Topologies in the set of rapidly decreasing distributions

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To the memory of Professor Janusz Mika

Abstract

Two topologies are studied in the set of rapidly decreasing distributions on $\mathbb{R}^n$.

Introduction

We study the topologies $\tilde{b}$ and op in the set of rapidly decreasing distributions on $\mathbb{R}^n$. The topology $\tilde{b}$ is remarkable because a net of rapidly decreasing distributions is $\tilde{b}$-convergent if and only if it is convergent in the sense of the convergence space $\mathcal{O}'_C(\mathbb{R}^n)$ of L. Schwartz. The advantage of the topology op is that the Fourier transformation yields an isomorphism of the space of rapidly decreasing distributions onto $\mathcal{O}_M(\mathbb{R}^n)$.

1 The Fréchet spaces $S^1_\mu(\mathbb{R}^n)$, $\mu \in \mathbb{R}$, and the J. Horváth space $\mathcal{O}_C(\mathbb{R}^n) = \lim\operatorname{ind}_{\mu\to\infty} S^1_\mu$

Let $\mu \in \mathbb{R}$. Then $S^1_\mu(\mathbb{R}^n)$ is the space of infinitely differentiable complex functions ϕ on $\mathbb{R}^n$ such that

$$\pi^1_{\mu,\alpha}(\phi) < \infty \quad \text{for every multiindex } \alpha \in \mathbb{N}_0^n$$

where

$$\pi^1_{\mu,\alpha}(\phi) = \int_{\mathbb{R}^n} (1+|x|^2)^{-\mu/2} |(\partial^\alpha \varphi)(x)|\, dx.$$

Every $S^1_\mu(\mathbb{R}^n)$ is a Fréchet space whose topology is determined by the countable system of seminorms $\{\pi^1_{\mu,\alpha} : \alpha \in \mathbb{N}_0^n\}$. If $\mu, \nu \in \mathbb{R}$ and $\mu < \nu$, then $S^1_\mu(\mathbb{R}^n) \hookrightarrow S^1_\nu(\mathbb{R}^n)$.

Let $\mathcal{O}_C(\mathbb{R}^n) = \liminf\mathrm{ind}_{\mu\to\infty} S^1_\mu(\mathbb{R}^n)$. (The notion of inductive limit is explained in [B2, Sect. II.2.4], [R-R, Sect.V.2] and [Y, Sect. I, Definition 6].) The idea of using lim ind goes back to J. Horváth [H2, Sect. 2.12, Example 9]. Originally Horváth defined $\mathcal{O}_C(\mathbb{R}^n)$ as $\lim\mathrm{ind}_{\mu\to\infty} S_\mu(\mathbb{R}^n)$, where the spaces $S_\mu(\mathbb{R}^n)$ are distinct from but similar to $S^1_\mu(\mathbb{R}^n)$. The fact that replacing $S_\mu(\mathbb{R}^n)$ by $S^1_\mu(\mathbb{R}^n)$ does not affect $\mathcal{O}_C(\mathbb{R}^n)$ is a consequence of [K3, Sects. I–III].

In what follows it will be important that $\mathcal{S}(\mathbb{R}^n)$ is sequentially dense in each $S^1_\mu(\mathbb{R}^n)$ and in $\mathcal{O}_C(\mathbb{R}^n)$. For $S^1_\mu(\mathbb{R}^n)$ this can be proved by routine analytic tools, while the denseness of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{O}_C(\mathbb{R}^n)$ can be proved as follows. If $p \in \mathcal{O}_C(\mathbb{R}^n)$, then $p \in S^1_{\mu_0}(\mathbb{R}^n)$ for some μ_0. By sequential denseness of $\mathcal{S}(\mathbb{R}^n)$ in $S^1_{\mu_0}(\mathbb{R}^n)$, there is a sequence $(p_k)_{k\in\mathbb{N}} \subset C^\infty_C(\mathbb{R}^n)$ converging to p in the topology of $S^1_{\mu_0}(\mathbb{R}^n)$. A fortiori $(p_k)_{k\in\mathbb{N}}$ converges to p in the topology of $\mathcal{O}_C(\mathbb{R}^n)$.

2 The isomorphisms of Horváth

Theorem 2.1 (variant for $S^1_\mu(\mathbb{R}^n)$ of a result stated in [H2, Sect. 2.5, Example 8])**.** *Let $\mu, \lambda \in \mathbb{R}$, $\phi \in S^1_\mu(\mathbb{R}^n)$ and $\Psi \in C^\infty(\mathbb{R}^n)$ be a function with complex values. Then $\Psi \in S^1_\lambda(\mathbb{R}^n)$ if and only if*

$$\Psi(x) = (1 + |x|^2)^{-(\lambda-\mu)/2}\phi(x) \quad \text{for every } x \in \mathbb{R}^n. \tag{2.1}$$

Moreover the equality (2.1) *yields an isomorphism $I_{\mu,\lambda} : S^1_\mu(\mathbb{R}^n) \to S^1_\lambda(\mathbb{R}^n)$ of locally convex spaces.*

If $I'_{\lambda,\mu}$ is the mapping adjoint to $I_{\lambda,\mu}$, then [B2, Sect. IV.4.2, Proposition 6] implies

Corollary 2.2. *$I'_{\lambda,\mu} : (S^1_\lambda(\mathbb{R}^n))'_b \to (S^1_\mu(\mathbb{R}^n))'_b$ is an isomorphism of the strong dual spaces $(S^1_\lambda(\mathbb{R}^n))'_b$ and $(S^1_\mu(\mathbb{R}^n))'_b$.*

An analogous assertion for weak dual spaces is a trivial consequence of Theorem 2.1.

