Global regular periodic solutions to equations of weakly compressible barotropic fluid motions. Case B

Wojciech M. Zajączkowski

[1] Institute of Mathematics, Polish Academy of Sciences,
Śniadeckich 8, 00-656 Warsaw, Poland
e-mail:wz@impan.gov.pl
Institute of Mathematics and Cryptology, Cybernetics Faculty,
Military University of Technology,
S. Kaliskiego 2, 00-908 Warsaw, Poland

Abstract

We consider barotropic motions described by the compressible Navier-Stokes equations in a box with periodic boundary conditions. We are looking for density ϱ in the form $\varrho = a + \eta$, where a is a constant and $\eta|_{t=0}$ is sufficiently small in H^2-norm. We assume existence of potentials φ and ψ such that $v = \nabla\varphi + \operatorname{rot}\psi + \fint v dx$. Next we assume that $\nabla\varphi|_{t=0}$ is sufficiently small in H^2-norm too. Finally, we assume that the second viscosity coefficient ν is sufficiently large. Then we prove long time existence of solutions such that $v \in L_\infty(0,T;H^2(\Omega)) \cap L_2(0,T;H^3(\Omega))$, $v_{,t} \in L_\infty(0,T;H^1(\Omega)) \cap L_2(0,T;H^2(\Omega))$, where the existence time T is proportional to ν. Next for T sufficiently large we obtain that $v(T)$ is correspondingly small so global existence is proved using the methods appropriate for problems with small data.

Mathematical Subject Classification (2010): 35A01, 35B10, 76N10

Key words and phrases: compressible Navier-Stokes equations, periodic solutions, density close to a constant, gradient part of velocity small, large second viscosity, existence of global regular solutions

1 Introduction

We are looking for existence of global regular periodic solutions to the following problem

$$
\begin{aligned}
&\varrho v_t + \varrho v\cdot\nabla v - \mu\Delta v - \nu\nabla\operatorname{div} v + \nabla p = \varrho f \quad && \text{in } \Omega\times\mathbb{R}_+,\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \Omega\times\mathbb{R}_+,\\
&v|_{t=0} = v_0, \quad \varrho|_{t=0} = \varrho_0 && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity of the fluid, $\varrho = \varrho(x,t) \in \mathbb{R}_+$ is density, $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ is the external force field, $p = p(\varrho)$ and μ, ν are constant positive viscosity coefficients. $\Omega \subset \mathbb{R}^3$ is a box and the periodic boundary conditions are assumed on $\partial\Omega$.

Looking for weakly compressible motions we assume that

$$
\varrho = a + \eta,
\tag{1.2}
$$

where a is a positive constant, $\eta_0 = \eta|_{t=0}$ is sufficiently small and the second viscosity coefficient ν is sufficiently large.

In view of (1.2) we write (1.1) in the form

$$
\begin{aligned}
&(a+\eta)(v_t + v\cdot\nabla v) - \mu\Delta v - \nu\nabla\operatorname{div} v + a_0\nabla\eta\\
&= (p_\varrho(a)) - p_\varrho(a+\eta)\nabla\eta + (a+\eta)f,\\
&v|_{t=0} = v_0,
\end{aligned}
\tag{1.3}
$$

where $p_\varrho = \frac{dp}{d\varrho}$, $a_0 = p_\varrho(a)$ and

$$
\begin{aligned}
&\eta_t + v\cdot\nabla\eta + a\operatorname{div} v + \eta\operatorname{div} v = 0,\\
&\eta|_{t=0} = \eta_0.
\end{aligned}
\tag{1.4}
$$

Sometimes it is convenient to consider (1.3) in the form

$$
\begin{aligned}
&v_t + v\cdot\nabla v - \frac{\mu}{a}\Delta v - \frac{\nu}{a}\nabla\operatorname{div} v + \frac{a_0}{a}\nabla\eta\\
&= -\frac{\mu}{a}\frac{\eta}{a+\eta}\Delta v - \frac{\nu}{a}\frac{\eta}{a+\eta}\nabla\operatorname{div} v + \frac{a_0}{a}\frac{\eta}{a+\eta}\nabla\eta\\
&\quad + \frac{1}{a+\eta}(p_\varrho(a) - p_\varrho(a+\eta))\nabla\eta + f,\\
&v|_{t=0} = v_0.
\end{aligned}
\tag{1.5}
$$

Since we are looking for solutions to (1.1) with large ν it is natural to introduce periodic potentials φ and ψ such that

$$
v = \nabla\varphi + \operatorname{rot}\psi + G,
\tag{1.6}
$$

where $G=\frac{1}{|\Omega|}\int_\Omega vdx$.
From $(1.1)_{1,2}$ and assumption (1.2) we obtain

$$\frac{d}{dt}\int\limits_\Omega (a+\eta)vdx=\int\limits_\Omega (a+\eta)fdx.$$

Hence

$$\int\limits_\Omega vdx=\frac{1}{a}\bigg[-\int\limits_\Omega \eta vdx+\int\limits_{\Omega^t}(a+\eta)fdxdt'+\int\limits_\Omega (a+\eta_0)v_0dx\bigg].$$

Therefore, —

$$G=\frac{1}{a|\Omega|}\bigg[-\int\limits_\Omega \eta vdx+\int\limits_{\Omega^t}(a+\eta)fdxdt'+\int\limits_\Omega (a+\eta_0)v_0dx\bigg].$$

In this case equations (1.3), (1.4), (1.5) take the form

$$\begin{aligned}&(a+\eta)(\nabla\varphi_t+\operatorname{rot}\psi_t+G_t+(\nabla\varphi+\operatorname{rot}\psi+G)\cdot\nabla(\nabla\varphi+\operatorname{rot}\psi))\\ &\quad-\mu\Delta(\nabla\varphi+\operatorname{rot}\psi)-\nu\nabla\Delta\varphi+a_0\nabla\eta\\ &=(p_\varrho(a)-p_\varrho(a+\eta))\nabla\eta+(a+\eta)f,\\ &\nabla\varphi|_{t=0}=\nabla\varphi_0,\quad \operatorname{rot}\psi|_{t=0}=\operatorname{rot}\psi_0.\end{aligned}\tag{1.7}$$

Moreover, we assume that

$$f=f_r+f_g,\tag{1.8}$$

where f_r is divergence free and f_g is the gradient part. Then $\operatorname{div} f=\operatorname{div} f_g$, $\operatorname{rot} f=\operatorname{rot} f_r$. Next we have

$$\begin{aligned}&\eta_t+v\cdot\nabla\eta+a\Delta\varphi+\eta\Delta\varphi=0\\ &\eta|_{t=0}=\eta_0.\end{aligned}\tag{1.9}$$

Finally, (1.5) takes the form
(1.10)

$$\begin{aligned}&\nabla\varphi_t+\operatorname{rot}\psi_t+G_t+(\nabla\varphi+\operatorname{rot}\psi+G)\cdot\nabla(\nabla\varphi+\operatorname{rot}\psi)\\ &\quad-\frac{\mu}{a}\Delta(\nabla\varphi+\operatorname{rot}\psi)-\frac{\nu}{a}\nabla\Delta\varphi+\frac{a_0}{a}\nabla\eta=-\frac{\mu}{a}\frac{\eta}{a+\eta}\Delta v\\ &\quad-\nu\frac{\eta}{a+\eta}\nabla\operatorname{div} v+a_0\frac{\eta}{a+\eta}\nabla\eta+\frac{1}{a+\eta}(p_\varrho(a)-p_\varrho(a+\eta))\nabla\eta+f_r+f_g,\\ &v|_{t=0}=v_0.\end{aligned}$$

In this paper the following barotropic motions are considered

$$p = A\varrho^{\varkappa}, \quad \varkappa > 1, \quad A - \text{const.} \tag{1.11}$$

We need an equation for $\nabla\eta$. To derive it we multiply (1.7) by $\frac{a}{\nu}$, apply operator ∇ to (1.9) and sum up the results. Then we have

$$\begin{aligned} &\nabla\eta_t + \frac{a_0}{\nu}\nabla\eta = -\nabla(v\cdot\nabla\eta) - \nabla(\eta\Delta\varphi) - \frac{a}{\nu}(\varrho v_t + \varrho v\cdot\nabla v) \\ &\quad + \frac{\mu a}{\nu}\Delta v + \frac{a}{\nu}(p_\varrho(a) - p_\varrho(a+\eta))\nabla\eta + \frac{a}{\nu}\varrho f, \\ &\eta|_{t=0} = \eta_0. \end{aligned} \tag{1.12}$$

The aim of this paper is to prove existence of global regular periodic solutions to problem (1.7)–(1.9). To show existence of such solutions we assume that the initial density is close to a constant assuming that the norms $\|\eta(0)\|_{H^2(\Omega)}$, $\|\eta_t(0)\|_{H^1(\Omega)}$ are sufficiently small.
Moreover, we assume that the second viscosity coefficient ν is sufficiently large. This implies that velocity v must be considered in form (1.6) because divergence free and potential parts have to be treated differently. Therefore we are looking for such solutions that $\nabla\varphi$ in some norms is small but $\operatorname{rot}\psi$ in these norms not.
The natural way to derive necessary estimates is the energy method. The method was first applied to equations of viscous compressible heat-conducting fluids by Matsumura and Nishida in [MN1, MN2, MN3]. Next by Valli and Zajączkowski in [V, VZ]. The free boundary barotropic case was considered in [Z1, Z2]. Finally, a free boundary viscous compressible heat-conducting case was considered by Zadrzyńska [Za]. The method is natural in the problem because v is considered in form (1.6) and the second viscosity ν is very large comparing to μ so the anisotropic approach to velocity is necessary.
Since $\|\eta(0)\|_{H^2(\Omega)}$ and $\|\eta_t(0)\|_{H^1(\Omega)}$ are small we denote the motion a weakly compressible. However, we consider the periodic problem the proof can be extended to motions with different boundary conditions.
In Section 2 some preliminary results are formulated and proved. In Section 3 the main estimates for large ν and time are shown. The estimates are of an a priori type. However, for the local solutions they are real estimates. The estimates are made without smallness assumptions on norms of $\operatorname{rot}\psi$. This is possible thanks to the following two estimates: $\|v\|_{L_{6,\infty}(\Omega^t)}$ (see Lemma 2.2, Remark 2.3 and Remark 2.4) and $\|v_t\|_{L_{2,\infty}(\Omega^t)}$ (see Lemma 2.9). These estimates are crucial for proofs of Lemma 4.1, Corollary 4.2, Theorem 4.3 and Theorem 5.7.

