivalent to the condition that there are $K \in [0, \infty[$ and $k \in \mathbb{R}$ such that

$$\|\exp \hat{T}(x)\|_{M_{m\times m}} \leq K(1 + |x|)^k \quad \text{for every } x \in \mathbb{R}^n. \tag{12.1$'$}$$

The proof of the equivalence is not trivial. See [K2, remarks after the main theorem of Sect. 2] and [K3, Sect. 3.1, proof of Proposition 5].

The exact definition of the convolution algebra $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is as follows:

- as a set, $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ consists of $m \times m$ matrices whose entries belong to $\mathcal{O}'_C$, i.e. are rapidly decreasing distributions on $\mathbb{R}^n$,
- whenever $(T_{i,k})_{i,k=1}^m$ and $(U_{k,j})_{k,j=1}^m$ belong to $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$, then

$$(T_{i,k})_{i,k=1}^m \,\text{☼}\, (U_{k,j})_{k,j=1}^m := \Big(\sum_{k=1}^m T_{i,k} \,\text{☼}\, U_{k,j}\Big)_{i,j=1}^m. \tag{12.2}$$

The star on the right side of (12.2) denotes convolution in the sense of Theorem 6.4.1. In (12.1) the function $\mathbb{R}^n \ni x \mapsto \hat{T}(x) \in M_{m\times m}$ is the Fourier image of the matricial distribution $T \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$, and therefore this function belongs to $(\mathcal{O}_M)(\mathbb{R}^n; M_{m\times m})$. Whenever $x \in \mathbb{R}^n$, then $\operatorname{Re}\sigma(\hat{T}(x))$ denotes the set of the real parts of eigenvalues of the matrix $\hat{T}(x)$. Let us mention that the distributions T_t, $t \in [0, \infty]$, constituting the convolution semigroup $(T_t)_{t\geq 0}$ occur in explicit form in [K1, Sect. 2, Remark 4] and in [F, Sect. 7.8, formulas (8.5) and (8.6)].

If $T \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ satisfies (12.1), then the formula

$$S_t(U) = T_t \,\text{☼}\, U, \quad t \in [0, \infty[,\ U \in \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m), \tag{12.3}$$

defines a semigroup $(S_t)_{t\geq 0} \subset L(\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m), \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$ with infinitesimal generator equal to the convolution operator $[T\,\text{☼}]|_{\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)}$. The semigroup property of the family $(S_t)_{t\geq 0}$ of distributions follows from the fact that $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ acts in $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ by the convolution ☼, and the latter is a consequence of associativity on $\mathcal{O}'_C \times \mathcal{O}'_C \times \mathcal{S}'$ of the convolution $\star$ from Theorem 6.4.1. The associativity is also an easy consequence of Theorem 10.1 and the associativity of multiplication on $\mathcal{O}_M \times \mathcal{O}_M \times \mathcal{S}'$.

We now briefly review the results of [K1] and [K2] concerning the locally convex spaces which are universally invariant with respect to one-parameter operator semigroups on $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$ generated by the operators $[T\,\star]|_{\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)}$ for $T \in \mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$ satisfying condition (12.1). These results are related to semigroup-theoretical sense of the *Petrovskiĭ weak condition for forward evolutionarity.*

It follows from [K1, Sect. 2, Theorem 2] and [K2, Sect. 2] that whenever $T \in \mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$ and E is one of the l.c.v.s. $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$, $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$, $\mathcal{D}_{L^2}(\mathbb{R}^n;\mathbb{C}^m)$ or $\tilde{S}_\mu(\mathbb{R}^n;\mathbb{C}^m)$, $\mu \in [0,\infty[$, then (12.1) is equivalent to the following property of T:

$$[T\,\star]|_E \in L(E;E) \text{ and } [T\,\star]|_E \text{ is the generator of}$$
$$\text{a one-parameter operator semigroup of class } (C_0) \text{ on } E.$$

The above mentioned one-parameter operator semigroup generated by $[T\,\star]|_E$ has the form $(S_t|_E)_{t\geq 0}$ where $(S_t)_{t\geq 0} \subset L(\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m),\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m))$ is the semigroup occurring in (12.2).

By [K1, Sect. 2, Theorem 2], if $T \in \mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$ satisfies the condition (12.1), then the l.c.v.s. $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$, $\mathcal{D}_{L^p}(\mathbb{R}^n;\mathbb{C}^m)$, $p \in [1,\infty]$, $\tilde{S}_\mu(\mathbb{R}^n;\mathbb{C}^m)$, $\mu \in [0,\infty[$, $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$, $\mathcal{O}'_C(\mathbb{R}^n;\mathbb{C}^m)$, $\mathcal{D}'_{L^q}(\mathbb{R}^n;\mathbb{C}^m)$, $q \in\,]1,\infty]$, are invariant with respect to every one-parameter operator semigroup $(S_t)_{t\geq 0} \subset L(\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m),\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m))$ generated by the operator $[T\,\star]|_{\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)}$.

Assume that $T = (T_{i,j})_{i,j=1}^m \in \mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$ and consider the system of convolution equations

$$\partial_t u_i(t,x) = \sum_{j=1}^m T_{i,j} \underset{(x)}{\star} u_j(t,x), \qquad i = 1,\dots,m, \tag{12.4}$$

with initial conditions

$$u_i(0,x) = \mathring{u}_i(x), \qquad i = 1,\dots,m, \tag{12.5}$$

where $(u_1(t,\cdot),\dots,u_m(t,\cdot)) \in E$ for every $t \in [0,\infty[$, E being one of $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$, $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$, $\mathcal{D}_{L^2}(\mathbb{R}^n;\mathbb{C}^m)$ or $\tilde{\mathcal{S}}_\mu(\mathbb{R}^n;\mathbb{C}^m)$, $\mu \in [0,\infty[$. *Then the weak Petrovskiĭ forward evolutionarity condition for the system* (12.4)–(12.5) *coincides with* (12.1) *and is a necessary and sufficient condition for the operator* $[T\,\star]|_E$ *to be the generator of an infinitely differentiable one-parameter operator semigroup on* E. Thus the weak Petrovskiĭ forward evolutionarity condition (12.1) turns out to be a necessary and sufficient condition for the existence of a governing semigroup for the forward Cauchy problem (12.4)–(12.5).

If E is one of the other l.c.v.s. listed before, then we can only assert that (12.1) is a sufficient condition for the existence of a governing semigroup. Notice that I. G. Petrovskiĭ analyzed in [P] the case of $E = C_b^\infty(\mathbb{R}^n;\mathbb{C}^m) = \tilde{S}_0(\mathbb{R}^n;\mathbb{C}^m)$ assuming that all $T_{i,j}$ are differential operators with constant coefficients, without using the theory of distributions.

It follows that if E is any of the spaces listed above and the condition (12.1) is not satisfied, then the initial value forward system (12.4)–(12.5) is not governed by a semigroup. The question then arises about phenomena related to the system (12.4)–(12.5) which prevent the existence of a governing semigroup. An answer for $E = \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ can be found in [R, Sect. 3.10, Theorem 4].

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