3 Schwartz's convergence space of rapidly decreasing distributions on $\mathbb{R}^n$

It will sometimes be useful to distinguish clearly between a topological space or a convergence space and the set of elements of this space, without any topology. So, we shall denote by $[E]$ the set of all elements of a topological space E or a convergence space E. We say that a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is bounded if the set of all translations of T is bounded in the strong dual topology of $\mathcal{D}'(\mathbb{R}^n)$. Every distribution belonging to the space $(\mathcal{D}_{L^1})'_b$, the strong dual of $\mathcal{D}_{L^1}$ in the sense of the theory of linear topological spaces, is a bounded distribution. Since $\mathcal{D}_{L^1} = S^1_0(\mathbb{R}^n)$, it follows that the space $(S^1_0(\mathbb{R}^n))'_b$ is equal to the set of all bounded distributions. Schwartz's definition of the convergence space $\mathcal{O}'_C(\mathbb{R}^n)$ says that two conditions have to be satisfied:

(a) $[\mathcal{O}'_C(\mathbb{R}^n)] = \bigcap_{\mu\in\mathbb{R}}(1+|\cdot|^2)^{-\mu/2}[(S^1_0(\mathbb{R}^n))'_b]$

where $(S^1_0(\mathbb{R}^n))'_b$ is the strong dual of $S^1_0(\mathbb{R}^n)$ in the sense of the theory of linear topological spaces, and

(b) a net $(\overset{\iota}{T})_{\iota\in J}$ of elements of $[\mathcal{O}'_C(\mathbb{R}^n)]$ converges by definition to $\overset{0}{T} \in [\mathcal{O}'_C(\mathbb{R}^n)]$ if and only if whenever $\nu \in [0,\infty[$, then the net $((1+|\cdot|^2)^{\nu/2}\overset{\iota}{T})_{\iota\in J}$ converges to $(1+|\cdot|^2)^{\nu/2}\overset{0}{T}$ in the topology of $(S^1_0(\mathbb{R}^n))'_b$.

Convergence spaces have some connections with the theory of locally convex spaces. See [J, Sects. 9.9 and 10.9].

It follows from Corollary 2.2 that conditions (a) and (b) can be equivalently written in the form:

(A) $[\mathcal{O}'_C(\mathbb{R}^n)] = \bigcap_{\mu\in\mathbb{R}}[(S^1_\mu(\mathbb{R}^n))'_b]$,

(B) a net $(U_\iota)_{\iota\in J}$ of elements of $[\mathcal{O}'_C(\mathbb{R}^n)]$ converges by definition to zero if and only if, *for every $\mu \in \mathbb{R}$, it converges to zero in the topology of the space $(S^1_\mu(\mathbb{R}^n))'_b$.*

In what follows, without special mention, we shall use the language of the theory of locally convex spaces. The part of condition (B) written in italics means that the net $(U_\iota)_{\iota\in J}$ converges to zero in the so called topology of intersection (see [B1, Sect. I.4] or [Sf, Sect. II.5]) applied to $\bigcap_{\mu\in\mathbb{R}^n}(S^1_\mu(\mathbb{R}^n))'_b$. This topology is defined as the weakest locally convex topology τ in $\bigcap_{\mu\in\mathbb{R}^n}[(S^1_\mu(\mathbb{R}^n))'_b]$ such that for every $\mu \in \mathbb{R}$ the natural projection $pr_\mu : (\bigcap_{\mu\in\mathbb{R}^n}[(S^1_\mu(\mathbb{R}^n))'_b],\tau) \to (S^1_\mu(\mathbb{R}^n))'_b$ is continuous. Thus the topology of $\bigcap_{\mu\in\mathbb{R}^n}(S^1_\mu(\mathbb{R}^n))'_b$ is a projective topology, and condition (B) can be equivalently formulated as

(B)′ a net $(U_\iota)_{\iota\in J}$ of elements $[\mathcal{O}'_C(\mathbb{R}^n)]$ converges to zero in the convergence space $\mathcal{O}'_C(\mathbb{R}^n)$ if and only if it converges to zero in the topology of the intersection $\bigcap_{\mu\in\mathbb{R}^n}(S^1_\mu(\mathbb{R}^n))'_b$.