In Section 4 we prove existence of long time solutions, where the existence time T is proportional to ν. The existence is proved in the following way. Having long time esimate proved in Corollary 4.2 we have existence by time extension of local solutions.
The proof of Theorem 4.3 is based on the time extension of a local solution and the derived long time estimate. The proof is understandable but not explicit. To have an explicit proof we have to use the method of successive approximations in the time interval $[0,T]$. But looking for considerations in Section 3 such proof will be very complicated.
However, the long time solutions are not global. Therefore in Section 5 we prove existence of global regular solutions. For this we need some decay estimates (see (5.51), (5.52))which imply smallness of data at time T (see (5.53)). Then considering problem (1.7)–(1.9) with small initial data at time T (see (5.53)) we use the technique from [BSZ, VZ, Z1, Z2] to prove existence of global regular solutions (see Theorem 5.7).
Now, we formulate the main results of this paper

Theorem A (local existence). *Let $\nu > 0$ be given sufficiently large. Let $v = \nabla\varphi + \operatorname{rot}\psi$, $\varrho = a + \eta$, a-positive constant. Let $\eta(0) \in L_\infty(\Omega)$, $\eta(0), \nabla\varphi(0), \operatorname{rot}\psi(0) \in \Gamma_1^2(\Omega)$, $f \in L_2(0,T;\Gamma_1^1(\Omega))$, $\|\nabla\varphi(0)\|_{\Gamma_1^2(\Omega)} \le \frac{c}{\sqrt{\nu}}$, $\|\operatorname{rot}\phi(0)\|_{\Gamma_1^2(\Omega)} \le c$, $\|\eta(0)\|_{\Gamma_1^2(\Omega)} \le \frac{c}{\nu}$, $|f_g|_{6/5,2,\Omega^t} \le \frac{c}{\nu}$, $f \in L_2(0,T;\Gamma_1^1(\Omega))$, $f \in L_6(0,T;L_3(\Omega)) \cap L_1(0,T;L_\infty(\Omega))$. Assume that there exist positive constants φ_* and c such that $c\nu^\varkappa \le \varphi_* \le \varphi(0)$, where $\varkappa \in (1/2,1)$. Then there exists a regular long time solution to problem (1.1) expressed in the form (1.6)–(1.9) such that $\sqrt{\nu}\nabla\varphi, \operatorname{rot}\psi \in L_\infty(0,T;\Gamma_1^2(\Omega)) \cap L_2(0,T;\Gamma_1^3(\Omega))$, $\nu\nabla\varphi \in L_2(0,T;\Gamma_1^3(\Omega))$, $\nu\eta \in L_\infty(0,T;\Gamma_1^2(\Omega))$, and $v \in \mathfrak{N}(\Omega^t)$, $t \le T \le \nu$, where T is the time of local existence and*

$$
\begin{aligned}
&\|v\|_{\mathfrak{N}(\Omega^t)} \le \phi(\nu\|\eta(0)\|_{\Gamma_1^2(\Omega)}, \quad \nu^{1/2}\|\nabla\varphi(0)\|_{\Gamma_1^2(\Omega)},\\
&\|\operatorname{rot}\psi(0)\|_{\Gamma_1^2(\Omega)}, \quad \nu\|\varphi(0)\|_1, \quad \nu|f_g|_{6/5,2,\Omega^t}, \quad \|f\|_{L_2(0,t;\Gamma_1^1(\Omega))},\\
&\|f\|_{L_6(0,t;L_3(\Omega))\cap L_1(0,t;L_\infty(\Omega))}),
\end{aligned}
\tag{1.13}
$$

where $t \le T$, ϕ is an increasing positive function of its arguments and the space $\mathfrak{N}(\Omega^t)$ is defined by

$$
\begin{aligned}
\|v\|_{\mathfrak{N}(\Omega^t)} &= \nu^{1/2}\|\nabla\varphi\|_{L_\infty(0,t;\Gamma_1^2(\Omega))} + \|\operatorname{rot}\psi\|_{L_\infty(0,t;\Gamma_1^2(\Omega))}\\
&\quad + \|\operatorname{rot}\psi\|_{L_2(0,t;\Gamma_1^3(\Omega))} + \nu\|\nabla\varphi\|_{L_2(0,t;\Gamma_1^3(\Omega))}.
\end{aligned}
$$

For $T > \nu$ we have a global existence of such solutions that.
To prove global existence of solutions to problem (1.1) we use the step by step in time extension.

The existence for $t \le T \le \nu$ is shown in Section 4 but for $t \ge T > \nu$ in Section 5.
Therefore, we have

Theorem B (Global existence). *Let the assumptions of the Theorem A hold. Let $\|f_g(t)\|_1 \le ce^{-\alpha t}$, $\alpha > 0$. Let $f \in L_2(kT,(k+1)T;\Gamma_1^2(\Omega)) \cap L_6(kT,(k+1)T;L_3(\Omega)) \cap L_1(kT,(k+1)T;L_\infty(\Omega))$.*
Then $v \in \mathfrak{N}(\Omega\times(kT,(k+1)T))$, $k \in \mathbb{N}_0$ and (1.13) holds with interval $(0,T)$ replaced by $(kT,(k+1)T)$, $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

Now we describe the idea of proofs of Theorems A and B.

Our aim is to derive a global estimate for regular solutions to (1.3), (1.4) using presentation (1.9), (1.10). By the regular velocity we mean such velocity that $v \in L_2(0,T;\Gamma_1^3(\Omega))$. Then we have a corresponding regularity for $\eta \in L_\infty(0,T;\Gamma_1^2(\Omega))$. This kind of regularity is necessary to estimate nonlinear terms.

First we derive the inequality for $v \in L_\infty(0,T;L_6(\Omega))$. Multiplying $(1.3)_1$ by $v|v|^{r-2}$, integrating over Ω^t we get (see Lemma 2.2)

$$
\begin{aligned}
(1.14)\qquad & |v(t)|_r + \bigg(\int_0^t |\nabla|v|^{r/2}|^2dxdt'\bigg)^{1/r} \le c|\eta|_{\frac{3r}{r+1},r,\Omega^t} + c\nu|\Delta\varphi|_{\frac{3r}{r+1},r,\Omega^t}\\
&\quad + c|f|_{\frac{3r}{2r+1},r,\Omega^t} + c|\varrho_0|_\infty^{1/r}|v_0|_r,
\end{aligned}
$$

where $r \le 6$.
The second term on the r.h.s. of (1.14) is not controlled for large ν. Hence to control it we use the interpolation in the case $r=6$

$$
(1.15)\qquad |\Delta\varphi|_{18/7,6,\Omega^t} \le c|\nabla\Delta\varphi|_{2,\infty,\Omega^t}^{2/3}|\nabla\varphi|_{2,\Omega^t}^{1/3},
$$

where the mean values of $\Delta\varphi$, $\nabla\Delta\varphi$, $\nabla\varphi$ are zero in view of periodic boundary conditions. To estimate the last factor we derive the equation obtained from $(1.3)_1$ by applying the div operator

$$
\begin{aligned}
(1.16)\qquad & a\Delta\varphi_t - (\mu+\nu)\Delta^2\varphi + a_0\Delta\eta = -a\mathrm{div}\,(v\cdot\nabla v) + \mathrm{div}\,[-\eta v_t\\
&\quad - \eta v\cdot\nabla v + (p_\varrho(a) - p_\varrho(a+\eta))\nabla\eta + (a+\eta)f].
\end{aligned}
$$

Applying operator Δ^{-1} to (1.16) yields
(1.17)

$$
\begin{aligned}
& a\varphi_t - (\mu+\nu)\Delta\varphi = -a\Delta^{-1}\partial_{x_i}\partial_{x_j}(v_iv_j) + a\Delta^{-1}\partial_{x_i}(\Delta\varphi v_i)\\
&\quad + \Delta^{-1}\mathrm{div}\,[-\eta v_t - \eta v\cdot\nabla v + (p_\varrho(a) - p_\varrho(a+\eta))\nabla\eta + af_g\\
&\quad + \eta f] - (\eta - \fint_\Omega \eta dx) + a\fint_\Omega \varphi_t dx \equiv D_1 + D_2 + F - \bar\eta + a\fint_\Omega \varphi_t dx,
\end{aligned}
$$

where $\fint_\Omega = \frac{1}{|\Omega|}\int_\Omega$, $\bar\eta = \eta - \fint_\Omega \eta dx$.