Now let us pass to the space $(\mathcal{O}_C(\mathbb{R}^n))'_{\tau_b}$ constructed in [K3, Sect. IV]. This space is defined as $([(\mathcal{O}_C(\mathbb{R}^n))'], \tau_b)$ where $[(\mathcal{O}_C(\mathbb{R}^n))']$ is the set of all continuous linear functionals on $\mathcal{O}_C(\mathbb{R}^n)$, and τ_b is the $\mathfrak{S}$-topology in $[(\mathcal{O}_C(\mathbb{R}^n))']$ corresponding to the covering $\bigcup_{\mu\in\mathbb{R}} \mathcal{B}_\mu$ of $\mathcal{O}_C(\mathbb{R}^n)$ in which $\mathcal{B}_\mu$ is the family of all bounded subsets of $S^1_\mu(\mathbb{R}^n)$. From [H2, Sect. 2.12, Proposition 2] or [Sf, Sect. II.6, Theorem 6.1] it follows that $[(\mathcal{O}_C(\mathbb{R}^n))'] = \bigcap_{\mu\in\mathbb{R}}[(S^1_\mu(\mathbb{R}^n))']$. Hence, by (A), $[(\mathcal{O}'_C(\mathbb{R}^n))] = [(\mathcal{O}_C(\mathbb{R}^n))']$. Moreover, the topology τ_b in $[(\mathcal{O}_C(\mathbb{R}^n))']$ is determined by the system of seminorms $\{p_{\mu,\mathcal{B}} : \mu \in \mathbb{R},\, B \in \mathcal{B}_\mu\}$ where $p_{\mu,B}(f) = \sup_{\phi\in B} |\langle f, \phi\rangle|$ for every $f \in [(\mathcal{O}_C(\mathbb{R}^n))']$. Furthermore, for any fixed $\mu \in \mathbb{R}^n$, the system of seminorms $\{p_{\mu,B} : B \in \mathcal{B}_\mu\}$ determines the topology $(S^1_\mu(\mathbb{R}^n))'_b$. Hence τ_b is the weakest locally convex topology in $[(\mathcal{O}_C(\mathbb{R}^n))']$ such that, for every $\mu \in \mathbb{R}$, the natural projection $pr_\mu : [(\mathcal{O}_C(\mathbb{R}^n))'] \to (S^1_\mu(\mathbb{R}^n))'_b$ is continuous. This proves that $(\mathcal{O}_C(\mathbb{R}^n))'_{\tau_b}$ is equal to $\bigcap_{\mu\in\mathbb{R}^n}(S^1_\mu(\mathbb{R}^n))'_b$. Therefore, by (A) and (B)$'$, we have

1° $[\mathcal{O}'_C(\mathbb{R}^n)] = [(\mathcal{O}_C(\mathbb{R}^n))']$,

2° a net $(U_\iota)_{\iota\in J}$ of elements of $[\mathcal{O}'_C(\mathbb{R}^n)]$ converges to zero in the sense of the convergence space $\mathcal{O}'_C(\mathbb{R}^n)$ of Schwartz if and only if this net is τ_b-convergent.

The topology τ_b in $[\mathcal{O}'_C(\mathbb{R}^n)]$ having property 2° is unique because, according to [K-A, Sect. I.2.6, Proposition 3], a subset of $[(\mathcal{O}_C(\mathbb{R}^n))']$ is τ_b-closed if and only if it contains the limit of any τ_b-convergent net of elements of this subset.

4 The subset RD of $[\mathcal{S}'(\mathbb{R}^n)]$ and the topology $\tilde{b}$ in RD

Define

$$RD = \{T \in \mathcal{S}'(\mathbb{R}^n) : T \text{ is continuous in the topology of } \mathcal{O}_C(\mathbb{R}^n)\}.$$

It follows that $RD = [\mathcal{S}'(\mathbb{R}^n)] \cap [(\mathcal{O}_C(\mathbb{R}^n))'] = [\mathcal{S}'(\mathbb{R}^n)] \cap [\mathcal{O}'_C(\mathbb{R}^n)]$. By [H2, Sect. 2.12, Proposition 2] or [Sf, Sect. II.6, Theorem 6.1] we have

$$RD = \{T \in [\mathcal{S}'(\mathbb{R}^n)] : \text{if } \mu \in \mathbb{R}, \text{ then } T \text{ is continuous in the topology of } S^1_\mu(\mathbb{R}^n)\}.$$

Since $\mathcal{S}(\mathbb{R}^n)$ is (sequentially) dense in every $S^1_\mu(\mathbb{R}^n)$, $\mu \in \mathbb{R}$, it follows that

$$RD = \{T \in [\mathcal{S}'(\mathbb{R}^n)] : \text{if } \mu \in \mathbb{R}, \text{ then } T \text{ extends uniquely to a continuous functional } T_\mu \text{ on } S^1_\mu(\mathbb{R}^n)\}.$$

The topology $\tilde{b}$ in the set RD of distributions is defined as the initial topology defined by the inclusion $RD \subset ((\mathcal{O}_C(\mathbb{R}^n))', \tau_b)$.

5 The subset RD of $\mathcal{S}'(\mathbb{R}^n)$

Define

$$\mathbf{RD} := \{T \in \mathcal{S}'(\mathbb{R}^n) : [T\,*]|_{\mathcal{S}(\mathbb{R}^n)} \in L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))\}$$

where $L(\cdot,\cdot)$ stands for the set of continuous linear mappings. The above definition bases on $T * \varphi$, convolution of a distribution with a test function, which is a function belonging to $C^\infty(\mathbb{R}^n)$ whose value at $x \in \mathbb{R}^n$ is $[T * \varphi](x) = T((\varphi^\vee)_{-x})$. Our Proposition 7.4 shows that requiring a distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ to satisfy the condition $[T\,*]|_{\mathcal{S}(\mathbb{R}^n)} \in L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$ is a severe restriction. Theorem 5.2 shows that a similar condition $[T\,*]|_{\mathcal{S}(\mathbb{R}^n)} \in L(\mathcal{S}(\mathbb{R}^n), \mathcal{O}_C(\mathbb{R}^n))$ is not a restriction at all.