The aim of this paper is to show that the quantity $\Psi = \nu|\nabla\varphi|_{3,1,2,\Omega^t}$ is bounded for any $t\in\mathbb{R}_+$. Then we see that $|\nabla\Delta\varphi|_{2,\infty,\Omega^t}\le c\|\Delta\varphi\|_{W_2^{2,1}(\Omega^t)}\le c|\nabla\varphi|_{3,1,2,\Omega^t}$. Hence we have to derive that $\nu^{1/3}|\nabla\varphi|_{2,\Omega^t}^{1/3}\le\frac{\Phi^\alpha}{\nu^\beta}$, α, β positive constant numbers. This is the aim of Lemmas 2.3, 2.4 and Remark 2.2.
First we show the idea of the proof of Lemma 2.3. Multiplying (1.17) by φ and integrating over Ω yields

$$\begin{aligned}&\frac{a}{2}\frac{d}{dt}|\varphi|_2^2+(\mu+\nu)|\nabla\varphi|_2^2=\int_\Omega D_1\varphi dx+\int_\Omega D_2\varphi dx+\int_\Omega F\varphi dx\\&-\int_\Omega\bar\eta\varphi dx+a\fint_\Omega\varphi_t dx\int_\Omega\varphi dx.\end{aligned}\tag{1.18}$$

We need to obtain such estimate for $|\nabla\varphi|_{2,\Omega^t}$ that

$$|\nabla\varphi|_{2,\Omega^t}\le\frac{c}{\nu^\alpha},\quad\alpha>1,\tag{1.19}$$

where c depends on such norms of v, $\nabla\varphi$, η that $\nabla\varphi, v\in L_2(0,T;\Gamma_1^3(\Omega))\cap L_\infty(0,T;\Gamma_1^2(\Omega))$, $\eta\in L_\infty(0,T;\Gamma_1^2(\Omega))$.
The dependence can be strongly nonlinear because small parameter $1/\nu^\alpha$ (ν is assumed to be large) helps to get an estimate by a perturbation argument. But the first and the last terms from the r.h.s. od (1.18) will not imply estimate (1.19). They must be treated in the following different way

$$\begin{aligned}\bigg|\int_\Omega D_1\varphi dx\bigg|&\le\int_\Omega\frac{|D_1|\varphi^2}{\varphi_*}dx\le\frac{|D_1|_{p/(p-2)}}{\varphi_*}|\varphi|_p^2\\&\le\frac{1}{2}(\mu+\nu)|\nabla\varphi|_2^2+\frac{c|D_1|_{p/(p-2)}^{1/(1-\varkappa)}}{[(\mu+\nu)^\varkappa\varphi_*]^{1/(1-\varkappa)}}|\varphi|_2^2,\quad\varkappa\in(1/2,1)\end{aligned}$$

and

$$a\fint_\Omega\varphi_t dx\int_\Omega\varphi dx=0,$$

because we assume that $\varphi=\varphi'+L$, $\int\varphi' dx=0$ and L is some arbitrary constant, where $0<\varphi_*=\min_\Omega\varphi$. φ can be chosen positive because it is determined up to an arbitrary positive constant L.
Using that $f=\operatorname{div}F'$ we get

$$\bigg|\int_\Omega F\varphi dx\bigg|=\bigg|\int_\Omega F'\nabla\varphi dx\bigg|\le\varepsilon|\nabla\varphi|_2^2+c/\varepsilon|F'|_2^2.$$

Since $|D_1| \le c|v|^2$ we derive (2.31) in the form

$$
\begin{aligned}
&a|\varphi(t)|_2^2 + (\mu+\nu)|\nabla\varphi|_{2,\Omega^t}^2 \le \exp\left(\frac{c|v|_{2p/(p-2),2/(1-\varkappa),\Omega^t}^{2/(1-\varkappa)}}{[(\mu+\nu)^\varkappa\varphi_*]^{1/(1-\varkappa)}}\right)\\
&\cdot\bigg[\frac{c}{\mu+\nu}\{|\Delta\varphi|_{3/2,\infty,\Omega^t}^2 A_1^2 + |\eta|_{6,\infty,\Omega^t}^2|v_t|_{2,\Omega^t}^2\\
&+|\eta|_{6,\infty,\Omega^t}^2|v|_{6,\infty,\Omega^t}^2A_1^2 + |\eta|_{3,\infty,\Omega^t}^2|\nabla\eta|_{2,\Omega^t}^2 + |\eta|_{3,\infty,\Omega^t}^2|f|_{2,\Omega^t}^2\\
&+|f_g|_{2,\Omega^t}^2 + |\eta|_{2,\Omega^t}^2\} + a|\varphi(0)|_2^2\bigg].
\end{aligned}
\tag{1.20}
$$

Lemma 2.4 implies the bound for φ.
Let $\hat{k} = \frac{c_1}{\nu^\varkappa}$, $\eta \in L_\infty(0,T;H^2(\Omega))$, $v \in L_\infty(\Omega^T)$, $f \in L_3(\Omega^T)$. Then

$$
\begin{aligned}
|\varphi|_{\infty,\Omega^t} \le 2\hat{k}\bigg[&1 + ct^{(1+\varkappa)/r_0}\mathrm{meas}^{(1+\varkappa)/p_0}\Omega\left(\frac{1}{\mu+\nu}\right)^{(2/q)(1+1/\varkappa)}\cdot\\
&\cdot\phi(\|\eta\|_{2,\infty,\Omega^t}, |v|_{\infty,\Omega^t}, |f|_{3,\Omega^t})\bigg] \equiv \gamma_*,
\end{aligned}
\tag{1.21}
$$

where $t \le T$, $\varkappa \in (0,4/3)$, $3/p_0 + 2/r_0 = 3/2$ and ϕ is an increasing positive function. Next Remark 2.2 gives

$$
\frac{c_*}{\nu^\varkappa} = \varphi_* \le \varphi,
\tag{1.22}
$$

where φ is defined up to an arbitrary constant L but γ_* not because γ_* depends on $\hat{k} = \max\{|\bar\varphi(0)|_\infty, 1\}$ which is independent of L.

Finally we have to estimate the second term on the r.h.s. of (1.14). In view of (1.15) we have

$$
\nu|\Delta\varphi|_{18/7,6,\Omega^t} \le c\nu^{2/3}|\nabla\Delta\varphi|_{2,\infty,\Omega^t}^{2/3}\nu^{1/3}|\nabla\varphi|_{2,\Omega^t}^{1/3} \equiv I_1.
\tag{1.23}
$$

Recall that $\Psi(t) = \nu|\nabla\varphi|_{3,1,2,\Omega^t}$. Our aim is to show that there exist positive constants α_k, β_k, $k = 1,2,\dots$, such that

$$
I_1 \le \sum_k \frac{\Psi^{\alpha_k}}{\nu^{\beta_k}}.
\tag{1.24}
$$

By imbedding we already have that

$$
I_1 \le c\Psi^{2/3}\nu^{1/3}|\nabla\varphi|_{2,\Omega^t}^{1/3}.
\tag{1.25}
$$

Hence, we have to use (1.20). First we have to estimate the argument of exp function. We assume that

$$
c_1 \le (\mu+\nu)^\varkappa\varphi_*.
\tag{1.26}
$$

Then the term under exp in (1.20) is bounded by

$$c|v|_{6,\infty,\Omega^t}^{2/(1-\varkappa)},\quad \varkappa=3/2-3/p\in(0,1)\quad\text{for}\quad p\in(3,6).$$

Assuming that

$$|\eta(0)|\le\frac{c}{\nu},|f_g|_{2,\Omega^t}\le\frac{c}{\nu}\tag{1.27}$$

we obtain (see (2.62) and (2.64))

$$|\eta|_{r,\infty,\Omega^t}\le\frac{c}{\nu}(\Psi+c)\tag{1.28}$$

Then (1.20) implies

$$\begin{aligned}|\nabla\varphi|_{2,\Omega^t}&\le\phi(|v|_{6,\infty,\Omega^t},|v_t|_{2,\Omega^t},|\nabla\eta|_{2,\Omega^t},|f|_{2,\Omega^t})\cdot\\&\cdot\left[\frac{1}{\nu^2}(\Psi+c)+\frac{c}{\nu^{1/2}}|\varphi(0)|_2\right].\end{aligned}\tag{1.29}$$

Assuming additionally that

$$|\varphi(0)|_2\le\frac{c}{\nu^\varkappa},\quad \varkappa>1/2\tag{1.30}$$

and using (1.30) in (1.29) and the result in (1.23) yields

$$\begin{aligned}I_1&\le\phi(|v|_{6,\infty,\Omega^t},|v_t|_{2,\Omega^t},|\nabla\eta|_{2,\Omega^t},|f|_{2,\Omega^t})\cdot\\&\cdot\left[\frac{\Psi}{\nu^{1/3}}+\frac{\Psi^{2/3}}{\nu^{1/3}}+\frac{\Psi^{2/3}}{\nu^{\varkappa/3-1/6}}\right].\end{aligned}\tag{1.31}$$