Theorem 5.1. *$RD \subset$ **RD**.*

Proof. Recall that

$$\rho_{\mu,\alpha}(\varphi) := \sup_{x\in\mathbb{R}^n} (1+|x|)^\mu |\partial^\alpha \varphi(x)|$$

for every $\mu \in [0,\infty[$, $\alpha \in \mathbb{N}_0^n$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The system of seminorms $\{\rho_{\mu,\alpha} : \mu \in [0,\infty[, \alpha \in \mathbb{N}_0^n\}$ determines the locally convex topology in $\mathcal{S}(\mathbb{R}^n)$ (as also does any subsystem $\{\rho_{\mu,\alpha} : \mu \in M, \alpha \in \mathbb{N}_0^n\}$ where M is an unbounded subset of $[0,\infty[$).

Let $T \in RD$. To prove that $T \in \mathbf{RD}$, fix some $\lambda \in]\mu+n,\infty[$. Since T_λ is a continuous linear functional on $S^1_\lambda(\mathbb{R}^n)$, it follows that $S^1_\lambda(\mathbb{R}^n) \ni \phi \mapsto |T_\lambda(\phi)| \in \mathbb{C}$ is a continuous seminorm on $S^1_\lambda(\mathbb{R}^n)$, so that there are constants $C_\lambda \in]0,\infty[$ and $b_\lambda \in \mathbb{N}$ such that

$$|T_\lambda(\phi)| \le C_\lambda \max_{|\beta|\le b_\lambda} \pi^1_{\lambda,\beta}(\phi) \quad \text{for every } \phi \in S^1_\lambda(\mathbb{R}^n).$$

Since $[\mathcal{S}(\mathbb{R}^n)] \subset [S^1_\lambda(\mathbb{R}^n)]$, it follows that

$$|T(\psi)| = |T_\lambda(\psi)| \le C_\lambda \max_{|\beta|\le b_\lambda} \pi^1_{\lambda,\beta}(\psi) \quad \text{for every } \psi \in \mathcal{S}(\mathbb{R}^n).$$

Consequently, for every $x \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
|\partial^\alpha([T*\varphi](x))| &= |[T*\partial^\alpha](x)| = |T_{(y)}((\partial^\alpha\varphi)(x-y))| \\
&\le C_\lambda \max_{|\beta|\le b_\lambda} [\pi^1_{\lambda,\beta}]_{(y)}((\partial^\alpha\varphi)(x-y)) \\
&= C_\lambda \max_{|\beta|\le b_\lambda} \int_{\mathbb{R}^n} (1+|y|)^{-\lambda} |(\partial^{\alpha+\beta}\varphi)(x-y)|\,dy.
\end{aligned}$$

Since $\rho_{\mu,\alpha+\beta}(\varphi) = \sup_{x\in\mathbb{R}}(1+|x|)^{\mu}|(\partial^{\alpha+\beta}\varphi)(x)|$, it follows that

$$|(\partial^{\alpha+\beta}\varphi)(x-y)| \le \rho_{\mu,\alpha+\beta}(\varphi)\cdot(1+|x-y|)^{-\mu},$$

so

$$\begin{aligned}|\partial^{\alpha}(T*\varphi)(x) &\le C_\lambda \max_{|\beta|\le b_\lambda} \rho_{\mu,\alpha+\beta}(\varphi)\int_{\mathbb{R}^n}(1+|y|)^{-\lambda}(1+|x-y|)^{-\mu}\,dy\\ &= C_\lambda \max_{|\beta|\le b_\lambda} \rho_{\mu,\alpha+\beta}(\varphi)\int_{\mathbb{R}^n}(1+|y|)^{-(\lambda-\mu)}(1+|x-y|)^{-\mu}(1+|y|)^{-\mu}\,dy.\end{aligned}$$

. Since $(1+|x-y|)(1+|y|)\ge 1+|x-y|+|y|\ge 1+|x|$,

$$|\partial^{\alpha}(T*\varphi)(x) \le C_\lambda \max_{|\beta|\le b_\lambda} \rho_{\mu,\alpha+\beta}(\varphi)\cdot\left(\int_{\mathbb{R}^n}(1+|y|)^{-(\lambda-\mu)}\,dy\right)\cdot(1+|x|)^{-\mu}$$

where the integral is finite because $\lambda \in\,]\mu+n,\infty[$. The last inequality implies that

$$\rho_{\mu,\alpha}(T*\varphi) \le C_\lambda \max_{|\beta|\le b_\lambda} \rho_{\mu,\alpha+\beta}(\varphi)\int_{\mathbb{R}^n}(1+|y|)^{-(\lambda-\mu)}\,dy.$$

It follows that $T*\varphi \in \mathcal{S}(\mathbb{R}^n)$ whenever $\varphi\in\mathcal{S}(\mathbb{R}^n)$, and the mapping $\mathcal{S}(\mathbb{R}^n)\ni\varphi\mapsto T*\varphi\in\mathcal{S}(\mathbb{R}^n)$ is continuous. □