Then from (1.14) for $r=6$ we get (see (2.82))

$$\begin{aligned}&|v(t)|_6+\left(\int_0^t|\nabla|v|^3|^2dxdt'\right)^{1/6}\le c|\eta|_{18/7,6,\Omega^t}\\&+\phi(|v|_{6,\infty,\Omega^t},|v_t|_{2,\Omega^t},|\nabla\eta|_{2,\Omega^t},|f|_{2,\Omega^t})[\Psi/\nu^{1/3}+\Psi^{2/3}/\nu^{1/3}\\&+\Psi^{2/3}/\nu^{\varkappa/3-1/6}]+c|f|_{18/7,6,\Omega^t}+c|\varrho_0|_\infty^{1/6}|v_0|_6\equiv D_1.\end{aligned}\tag{1.32}$$

Finally, Lemma 2.9 implies

$$\begin{aligned}&|v_t(t)|_2^2+\mu|\nabla v_t|_{2,\Omega^t}^2+\nu|\Delta\varphi_t|_{2,\Omega^t}^2\le\exp(|\eta_t|_{3,2,\Omega^t}^2\\&\quad+(\|\eta\|_{2,\infty,\Omega^t}^2+D_1^2+1)A_1^2)\cdot[|\eta_t|_{2,\Omega^t}^2+\|\eta_t\|_{1,\infty,\Omega^t}^2(D_1^2A_1^2\\&\quad+|\nabla\varphi|_{3,1,2,\Omega^t}^2A_1^2+|f|_{2,\Omega^t}^2)+|f_t|_{2,\Omega^t}^2+|\varrho_0|_\infty|v_t(0)|_2^2]\\&\equiv D_2^2.\end{aligned}\tag{1.33}$$

Let us introduce the quantities

$$\begin{aligned}
&\Psi(t) = \nu|\nabla\varphi|_{3,1,2,\Omega^t},\\
&\chi_1^2(t) = \nu|\nabla\varphi|^2_{2,1,\infty,\Omega^t} + |\mathrm{rot}\,\psi|^2_{2,1,\infty,\Omega^t},\\
&\chi_2^2(t) = \nu|\nabla\varphi|^2_{3,1,2,\Omega^t} + |\mathrm{rot}\,\psi|^2_{3,1,2,\Omega^t},\\
&\Phi_1^2(t) = \nu(|\nabla\varphi|^2_{2,1,\infty,\Omega^t} + |\nabla\varphi|^2_{3,1,2,\Omega^t}),\\
&\Phi_2^2(t) = |\mathrm{rot}\,\psi|^2_{2,1,\infty,\Omega^t} + |\mathrm{rot}\,\psi|^2_{3,1,2,\Omega^t}.
\end{aligned}$$

From (4.2) we have inequality

$$\begin{aligned}
\chi_1^2(t) + \chi_2^2(t) + \Psi^2(t) &\le \phi\bigg(D_1, D_2, A_1, \frac{\Psi}{\nu}, \frac{\Phi_1}{\sqrt{\nu}}, \frac{\Psi}{\nu}, \Phi_2,\\
&|f|_{1,1,2,\Omega^t}, |\eta|_{2,1,\infty,\Omega^t}, |\eta|_{2,1,2,\Omega^t}\bigg)\\
&+ c|\eta|^2_{2,1,\infty,\Omega^t}(1 + |\eta|^2_{2,1,\infty,\Omega^t})\Psi^2 + \nu|\varphi(0)|^2_{2,1} + |\mathrm{rot}\,\psi(0)|^2_{2,1}.
\end{aligned} \tag{1.34}$$

Hence for ν sufficiently large there exists a constant A such that

$$X(t) \equiv \chi_1(t) + \chi_2(t) + \Psi(t) \le A. \tag{1.35}$$

there is such restriction on time that t is proportional to some positive increasing function of ν. But large time is not convenient because strong restrictions on v, f follow from time-integrals norms. Therefore to derive estimate (1.35) for all $t \in \mathbb{R}_+$ we perform the procedure step by step in time. In Theorem 5.7 we prove that

$$X(t) \le A, \quad t \in [kT, (k+1)T], \quad k \in \mathbb{N}_0 = \| \cup \{0\}, \tag{1.36}$$

where A does not depend on k.
The derivation of such estimate is possible thanks to sufficiently large ν, T and sufficiently small $|\nabla\varphi(0)|_{2,1}|$, $|\eta(0)|_{2,1}$.
In the proof is used natural dissipation mechanism in the compressible Navier-Stokes equations connected with viscosity coefficients μ and ν.

2 Notation and auxiliary results

We use the simplified notation

$$\|u\|_{L_p(\Omega)} = |u|_p, \quad \|u\|_{H^s(\Omega)} = \|u\|_s, \quad \|u\|_{W_p^s(\Omega)} = \|u\|_{s,p},$$

$$|u|_{k,l}^2 = \sum_{i=0}^{l} \|\partial_t^i u\|_{k-i}^2, \quad |u|_{k,l,r,\Omega^t} = \bigg(\int_0^t |u(t')|_{k,l}^r dt'\bigg)^{1/r},$$

$$|u|_{r,q,\Omega^t} = \bigg(\int_0^t |u(t')|_r^q dt'\bigg)^{1/q},$$

$$\|u\|_{L_r(0,t;H^s(\Omega))} = \|u\|_{s,r,\Omega^t}, \quad \|u\|_{L_r(0,t;W_p^s(\Omega))} = \|u\|_{s,p,r,\Omega^t}.$$

Introduce the spaces

$$\Gamma_l^k(\Omega) = \{u : |u|_{k,l} < \infty\}, \quad l \le k, \quad l,k \in \mathbb{N}_0.$$

By ϕ, ϕ_σ, $\bar\phi_\sigma$, $\sigma \in \mathbb{N}$, we denote always increasing positive functions of their arguments.
First we obtain the energy type estimate for solutions to problem (1.1).

Lemma 2.1. *Assume that (ϱ, v) is a solution to problem (1.1). Assume that $p = p(\varrho) = A\varrho^\varkappa$, $\varkappa > 1$. Assume the periodic boundary conditions and that $f \in L_{\infty,2}(\Omega \times (kT,(k+1)T))$, $k \in \mathbb{N}_0$, $a/2 \le \varrho \le 3a/2$, $\varrho_0 \in L_2(\Omega)$, $f \in L_{\infty,1}(\Omega^t)$ and $\mu' > 0$. Let $\sup_{k\in\mathbb{N}_0} \int_{kT}^{(k+1)T} |\eta(t)|_2^2 dt \le \mu T/2$ and*

$$B(T) = c \sup_{k\in\mathbb{N}_0} \exp\bigg[c\int_{kT}^{(k+1)T} |\eta(t)|_2^2 dt\bigg][|\varrho_0|_1^2 |f|_{\infty,2,\Omega\times(kT,(k+1)T)}^2 + \mu' a^\varkappa + |\varrho_0|_1^2 |f|_{\infty,1,\Omega^t}^2 + |\varrho_0|_2^2 |v_0|_2^2].$$

$\alpha(0) = \int_\Omega \big(\frac{1}{2}\varrho_0 v_0^2 + \frac{A}{\varkappa-1}\varrho_0^\varkappa\big)dx < \infty$

$$\varrho_0 \in L_2(\Omega), \quad v_0 \in L_2(\Omega).$$

Then there exist positive numbers μ', μ'' less than μ such that

$$\int_\Omega \bigg(\frac{1}{2}\varrho v^2 + \frac{A}{\varkappa - 1}\varrho^\varkappa\bigg)dx + \mu'' \int_0^t \|v(t')\|_1^2 dt' + \nu \int_0^t |\operatorname{div} v(t')|_2^2 dt' \le c e^{\mu' T}\bigg(\frac{B(T)}{1 - e^{-\mu' T/2}} + \alpha(0)\bigg) \equiv A_1^2. \tag{2.1}$$

Proof. Multiplying $(1.1)_1$ by v, integrating over Ω and using the periodic boundary conditions, we obtain

$$
\begin{aligned}
(2.2)\qquad &\frac{1}{2}\int_\Omega (\varrho\partial_t v^2 + \varrho v\cdot\nabla v^2)dx + \mu|\nabla v|_2^2 + \nu|\mathrm{div}\, v|_2^2 \\
&\quad + \int_\Omega \nabla p(\varrho)\cdot v dx = \int_\Omega \varrho f\cdot v dx
\end{aligned}
$$

Adding the identity

$$\frac{1}{2}\int_\Omega [\varrho_t + \mathrm{div}\,(\varrho v)]v^2 dx = 0$$

we derive from (2.2) the equality

$$(2.3)\qquad \frac{1}{2}\frac{d}{dt}\int_\Omega \varrho v^2 dx + \mu|\nabla v|_2^2 + \nu|\mathrm{div}\, v|_2^2 + \int_\Omega \nabla p(\varrho)\cdot v dx = \int_\Omega \varrho f\cdot v dx.$$

Using that $p = A\varrho^\varkappa$, $\varkappa > 1$, the last term on the l.h.s. of (2.3) equals

$$(2.4)\qquad \int_\Omega \nabla p(\varrho)\cdot v dx = A\int_\Omega v\cdot\nabla\varrho^\varkappa dx.$$

To use (2.4) in (2.3) we multiply $(1.1)_2$ by $\varrho^{\varkappa-1}$. Then we get

$$\varrho^{\varkappa-1}(\varrho_t + v\cdot\nabla\varrho) + \varrho^\varkappa \mathrm{div}\, v = 0.$$

Continuing, we have

$$(2.5)\qquad \frac{1}{\varkappa}\partial_t\varrho^\varkappa + \frac{1}{\varkappa}v\cdot\nabla\varrho^\varkappa + \varrho^\varkappa\mathrm{div}\, v = 0.$$

Adding $\frac{\varkappa-1}{\varkappa}v\cdot\nabla\varrho^\varkappa$ to both sides of (2.5) yields

$$\frac{1}{\varkappa}\frac{\partial}{\partial t}\varrho^\varkappa + \mathrm{div}\,(v\varrho^\varkappa) = \frac{\varkappa-1}{\varkappa}v\cdot\nabla\varrho^\varkappa.$$

Integrating the equality over Ω and using boundary conditions gives

$$\frac{A}{\varkappa}\frac{d}{dt}\int_\Omega \varrho^\varkappa dx = \frac{A(\varkappa-1)}{\varkappa}\int_\Omega v\cdot\nabla\varrho^\varkappa dx.$$

Hence

$$\frac{A}{\varkappa-1}\frac{d}{dt}\int_\Omega \varrho^\varkappa dx = A\int_\Omega v\cdot\nabla\varrho^\varkappa dx. \tag{2.6}$$

In view of (2.4) and (2.6) equality (2.3) takes the form

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho v^2 + \frac{A}{\varkappa-1}\varrho^\varkappa\right)dx + \mu|\nabla v|_2^2 + \nu|\operatorname{div} v|_2^2 = \int_\Omega \varrho f\cdot v dx. \tag{2.7}$$

Using that $\fint_\Omega v dx = G$, where G is defined between (1.6) and (1.7), and applying the Poincaré inequality we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho v^2 + \frac{A}{\varkappa-1}\varrho^\varkappa\right)dx + \bar\mu|v|_2^2 + \mu''\|v\|_1^2 + \nu|\operatorname{div} v|_2^2 \le \int_\Omega \varrho f\cdot v dx\\ + c|G|^2 \le \int_\Omega \varrho f\cdot v dx + c\left(|\eta|_2^2|v|_2^2 + \left|\int_{\Omega^t}\varrho f dx dt'\right|^2 + \left|\int_\Omega \varrho_0 v_0 dx\right|^2\right).\end{aligned}$$

Exploiting that $a/2\le\varrho\le\frac{3}{2}a$ and introducing the quantity

$$\alpha = \int_\Omega \left(\frac{1}{2}\varrho v^2 + \frac{A}{\varkappa-1}\varrho^\varkappa\right)dx$$

and using that

$$\left|\int_\Omega \varrho f\cdot v dx\right| = \left|\int_\Omega \sqrt{\varrho} f\sqrt{\varrho} v dx\right| \le \varepsilon\int_\Omega \varrho v^2 dx + \frac{c}{\varepsilon}|\varrho|_1^2|f|_\infty^2$$

we obtain

$$\frac{d}{dt}\alpha + \mu'\alpha + \mu''\|v\|_1^2 + \nu|\operatorname{div} v|_2^2 \le c|\eta|_2^2\alpha + c\beta, \tag{2.8}$$

where

$$\beta = |\varrho_0|_1^2|f|_\infty^2 + |\varrho_0|_1^2|f|_{\infty,1,\Omega^t}^2 + |\varrho_0|_2^2|v_0|_2^2 + \mu a^\varkappa.$$

Looking for the energy estimate with the bound independent of time we use the step by step in time argument. Then we consider (2.8) in the time interval $[kT,(k+1)T]$, $k\in\mathbb{N}$.

From (2.8) we have

$$\frac{d}{dt}\bigg[\alpha(t)\exp\bigg(\mu' t - c\int_{kT}^{t}|\eta(t')|_2^2dt'\bigg)\bigg]$$

$$+\exp\bigg(\mu' t - c\int_{kT}^{t}|\eta(t')|_2^2dt'\bigg)(\mu''\|v\|_1^2+\mu|\mathrm{div}\,v|_2^2)$$

$$\le \beta\exp\bigg(\mu' t - c\int_{kT}^{t}|\eta(t')|_2^2dt'\bigg).$$

Integrating with respect to time we finally get

(2.9)

$$\alpha(t)+e^{-\mu'(t-kT)}\int_{kT}^{t}(\mu''\|v\|_1^2+\nu|\mathrm{div}\,v|_2^2)dt'$$

$$\le \exp\bigg(-\mu' t + c\int_{kT}^{t}|\eta(t')|_2^2dt'\bigg)\int_{kT}^{t}\beta\exp\bigg(\mu' t' - c\int_{kT}^{t'}|\eta(t'')|_2^2dt''\bigg)dt'$$

$$+\alpha(kT)\exp\bigg(-\mu'(t-kT)+c\int_{kT}^{t}|\eta(t')|_2^2dt'\bigg).$$

Omitting the second integral on the l.h.s. of (2.9) and performing integration with respect to time we obtain the inequality

(2.10)

$$\alpha(t)\le\exp\bigg[c\int_{kT}^{t}|\eta(t')|_2^2dt'\bigg](c\beta_0)$$

$$+\alpha(kT)\exp\bigg(-\mu'(t-kT)+c\int_{kT}^{t}|\eta(t')|_2^2dt'\bigg),$$

where $\beta_0=|\varrho_0|_1^2|f|^2_{\infty,2,\Omega\times(kT,t)}+\mu' a^{\varkappa}+|\varrho_0|_1^2|f|^2_{\infty,1,\Omega^t}+|\varrho_0|_2^2|v_0|_2^2$.

Setting $t = (k+1)T$ yields

$$\alpha((k+1)T) \leq \exp\left[c \int_{kT}^{(k+1)T} |\eta(t')|_2^2 dt'\right][c\beta_0]$$

$$+ \alpha(kT)\exp\bigg(-\mu' T + c\int_{kT}^{(k+1)T} |\eta(t)|_2^2 dt\bigg)$$

$$\leq B(T) + \alpha(kT)\exp\bigg(-\mu' T + c\int_{kT}^{(k+1)T} |\eta(t)|_2^2 dt\bigg),$$

where

$$B(T) = \sup_{k\in\mathbb{N}_0} \exp\left[c\int_{kT}^{(k+1)T} |\eta(t)|_2^2 dt\right] c\beta_0.$$

Assuming that $\sup_{k\in\mathbb{N}_0}\int_{kT}^{(k+1)T} \eta^t(t)dt \leq \mu' T/2$. We have by iteration

$$\alpha(kT) \leq \frac{B(T)}{1-e^{-\mu' T/2}} + \alpha(0)e^{-\mu' kT/2}, \quad k \in \mathbb{N}_0.$$

Using the estimate in (2.10) yields

$$\alpha(t) \leq \frac{2B(T)}{1-e^{-\mu' T/2}} + \alpha(0), \quad t \in [kT,(k+1)T]. \tag{2.11}$$

From (2.9) for $t \in (kT,(k+1)T]$, $k \in \mathbb{N}_0$, and (2.11) we obtain

$$\int_\Omega \bigg(\frac{1}{2}\varrho v^2 + \frac{A}{\varkappa - 1}\varrho^\varkappa\bigg)dx + \mu''\int_{kT}^t \|v(t')\|_1^2 dt' + \nu\int_{kT}^t |\operatorname{div} v(t')|_2^2 dt'$$

$$\leq \left[ce^{\mu' T/2}\beta_0 + \frac{B(T)}{1-e^{-\mu T/2}} + \alpha(0)\right]e^{\mu' T}, \quad t \in [kT,(k+1)T].$$

The above inequality implies (2.1). This concludes the proof. □

Lemma 2.2. *Assume that $r>2$, $\varrho f\in L_{\frac{3r}{2r+1}}(\Omega)$, $p(\varrho)-p(a)\in L_{\frac{3r}{r+1}}(\Omega)$, $\operatorname{div} v\in L_{\frac{3r}{r+1}}$, $\nu=\nu_1+\nu_2$, $\frac{\nu_1^2}{\nu}$ sufficiently small. Then*

$$
\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int\limits_\Omega \varrho|v|^r dx+\frac{k_0}{2}\int\limits_\Omega |\nabla|v|^{r/2}|^2dx+\frac{\mu}{2}\int\limits_\Omega |v|^r\bigg|\nabla\frac{v}{|v|}\bigg|^2dx\\
&\quad+\frac{\nu}{2}\int\limits_\Omega |\operatorname{div} v|^2|v|^{r-2}dx\le \frac{c}{\nu^{r/2}}|p(\varrho)-p(a)|^r_{\frac{3r}{r+1}}+c|p(\varrho)\\
&\quad-p(a)-\nu_2\operatorname{div} v|^r_{\frac{3r}{r+1}}+c|\varrho f|^r_{\frac{3r}{2r+1}}+c|v|_2^r,
\end{aligned}
\tag{2.12}
$$

where $k_0=k_0\big(\mu,r,\frac{\nu_1^2}{\nu}\big)$, $c=c(r,k_0)$.