Remark. In [K3] it is proved that $RD = \mathbf{RD}$. Moreover, if $T\in\mathcal{S}'(\mathbb{R}^n)$, then the equivalent conditions (a) $T\in RD$ and (b) $T\in\mathbf{RD}$ are equivalent to

(c) for every $\mu\in[0,\infty[$, $\phi\in S^1_\mu(\mathbb{R}^n)$ and $\varphi\in\mathcal{D}(\mathbb{R}^n)$ the function $\mathbb{R}^n\ni z\mapsto T(\phi\cdot\varphi_z)\in\mathbb{C}$ belongs to $\mathcal{S}(\mathbb{R}^n)$.

It follows at once that if $T\in\mathcal{D}'(\mathbb{R}^n)$ is compactly supported then $T\in RD$.

Theorem 5.2. *If $T\in\mathcal{S}'(\mathbb{R}^n)$, then $[T*]|_{\mathcal{S}(\mathbb{R}^n)}\in L(\mathcal{S}(\mathbb{R}^n),\mathcal{O}_C(\mathbb{R}^n))$.*

The above result goes back to Horváth [H2, Sect. 4.11, Proposition 7] who defined $\mathcal{O}_C(\mathbb{R}^n)$ as $\liminf_{\mu\to\infty} S_\mu(\mathbb{R}^n)$, where the spaces $S_\mu(\mathbb{R}^n)$ are distinct from $S^1_\mu(\mathbb{R}^n)$, but similar. It follows from [K2, Sects. I–III] that the same $\mathcal{O}_C(\mathbb{R}^n)$ can be represented as $\liminf_{\mu\to\infty} S^1_\mu(\mathbb{R}^n)$. The proof of Theorem 5.2 then becomes much shorter. See [K2, Theorem 4.1(ii)].

Corollary 5.3. *If $T\in\mathcal{S}'(\mathbb{R}^n)$ and $T*\varphi\in\mathcal{S}(\mathbb{R}^n)$ for every $\varphi\in\mathcal{S}(\mathbb{R}^n)$, then the linear mapping $\mathcal{S}(\mathbb{R}^n)\ni\varphi\mapsto T*\varphi\in\mathcal{S}(\mathbb{R}^n)$ has closed graph.*

Proof. Suppose that $\mathcal{S}(\mathbb{R}^n)$-$\lim_{k\to\infty}\varphi_k = \varphi_0$ and $\mathcal{S}(\mathbb{R}^n)$-$\lim_{k\to\infty}(T*\varphi_k) = \psi_0$. Since $\mathcal{S}(\mathbb{R}^n)\hookrightarrow\mathcal{O}_C(\mathbb{R}^n)$, we have $\mathcal{O}_C(\mathbb{R}^n)$-$\lim_{k\to\infty}\varphi_k=\varphi_0$ and, by Theorem 5.2, $\mathcal{O}_C(\mathbb{R}^n)$-$\lim_{k\to\infty}(T*\varphi_k) = T*\psi_0$. Hence $\varphi_0=\psi_0$, which means that the graph of the mapping $\mathcal{S}(\mathbb{R}^n)\ni\varphi\mapsto T*\varphi\in\mathcal{S}(\mathbb{R}^n)$ is closed. □

Theorem 5.4 (part of [G-L, Theorem 7·2.2]). *If $T \in \mathcal{S}'(\mathbb{R}^n)$ and $T * \varphi \in \mathcal{S}(\mathbb{R}^n)$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then the mapping $\mathcal{S}(\mathbb{R}^n) \ni \varphi \mapsto T * \varphi \in \mathcal{S}(\mathbb{R}^n)$ is continuous.*

Proof. This is a consequence of Corollary 5.3 and the Closed Graph Theorem. The latter can be applied since the space $\mathcal{S}(\mathbb{R}^n)$ is metrizable and complete. □

6 The operator topology in RD

Let $\mathcal{A}$ be the family of all bounded closed (that is, compact) subsets of the Fréchet space $\mathcal{S}(\mathbb{R}^n)$, which is a Montel space. According to [Y, Sect. IV.7] the locally convex topology of bounded convergence in the set $L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$ is determined by the system of seminorms $\{p_{\mu,\alpha,A} : \mu \in [0,\infty[, \alpha \in \mathbb{N}_0^n, A \in \mathcal{A}\}$ where $p_{\mu,\alpha,A}(\mathcal{L}) = \sup_{\varphi\in A} \rho_{\mu,\alpha}(\mathcal{L}(\varphi))$ for every $\mathcal{L} \in L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$. The set $L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$ equipped with the topology of bounded convergence constitutes a locally convex space which is denoted by $L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))_b$.

We introduce in the set **RD** the locally convex topology *op* (operator topology) as the initial topology defined by the mapping $\mathbf{RD} \in T \mapsto [T\,*]|_{\mathcal{S}(\mathbb{R}^n)} \in L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))_b$. This means that the topology *op* in **RD** is determined by the system of seminorms $\{r_{\mu,\alpha,A} : \mu \in [0,\infty[, \alpha \in \mathbb{N}_0^n, A \in \mathcal{A}\}$ where $r_{\mu,\alpha,A}(T) = \sup_{\varphi\in A} \rho_{\mu,\alpha}(T * \varphi)$.