Proof. We express $(1.1)_1$ in the form

$$
\varrho v_t+\varrho v\cdot\nabla v-\mu\Delta v-\nu_1\nabla\operatorname{div} v+\nabla(p-p(a)-\nu_2\operatorname{div} v)=\varrho f, \tag{2.13}
$$

where $\nu=\nu_1+\nu_2$, $\nu_i>0$, $i=1,2$. Multiplying (2.13) by $v|v|^{r-2}$, integrating over Ω, using $(1.1)_2$ and boundary conditions yields

$$
\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int\limits_\Omega \varrho|v|^r dx+\mu\int\limits_\Omega \nabla v\cdot\nabla(v|v|^{r-2})dx+\nu_1\int\limits_\Omega \operatorname{div} v\operatorname{div}(v|v|^{r-2})dx\\
&\quad-\int\limits_\Omega (p-p(a)-\nu_2\operatorname{div} v)\operatorname{div}(v|v|^{r-2})dx=\int\limits_\Omega \varrho f v|v|^{r-2}dx.
\end{aligned}
\tag{2.14}
$$

First, we consider

$$
\begin{aligned}
J_1&=\mu\int\limits_\Omega \nabla v\cdot\nabla(v|v|^{r-2})dx\\
&=\mu\int\limits_\Omega |\nabla v|^2|v|^{r-2}dx+\mu\int\limits_\Omega v_k\nabla v_k\cdot(r-2)|v|^{r-3}\nabla|v|dx\equiv I_1+I_2.
\end{aligned}
$$

Using that $v_k\nabla v_k=\frac{1}{2}\nabla|v|^2=|v|\nabla|v|$ we have

$$
\begin{aligned}
I_2&=(r-2)\mu\int\limits_\Omega |v|^{r-2}|\nabla|v||^2dx=(r-2)\mu\int\limits_\Omega ||v|^{\frac{r}{2}-1}\nabla|v||^2dx\\
&=\frac{4(r-2)}{r^2}\mu\int\limits_\Omega |\nabla|v|^{r/2}|^2dx.
\end{aligned}
$$

To examine I_1 we use the formula

$$|\nabla u|^2 = |u|^2\left|\nabla\frac{u}{|u|}\right|^2 + \big|\nabla|u|\big|^2. \tag{2.15}$$

Then I_1 takes the form

$$\begin{aligned}
I_1 &= \mu\int\limits_\Omega \left(|v|^2\left|\nabla\frac{v}{|v|}\right|^2 + \big|\nabla|v|\big|^2\right)|v|^{r-2}dx\\
&= \mu\int\limits_\Omega \left[|v|^r\left|\nabla\frac{v}{|v|}\right|^2 + |v|^{r-2}\big|\nabla|v|\big|^2\right]dx\\
&= \mu\int\limits_\Omega \left[|v|^r\left|\nabla\frac{v}{|v|}\right|^2 + \big||v|^{\frac{r}{2}-1}\nabla|v|\big|^2\right]dx\\
&= \mu\int\limits_\Omega |v|^r\left|\nabla\frac{v}{|v|}\right|^2 dx + \frac{4\mu}{r^2}\int\limits_\Omega \big|\nabla|v|^{r/2}\big|^2 dx.
\end{aligned}$$

Hence

$$J_1 = \frac{4(r-1)\mu}{r^2}\int\limits_\Omega \big|\nabla|v|^{r/2}\big|^2 dx + \mu\int\limits_\Omega |v|^r\left|\nabla\frac{v}{|v|}\right|^2 dx.$$

Next, we consider

$$\begin{aligned}
J_2 &= \nu_1\int\limits_\Omega \operatorname{div} v \operatorname{div}(v|v|^{r-2})dx\\
&= \nu_1\int\limits_\Omega |\operatorname{div} v|^2|v|^{r-2}dx + \nu_1\int\limits_\Omega \operatorname{div} v\, v\cdot\nabla|v|^{r-2}dx \equiv I_3 + I_4,
\end{aligned}$$

where

$$I_4 = \nu_1(r-2)\int\limits_\Omega \operatorname{div} v\, v\cdot\nabla|v|\,|v|^{r-3}dx.$$

Then

$$|I_4| \le \nu_1(r-2)\int\limits_\Omega |\operatorname{div} v|\,|v|^{r-2}|\nabla|v||dx.$$

Employing the above expressions in (2.14) one gets

(2.16)

$$\begin{aligned}&\frac{1}{r}\frac{d}{dt}\int\limits_\Omega \varrho|v|^r dx+\frac{4(r-1)\mu}{r^2}\int\limits_\Omega \big|\nabla|v|^{r/2}\big|^2dx+\mu\int\limits_\Omega |v|^r\bigg|\nabla\frac{v}{|v|}\bigg|^2dx\\&\quad+\nu_1\int\limits_\Omega |\operatorname{div} v|^2|v|^{r-2}dx\le\nu_1(r-2)\int\limits_\Omega |\operatorname{div} v|\,|v|^{r-2}\big|\nabla|v|\big|dx\\&\quad+\int\limits_\Omega [p(\varrho)-p(a)-\nu_2\operatorname{div} v][\operatorname{div} v|v|^{r-2}+(r-2)|v|^{r-3}v\cdot\nabla|v|]dx\\&\quad+\bigg|\int\limits_\Omega \varrho f\cdot v|v|^{r-2}dx\bigg|.\end{aligned}$$

The first term on the r.h.s. of (2.16) is bounded by

$$\frac{\varepsilon_1}{2}\nu_1\int\limits_\Omega |\operatorname{div} v|^2|v|^{r-2}dx+\frac{1}{2\varepsilon_1}\nu_1(r-2)^2\int\limits_\Omega |v|^{r-2}\big|\nabla|v|\big|^2dx,$$

where the second integral equals

$$\frac{\nu_1}{2\varepsilon_1}(r-2)^2\int\limits_\Omega \big||v|^{\frac{r}{2}-1}\nabla|v|\big|^2dx=\frac{4\nu_1(r-2)^2}{2\varepsilon_1 r^2}\int\limits_\Omega |\nabla|v|^{r/2}|^2dx.$$

The second term on the r.h.s. of (2.16) can be expressed in the form

$$\begin{aligned}&\int\limits_\Omega (p(\varrho)-p(a))\operatorname{div} v|v|^{r-2}dx-\nu_2\int\limits_\Omega |\operatorname{div} v|^2|v|^{r-2}dx\\&\quad+\int\limits_\Omega [p(\varrho)-p(a)-\nu_2\operatorname{div} v](r-2)|v|^{r-3}v\cdot\nabla|v|dx\equiv J_1+J_2+J_3,\end{aligned}$$

where

$$|J_1|\le\frac{\varepsilon_2}{2}\int\limits_\Omega |\operatorname{div} v|^2|v|^{r-2}dx+\frac{1}{2\varepsilon_2}\int\limits_\Omega |p(\varrho)-p(a)|^2|v|^{r-2}dx$$

and

$$\begin{aligned}|J_3|&\le(r-2)\int\limits_\Omega |p(\varrho)-p(a)-\nu_2\operatorname{div} v|\,|v|^{r-2}|\nabla|v||dx\\&\le\frac{\varepsilon_3}{2}(r-2)\int\limits_\Omega |v|^{r-2}|\nabla|v||^2dx\\&\quad+\frac{r-2}{2\varepsilon_3}\int\limits_\Omega |p(\varrho)-p(a)-\nu_2\operatorname{div} v|^2|v|^{r-2}dx\equiv K_1.\end{aligned}$$

Hence we have

$$K_1 = 2\varepsilon_3 \frac{r-2}{r^2} \int\limits_\Omega \left|\nabla |v|^{r/2}\right|^2 dx$$
$$+ \frac{r-2}{2\varepsilon_3} \int\limits_\Omega |p(\varrho) - p(a) - \nu_2 \mathrm{div}\, v|^2 |v|^{r-2} dx.$$

Employing the above estimates in (2.16) yields

(2.17)
$$\frac{1}{r}\frac{d}{dt}\int\limits_\Omega \varrho |v|^r dx + \left[\frac{4(r-1)\mu}{r^2} - \frac{4\nu_1(r-2)^2}{2\varepsilon_1 r^2} - \frac{2\varepsilon_3(r-2)}{r^2}\right]\int\limits_\Omega \left|\nabla |v|^{r/2}\right|^2 dx$$
$$+ \mu \int\limits_\Omega |v|^r \left|\nabla \frac{v}{|v|}\right|^2 dx + \left[\nu - \frac{\nu_1\varepsilon_1}{2} - \frac{\varepsilon_2}{2}\right]\int\limits_\Omega |\mathrm{div}\, v|^2 |v|^{r-2} dx$$
$$\le \frac{1}{2\varepsilon_2}\int\limits_\Omega |p(\varrho) - p(a)|^2 |v|^{r-2} dx + \frac{r-2}{2\varepsilon_3}\int\limits_\Omega |p(\varrho) - p(a) - \nu_2 \mathrm{div}\, v|^2 |v|^{r-2} dx$$
$$+ \int\limits_\Omega \varrho f \cdot v |v|^{r-2} dx.$$

We set

$$\varepsilon_1 = \frac{\nu}{2\nu_1}, \quad \varepsilon_2 = \frac{\nu}{2}, \quad \varepsilon_3 = \frac{\mu}{r-2} \tag{2.18}$$

then coefficient near the second term on the l.h.s. of (2.17) equals

$$\begin{aligned} &\frac{4(r-1)\mu}{r^2} - \frac{4\nu_1^2(r-2)^2}{\nu r^2} - \frac{2\mu}{r^2} = \frac{2\mu(2r-3)}{r^2} - \frac{4\nu_1^2(r-2)^2}{\nu r^2} \\ &= \frac{4}{r^2}\left[\frac{(2r-3)\mu}{2} - \frac{\nu_1^2}{\nu}(r-2)^2\right] \equiv k_0\left(\mu, r, \frac{\nu_1^2}{\nu}\right) \end{aligned} \tag{2.19}$$

which is positive for $r \ge 2$ and ν_1^2/ν small.
Coefficients near the third and fourth terms are equal, respectively,

$$\mu \quad \text{and} \quad \frac{\nu}{2}.$$

To have $k_0 > 0$ we obtain the restriction on ν_1

$$\frac{\nu_1^2}{\nu} < \frac{\mu(2r-3)}{2(r-2)^2} \equiv d^2, \quad \text{so} \quad \nu_1 < d\sqrt{\nu}$$