7 Locally convex space $(\mathbf{RD}, op)$ and Fourier transformation

The Fourier transformation in $\mathcal{S}'(\mathbb{R}^n)$

The Fourier transformation $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear topological automorphism of the space $\mathcal{S}(\mathbb{R}^n)$ (see [G, Theorem 5.2.5], [S2, Sect. VII.6, Theorem XII] or [Y, Sect. VI.1]). Its transpose $\mathcal{F}'$ is a linear topological automorphism of $\mathcal{S}'(\mathbb{R}^n)$ equipped with the $*$-weak topology. $\mathcal{F}'$ is also a linear topological automorphism of $(\mathcal{S}(\mathbb{R}^n))'_b$ (see [B2, Sect. IV.4.2, Proposition 6]). Moreover, since $\mathcal{S}(\mathbb{R}^n)$ is (sequentially) dense in $\mathcal{S}'(\mathbb{R}^n)$, from the Parseval equality for $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ it follows that $\mathcal{F}'$ is equal to the extension of $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ onto by $\mathcal{S}'(\mathbb{R}^n)_b$ continuity. For this reason in what follows we shall write $\mathcal{F}$ instead $\mathcal{F}'$. Let us stress that we define $\mathcal{F}$ by the equalities $[\mathcal{F}(\varphi)](\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\varphi(x)\,dx$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\langle \mathcal{F}(U), \varphi\rangle = \langle U, \mathcal{F}(\varphi)\rangle$ for $U \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

$\mathcal{O}_M(\mathbb{R}^n)$ as the algebra of multipliers of $\mathcal{S}(\mathbb{R}^n)$

$\mathcal{O}_M(\mathbb{R}^n)$ is a locally convex space with the topology determined by Schwartz's system of seminorms $\{s_\alpha : \alpha \in \mathbb{N}_0^n, A \in \mathcal{A}\}$ where $s_{\alpha,A}(\phi) = \sup_{\varphi\in A,\, x\in\mathbb{R}^n} |\varphi(x)\partial^\alpha\phi(x)|$ for every $\phi \in \mathcal{O}_M(\mathbb{R}^n)$. See [S2, Sect. VII.5]. An equivalent system of seminorms is $\{s_{\mu,\alpha,A} : \mu \in [0,\infty[, \alpha \in \mathbb{N}_0^n, A \in \mathcal{A}\}$ where $s_{\mu,\alpha,A}(\phi) = \sup_{\varphi\in A} \rho_{\mu,\alpha}(\phi\cdot\varphi)$ for every $\phi \in \mathcal{O}_M(\mathbb{R}^n)$. The proof of equivalence is presented in [K1, Sects. 2.1 and 2.2]. The second system of seminorms corresponds to the initial topology defined by the mapping $[\mathcal{O}_M(\mathbb{R}^n)] \ni \phi \mapsto \phi\cdot \in L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))_b$ where $\phi\cdot$ denotes the operator of multiplication by ϕ. It is almost evident that if $\phi \in \mathcal{O}_M(\mathbb{R}^n)$, then $\phi\cdot \in L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$. Not obvious is the opposite implication: if $\phi \in C^\infty(\mathbb{R}^n)$ and $\phi\cdot \in L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$, then $\phi \in \mathcal{O}_M(\mathbb{R}^n)$; an ingenious short proof can be found in [Kh, Vol. 2, Chap. CA.III].

Theorem 7.1 ([Y, Sect. VI.3, equality (14) of Theorem 6])**.** *If $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\mathcal{F}(T * \varphi) = \mathcal{F}(\varphi)\cdot\mathcal{F}(T).$$

The proof presented by K. Yosida is short and elementary but refined.

Theorem 7.2. *If $T \in \mathbf{RD}$, then $\mathcal{F}(T) = (2\pi)^{n/2} e^{\frac{1}{2}|\cdot|}\mathcal{F}(T * e^{-\frac{1}{2}|\cdot|}) \in C^\infty(\mathbb{R}^n)$ and $\mathcal{F}|_{\mathbf{RD}}$ is a linear one-to-one mapping of $\mathbf{RD}$ onto $[\mathcal{O}_M(\mathbb{R}^n)]$.*

Proof. Since $\mathcal{F}(e^{-\frac{1}{2}|\cdot|^2}) = (2\pi)^{-n/2}e^{-\frac{1}{2}|\cdot|^2}$ (see [R, Sect. 2.2, Example 1] or [S-W, Sect. I.1, Theorem 1.13]), we infer from Theorem 7.1 that

$$\mathcal{F}(T * e^{-\frac{1}{2}|\cdot|^2}) = \mathcal{F}(T)\cdot\mathcal{F}(e^{-\frac{1}{2}|\cdot|^2}) = \mathcal{F}(T)(2\pi)^{-n/2}e^{-\frac{1}{2}|\cdot|^2},$$

so that

$$\mathcal{F}(T) = \phi_T$$

where

$$\phi_T = (2\pi)^{n/2}e^{\frac{1}{2}|\cdot|^2}\mathcal{F}(T * e^{\frac{1}{2}|\cdot|^2}) \in C^\infty(\mathbb{R}^n).$$