Then

$$\nu_2 > \nu - d\sqrt{\nu}.$$

Hence for ν large ν_2 is close to ν

Then (2.17) takes the form

(2.20)

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int\limits_\Omega \varrho|v|^r dx + k_0\int\limits_\Omega \left|\nabla|v|^{r/2}\right|^2 dx + \mu\int\limits_\Omega |v|^r\left|\nabla\frac{v}{|v|}\right|^2 dx\\
&\quad+\frac{\nu}{2}\int\limits_\Omega |\mathrm{div}\, v|^2|v|^{r-2}dx \le \frac{1}{\nu}\int\limits_\Omega |p(\varrho)-p(a)|^2|v|^{r-2}dx\\
&\quad+\frac{(r-2)^2}{2\mu}\int\limits_\Omega |p(\varrho)-p(a)-\nu_2\mathrm{div}\, v|^2|v|^{r-2}dx + \int\limits_\Omega |\varrho f\cdot v|\,|v|^{r-2}dx.
\end{aligned}$$

Consider the second term on the l.h.s. of (2.20). We use the Poincaré inequality

$$|\nabla|v|^{r/2}|_2^2 \ge c\bigg(|v|_r^r - \bigg|\ -\!\!\!\!\!\!\int\limits_\Omega |v|^{r/2}dx\bigg|^2\bigg)$$

and the estimate

$$|\nabla|v|^{r/2}|_2^2 + |v|_r^r \ge c|v|_{3r}^r.$$

Then the second term is bounded from below by

$$k_0|\nabla|v|^{r/2}|_2^2 \ge c|v|_r^r + \frac{c_0k_0}{2}|v|_{3r}^r + \frac{k_0}{2}|\nabla|v|^{r/2}|_2^2 - c\bigg|\int\limits_\Omega |v|^{r/2}dx\bigg|^2.$$

Finally, we use the interpolation

$$\bigg|\int |v|^{r/2}dx\bigg|^2 \le \varepsilon|v|_r^r + c/\varepsilon|v|_2^r.$$

Next, we have

$$\begin{aligned}
&\frac{1}{\nu}\int\limits_\Omega |p(\varrho)-p(a)|^2|v|^{r-2}dx \le \frac{1}{\nu}|p(\varrho)-p(a)|^2_{\frac{3r}{r+1}}|v|_{3r}^{r-2}\\
&\le \left[\frac{\varepsilon_4^{\frac{r}{r-2}}}{\frac{r}{r-2}}|v|_{3r}^r + \frac{1}{\frac{r}{2}\varepsilon_4^{r/2}\nu^{r/2}}|p(\varrho)-p(a)|^r_{\frac{3r}{r+1}}\right] \equiv L_1,
\end{aligned}$$

$$\frac{(r-2)^2}{2\mu}\int_\Omega |p(\varrho)-p(a)-\nu_2 \mathrm{div}\, v|^2|v|^{r-2}dx$$
$$\le \frac{(r-2)^2}{2\mu}|p(\varrho)-p(a)-\nu_2\mathrm{div}\, v|^2_{\frac{3r}{r+1}}|v|^{r-2}_{3r}$$
$$\le \frac{(r-2)^2}{2\mu}\left[\frac{\varepsilon_5^{\frac{r}{r-2}}}{\frac{r}{r-2}}|v|^r_{3r}+\frac{1}{\frac{r}{2}\varepsilon_5^{r/2}}|p(\varrho)-p(a)-\nu_2\mathrm{div}\, v|^r_{\frac{3r}{r+1}}\right]\equiv L_2,$$
$$\left|\int_\Omega \varrho f\cdot v|v|^{r-2}dx\right| \le \int_\Omega |\varrho f|\,|v|^{r-1}dx \le |\varrho f|_{\frac{3r}{2r+1}}|v|^{r-1}_{3r}$$
$$\le \frac{\varepsilon_6^{r/(r-1)}}{r/(r-1)}|v|^r_{3r}+\frac{1}{r\varepsilon_6^r}|\varrho f|^r_{\frac{3r}{2r+1}}\equiv L_3.$$

We set

$$\frac{\varepsilon_4^{r/(r-2)}}{r/(r-2)}\frac{1}{\nu}=\frac{k_0c_0}{6},\quad \frac{(r-2)^2}{2\mu}\frac{\varepsilon_5^{r/(r-2)}}{r/(r-2)}=\frac{k_0c_0}{6},\quad \frac{\varepsilon_6^{r/(r-1)}}{r/(r-1)}=\frac{k_0c_0}{6}$$

Then

$$L_1\le c_0\frac{k_0}{6}|v|^r_{3r}+\frac{c(r,k_0)}{\nu^{r/2}}|p(\varrho)-p(a)|^r_{\frac{3r}{r+1}},$$
$$L_2\le c_0\frac{k_0}{6}|v|^r_{3r}+c(r,k_0)|p(\varrho)-p(a)-\nu_2\mathrm{div}\, v|^r_{\frac{3r}{r+1}},$$
$$L_3\le c_0\frac{k_0}{6}|v|^2_{3r}+c(r,k_0)|\varrho f|^r_{\frac{3r}{2r+1}}.$$

Employing the estimates in (2.20) implies

$$\begin{aligned}&\frac{1}{r}\frac{d}{dt}\int_\Omega \varrho|v|^r dx+\frac{k_0}{2}\int_\Omega |\nabla|v|^{r/2}|^2dx+\mu\int_\Omega |v|^r\left|\nabla\frac{v}{|v|}\right|^2dx\\ &\quad+\frac{\nu}{2}\int_\Omega |\mathrm{div}\, v|^2|v|^{r-2}dx\le \frac{c}{\nu^{r/2}}|p(\varrho)-p(a)|^r_{\frac{3r}{r+1}}\\ &\quad+c|p(\varrho)-p(a)-\nu_2\mathrm{div}\, v|^r_{\frac{3r}{r+1}}+c|\varrho f|^r_{\frac{3r}{2r+1}}+|v|^r_2.\end{aligned}\tag{2.21}$$

This inequality implies (2.12) and concludes the proof. □

Remark 2.1. We integrate (2.12) with respect to time. Then we get

$$
\begin{aligned}
&\frac{1}{r}\int_\Omega \varrho|v|^r dx + \frac{k_0}{2}\int_{\Omega^t} \big|\nabla|v|^{r/2}\big|^2 dxdt' + \mu\int_{\Omega^t} |v|^r\Big|\nabla\frac{v}{|v|}\Big|^2 dxdt'\\
&\quad + \frac{\nu}{2}\int_{\Omega^t} |\operatorname{div} v|^2|v|^{r-2}dxdt' \le \frac{c}{\nu^{r/2}}\int_0^t |p(\varrho)-p(a)|^r_{\frac{3r}{r+1}}dt'\\
&\quad + c\int_0^t |p(\varrho)-p(a)-\nu_2\operatorname{div} v|^r_{\frac{3r}{r+1}}dt' + c\int_0^t |\varrho f|^r_{\frac{3r}{2r+1}}dt'\\
&\quad + \frac{1}{r}\int_\Omega \varrho_0|v_0|^r dx + cA_1^r\\
&\equiv A^r_{2,r}(t).
\end{aligned}
\tag{2.22}
$$

Since $\varrho = a + \eta$, we obtain from (2.22) for η so small that $|\eta| < a/2$ the inequality

$$
|v(t)|_r \le cA_{2,r}(t).
\tag{2.23}
$$

Since ϱ is bounded from below and above and since ν_2 is close to ν we obtain from (2.22) the inequality

$$
\begin{aligned}
&\frac{1}{r}|v|^r_r + \frac{k_0}{2}\int_{\Omega^t} |\nabla|v|^{r/2}|^2 dxdt' \le c\int_0^t |\eta|^r_{\frac{3r}{r+1}}dt' + c\nu^r\int_0^t |\Delta\varphi|^r_{\frac{3r}{r+1}}dt'\\
&\quad + c\int_0^t |f|^r_{\frac{3r}{2r+1}}dt' + \frac{1}{r}|\varrho_0|_\infty|v_0|^r_r + cA_1^r.
\end{aligned}
\tag{2.24}
$$

Simplifying, we get

$$
\begin{aligned}
&\frac{1}{\sqrt[r]{r}}|v|_r + \sqrt[r]{\frac{k_0}{2}}\bigg(\int_{\Omega^t} |\nabla|v|^{r/2}|^2 dxdt'\bigg)^{1/r} \le c|\eta|_{\frac{3r}{r+1},r,\Omega^t} + c\nu|\Delta\varphi|_{\frac{3r}{r+1},r,\Omega^t}\\
&\quad + c|f|_{\frac{3r}{2r+1},r,\Omega^t} + \frac{1}{\sqrt[r]{r}}|\varrho_0|_\infty^{1/r}|v_0|_r + cA_1.
\end{aligned}
\tag{2.25}
$$