In order to prove that if $T \in \mathbf{RD}$ then $\phi_T \in \mathcal{O}_M(\mathbb{R}^n)$, we shall use the implication: if $\phi \in C^\infty(\mathbb{R}^n)$ and $\phi\cdot\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, then $\phi \in \mathcal{O}_M(\mathbb{R}^n)$, that is, all the partial derivatives of ϕ grow slowly at infinity. This implication constitutes a hard part of the characterization of $\mathcal{O}_M(\mathbb{R}^n)$ as the function algebra of multipliers of $\mathcal{S}(\mathbb{R}^n)$. (Let us recall here the proof of V.-K. Khoan, mentioned earlier.) So, in order to prove that $\phi_T \in \mathcal{O}_M(\mathbb{R}^n)$ for every $T \in \mathbf{RD}$ we have only to check that $\phi_T \cdot \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, that is, if $T \in \mathbf{RD}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then $\phi_T \cdot \varphi \in \mathcal{S}(\mathbb{R}^n)$. But $\phi_T \cdot \varphi = \mathcal{F}(T) \cdot \varphi = \mathcal{F}(T) \cdot \mathcal{F}(\mathcal{F}^{-1}\varphi)$ and, by Theorem 7.1, $\mathcal{F}(T) \cdot \mathcal{F}(\mathcal{F}^{-1}\varphi) = \mathcal{F}(T * \mathcal{F}^{-1}\varphi) \in \mathcal{F}(\mathcal{S}(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n)$.

The above shows that $\mathcal{F}$ is a one-to-one mapping of $\mathbf{RD}$ into $\mathcal{O}_M(\mathbb{R}^n)$. It remains to prove that $\mathcal{F}$ maps $\mathbf{RD}$ onto $\mathcal{O}_M(\mathbb{R}^n)$, that is, $\mathcal{F}^{-1}(\phi) \in \mathbf{RD}$ for every $\phi \in \mathcal{O}_M(\mathbb{R}^n)$. But if $\phi \in \mathcal{O}_M(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then, by Theorem 7.1, $\mathcal{F}(\mathcal{F}^{-1}(\phi) * \varphi) = \phi \cdot \mathcal{F}(\varphi) \in \mathcal{S}(\mathbb{R}^n)$, whence $\mathcal{F}^{-1}(\phi) * \varphi \in \mathcal{S}(\mathbb{R}^n)$, so that $\mathcal{F}^{-1}(\phi) \in \mathbf{RD}$, by Theorem 5.4. □

Theorem 7.3. *The linear one-to-one surjection* $\mathcal{F}|_{\mathbf{RD}} : \mathbf{RD} \to [\mathcal{O}_M(\mathbb{R}^n)]$ *yields an isomorphism between the locally convex spaces* $(\mathbf{RD}, op)$ *and* $\mathcal{O}_M(\mathbb{R}^n)$.

Proof. In this proof we let (μ, α, A) range over $[0, \infty[\times \mathbb{N}_0^n \times \mathcal{A}$. Let $a, b \in \mathbb{Z}$. Then

$$\mathcal{F}_{a,b} : L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))_b \ni \mathcal{L} \mapsto \mathcal{F}^a \circ \mathcal{L} \circ \mathcal{F}^b \in L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))_b$$

is a continuous linear invertible mapping of $L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))_b$ onto itself with inverse $\mathcal{F}^{-a} \circ \mathcal{L} \circ \mathcal{F}^{-b} \mapsto \mathcal{L}$. Continuity of $\mathcal{F}_{a,b}$ is clear from continuity in $\mathcal{L}$ of the corresponding seminorms on $L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))_b$ which have the form $\sup_{\varphi \in A} \rho_{\mu,A}([\mathcal{F}^a \circ \mathcal{L} \circ \mathcal{F}^b](\varphi))$. Therefore, by [H2, Sect. 2.11, Proposition 2], the initial topologies in $\mathbf{RD}$ defined by the mappings $\mathbf{RD} \ni T \mapsto \mathcal{F}^a \circ (T\,*) \circ \mathcal{F}^b \in L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))_b$ are all equivalent for $a, b \in \mathbb{Z}$.

We are interested in the case $a = -1$, $b = 1$. Then we define

$$\sigma_{\mu,\alpha,A}(\mathcal{L}) := \sup_{\varphi \in A} \rho_{\mu,\alpha}([\mathcal{F}^{-1} \circ \mathcal{L} \circ \mathcal{F}](\varphi)), \qquad \mathcal{L} \in L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))_b.$$

By Theorem 7.1, for $T \in \mathbf{RD}$ we have

$$r_{\mu,\alpha,A}(T) = \sup_{\varphi \in A} \rho_{\mu,\alpha}(T * \varphi) = \sup_{\varphi \in A} \rho_{\mu,\alpha}([\mathcal{F}^{-1}(\mathcal{F}(T) \cdot \mathcal{F})](\varphi)) = \sigma_{\mu,\alpha,A}(\mathcal{F}(T)).$$

From the equality $r_{\mu,\alpha,A}(T) = \sigma_{\mu,\alpha,A}(\mathcal{F}(T))$, by [B2, Sect. II.5.6, Proposition 9] or [EDM 2, Sect. 424.F], it follows that $\mathcal{F}|_{\mathbf{RD}}$ is an isomorphism of the locally convex space $(\mathbf{RD}, op)$ onto the locally convex space $\mathcal{O}_M(\mathbb{R}^n)$. □

If $T \in \mathbf{RD}$ and $U \in \mathcal{S}'(\mathbb{R}^n)$ then we define the convolution $T \diamond U$ as a distribution belonging to $\mathcal{S}'(\mathbb{R}^n)$ and equal to ${}^t([T^\vee *]|_{\mathcal{S}(\mathbb{R}^n)})$ where the left superscript t stands for the transpose operator. All this means that

$$\langle T \diamond U, \varphi \rangle = \langle U, T^\vee * \varphi \rangle \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{7.1}$$

The last equality resembles [G, Sect. 4.4, Definition 4.4.1] and [H2, Sect. 4.11, Definition 3], and constitutes a provisional definition of convolution in $\mathcal{S}'(\mathbb{R}^n)$, limited to $T \in \mathbf{RD}$ and $U \in \mathcal{S}'(\mathbb{R}^n)$. Since our provisional convolution always leads to $T \diamond U \in \mathcal{S}'(\mathbb{R}^n)$, its disprovisionalization must be an $\mathcal{S}'(\mathbb{R}^n)$-convolution. The author knows only one $\mathcal{S}'(\mathbb{R}^n)$-convolution, namely the $\mathcal{S}'(\mathbb{R}^n)$-convolution of Y. Hirata and H. Ogata [H-O].

Theorem 7.4. *If* $T \in \mathbf{RD}$ *and* $U \in \mathcal{S}'(\mathbb{R}^n)$*, then*

$$\mathcal{F}(T \diamond U) = \mathcal{F}(T) \cdot \mathcal{F}(U).$$

Proof. Notice first that $\mathcal{F}(T \diamond U)$ makes sense, because $T \diamond U \in \mathcal{S}'(\mathbb{R}^n)$. Also $\mathcal{F}(T) \cdot \mathcal{F}(U)$ makes sense because $\mathcal{F}(T) = \phi_T \in \mathcal{O}_M(\mathbb{R}^n)$. To prove the equality of both expressions notice first that for every $T \in \mathbf{RD}$, $U \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, by Theorem 7.1, we have

$$\begin{aligned}\langle T \diamond U, \varphi\rangle = \langle U, T^\vee * \varphi\rangle &= \langle U, \mathcal{F}^{-1}(\mathcal{F}(T^\vee) \cdot \mathcal{F}(\varphi))\rangle \\ &= \langle \mathcal{F}^{-1}(U), \mathcal{F}(T^\vee) \cdot \mathcal{F}(\varphi)\rangle \\ &= \langle \mathcal{F}^{-1}(U) \cdot \mathcal{F}(T)^\vee, \mathcal{F}(\varphi)\rangle,\end{aligned}$$

whence

$$T \diamond U = \mathcal{F}(\mathcal{F}(T)^\vee \cdot \mathcal{F}^{-1}(U)).$$

From the last equality, by the Fourier inversion formula, it follows that

$$\begin{aligned}T \diamond U &= \mathcal{F}\big(\mathcal{F}(T)^\vee \cdot (2\pi)^n \mathcal{F}(U)^\vee\big) = (2\pi)^n \mathcal{F}\big((\mathcal{F}(T)^\vee \cdot \mathcal{F}(U))^\vee\big) \\ &= (2\pi)^n \mathcal{F}^\vee(\mathcal{F}(T) \cdot \mathcal{F}(U)) = \mathcal{F}^{-1}(\mathcal{F}(T) \cdot \mathcal{F}(U)),\end{aligned}$$

whence

$$\mathcal{F}(T \diamond U) = \mathcal{F}(T) \cdot \mathcal{F}(U). \quad \blacksquare$$

Remark. Theorem 7.4 means that our provisional convolution $\diamond$ has the property of Fourier exchange of convolution onto multiplication. The $\mathcal{S}'(\mathbb{R}^n)$ -convolution of Y. Hirata and H. Ogata also has this property, and this permits one to prove that $\diamond$ is the restriction of H-O-convolution to $\mathbf{RD} \times \mathcal{S}'(\mathbb{R}^n)$. Notice that H-O-convolution assigns to any convolvable pair $(T, U) \in \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ a distribution $T * U \in \mathcal{S}'(\mathbb{R}^n)$. Schwartz's convolution [S1] assigns to a convolvable pair $(T, U) \in \mathcal{D}'(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n)$ a distribution $T * U \in \mathcal{D}'(\mathbb{R}^n)$. It is proved in [H3, Example 6] that if $(T, U) \in \mathbf{RD} \times \mathcal{S}'(\mathbb{R}^n)$ then the pair (T, U) is Schwartz convolvable. In [D-V] an example is given of two measures $(T, U) \in \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ with Schwartz convolution $T * U$ in $\mathcal{D}'(\mathbb{R}^n) \setminus \mathcal{S}'(\mathbb{R}^n)$. This example shows that $\mathcal{S}'(\mathbb{R}^n)$-convolution is really needed.

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