Since $\Delta\varphi \in W_2^{2,1}(\Omega^t)$ then $\Delta\varphi \in L_{\frac{3r}{r+1},r}(\Omega^t)$ with arbitrary r, because

$$
\frac{5}{2} - \frac{3(r+1)}{3r} - \frac{2}{r} \le 2 \quad \text{so} \quad \frac{3}{2} - \frac{3}{r} \le 2 \text{ which holds for any } r.
$$

To derive a global estimate for solutions to problem (1.1) we need that the first two terms on the r.h.s. of (2.25) are estimated by quantities multiplied by the small parameter $\left(\frac{1}{\nu}\right)^\alpha$, $\alpha > 0$.
However, the second term on the r.h.s. of (2.25) contains the coefficient ν. This means that we need more delicate estimate to get the factor $\left(\frac{1}{\nu}\right)^\alpha$, $\alpha > 0$.
For this purpose we consider the interpolation

$$|\Delta\varphi|_{\frac{3r}{r+1}} \le c|\nabla\Delta\varphi|_2^\theta |\nabla\varphi|_2^{1-\theta}, \tag{2.26}$$

where θ is a solution to the equation

$$\frac{3(r+1)}{3r} - 1 = \frac{3}{2} - 2\theta.$$

Then $\theta = \frac{3}{4} - 1/2r$, $1-\theta = \frac{1}{4} + \frac{1}{2r}$. Therefore

$$|\Delta\varphi|_{\frac{3r}{r+1},r,\Omega^t} \le c|\nabla\Delta\varphi|_{2,\infty,\Omega^t}^{\frac{3}{4}-1/2r} \left(\int_0^t |\nabla\varphi|_2^{(1/4+1/2r)r} dt\right)^{1/r}, \tag{2.27}$$

where $(1/4+1/2r)r \le 2$ for $r \le 6$. For $r = 6$, (2.27) takes the form

$$|\Delta\varphi|_{18/7,6,\Omega^t} \le c|\nabla\Delta\varphi|_{2,\infty,\Omega^t}^{2/3} |\nabla\varphi|_{2,\Omega^t}^{1/3}. \tag{2.28}$$

To estimate the last factor on the r.h.s. of (2.28) we need the following equation derived from $(1.1)_1$ by applying the div operator

$$\begin{aligned} &a\Delta\varphi_t - (\mu+\nu)\Delta^2\varphi + a_0\Delta\eta = -a\mathrm{div}\,(v\cdot\nabla v) + \mathrm{div}\,[-\eta v_t - \eta v\cdot\nabla v \\ &\quad + (p_\varrho(a) - p_\varrho(a+\eta))\nabla\eta + (a+\eta)f]. \end{aligned} \tag{2.29}$$

Applying operator Δ^{-1} to (2.29) yields

$$\begin{aligned} a\varphi_t - (\mu+\nu)\Delta\varphi &= -a\Delta^{-1}\partial_{x_i}\partial_{x_j}(v_iv_j) + a\Delta^{-1}\partial_{x_i}(\Delta\varphi v_i) \\ &\quad + \Delta^{-1}\mathrm{div}\,[-\eta v_t - \eta v\cdot\nabla v + (p_\varrho(a) - p_\varrho(a+\eta))\nabla\eta + (a+\eta)f] \\ &\quad - (\eta - \fint_\Omega \eta dx) + a\fint_\Omega \varphi_t dx \\ &\equiv D_1 + D_2 + F - (\eta - \fint_\Omega \eta dx) + a\fint_\Omega \varphi_t dx, \end{aligned} \tag{2.30}$$

where $\fint_\Omega = \frac{1}{|\Omega|}\int$.

Integrating (2.30) over Ω we obtain identity in view of the periodic boundary conditions.

To obtain an estimate of the term $\nu|\Delta\varphi|_{\frac{3r}{r+1},r,\Omega^t}$, which appears on the r.h.s. of (2.24), in terms of the function Ψ^α/ν^β, where $\alpha>0$, $\beta>0$, $\Psi=\nu|\nabla\varphi|_{3,1,2,\Omega^t}$, we need the result

Lemma 2.3. *Let the assumptions of Lemma 2.1 hold. Let A_1 be defined in Lemma 2.1. Let $|\eta|\le a/2$. Let $0<\varphi_*=\min_\Omega\varphi$. Let $v\in L_{2p/(p-2),2/(1-x)}(\Omega^t)$, $\Delta\varphi\in L_{3/2,\infty}(\Omega^t)$, $\eta\in L_{6,\infty}(\Omega^t)$, $\nabla\eta\in L_2(\Omega^t)$, $v_t\in L_2(\Omega^t)$, $v\in L_{6,\infty}(\Omega^t)$, $f_g\in L_{6/5,2}(\Omega^t)$, $f\in L_2(\Omega^t)$, $\varphi(0)\in L_2(\Omega)$, $p\in(2,6)$, $\varkappa=\frac{3}{2}-\frac{3}{p}$.*
Then

$$
\begin{aligned}
&a|\varphi(t)|_2^2+(\mu+\nu)|\nabla\varphi|_{2,\Omega^t}^2\le\exp\bigg(\frac{c|v|_{2p/(p-2),2/(1-\varkappa),\Omega^t}^{2/(1-\varkappa)}}{[(\mu+\nu)^\varkappa\varphi_*]^{1/(1-\varkappa)}}\\
&+\frac{\int_0^t|\fint_\Omega\varphi_tdx|dt'}{\varphi_*}\bigg)\bigg[\frac{c}{\mu+\nu}(|\Delta\varphi|_{3/2,\infty,\Omega^t}^2A_1^2+|\eta|_{3,\infty,\Omega^t}^2|v_t|_{2,\Omega^t}^2\\
&+|\eta|_{6,\infty,\Omega^t}^2|v|_{6,\infty,\Omega^t}^2A_1^2+|\eta|_{6,\infty,\Omega^t}^2|\nabla\eta|_{2,\Omega^t}^2+|f_g|_{6/5,2,\Omega^t}^2+|\eta|_{2,\Omega^t}^2\\
&+|\eta|_{3,\infty,\Omega^t}^2|f|_{2,\Omega^t}^2)+a|\varphi(0)|_2^2\bigg].
\end{aligned}\tag{2.31}
$$

Proof. Multiplying (2.30) by φ and integrating over Ω yields

$$
\begin{aligned}
&\frac{a}{2}\frac{d}{dt}|\varphi|_2^2+(\mu+\nu)|\nabla\varphi|_2^2=\int_\Omega D_1\varphi dx+\int_\Omega D_2\varphi dx\\
&+\int_\Omega F\varphi dx-\int_\Omega\bar\eta\varphi dx+a\fint_\Omega\varphi_tdx\int_\Omega\varphi dx,
\end{aligned}\tag{2.32}
$$

where $\bar\eta=\eta-\fint_\Omega\eta dx$.

Now we estimate the particular terms from the r.h.s. of (2.32). The first term is bounded by

$$
\bigg|\int_\Omega D_1\varphi dx\bigg|\le\int_\Omega\frac{|D_1|\varphi^2}{\varphi_*}dx\le\frac{|D_1|_{p/(p-2)}}{\varphi_*}|\varphi|_p^2\equiv I_1,\tag{2.33}
$$

where $2<p<6$. Let $\alpha=\frac{|D_1|_{p/(p-2)}}{\varphi_*}$. Then we use the interpolation

$$
I_1^{1/2}=\alpha^{1/2}|\varphi|_p\le\alpha^{1/2}(\varepsilon^{1/\varkappa}|\nabla\varphi|_2+c\varepsilon^{-\frac{1}{1-\varkappa}}|\varphi|_2),
$$

where $\varkappa = 3/2 - 3/p$.
Setting $\varepsilon^{1/\varkappa}\alpha^{1/2} = (\mu+\nu)^{1/2}$ we have $\varepsilon = \left(\frac{\mu+\nu}{\alpha}\right)^{\varkappa/2}$.
Then

$$\alpha^{1/2}\varepsilon^{-1/(1-\varkappa)} = \frac{\alpha^{1/2(1-\varkappa)}}{(\mu+\nu)^{\varkappa/2(1-\varkappa)}}$$

Therefore

$$\begin{aligned} I_1 &\le \frac{1}{2}(\mu+\nu)|\nabla\varphi|_2^2 + \frac{c\alpha^{1/(1-\varkappa)}}{(\mu+\nu)^{\varkappa/(1-\varkappa)}}|\varphi|_2^2 \\ &= \frac{1}{2}(\mu+\nu)|\nabla\varphi|_2^2 + \frac{c|D_1|_{p/(p-2)}^{1/(1-\varkappa)}}{((\mu+\nu)^{\varkappa}\varphi_*)^{1/(1-\varkappa)}}|\varphi|_2^2, \end{aligned} \tag{2.34}$$

where $|D_1|_q \le c\sum_{i,j=1}^3 |v_l v_j|_q$ for any $q \in (1,\infty)$.
Consider the second term on the r.h.s. of (2.32). Integration by parts yields

$$\int_\Omega D_2\varphi dx = -\int_\Omega \Delta^{-1}(\Delta\varphi v)\cdot\nabla\varphi dx \equiv I_2.$$

Hence

$$\begin{aligned} |I_2| &\le \varepsilon_1|\nabla\varphi|_2^2 + c/\varepsilon_1|\Delta^{-1}(\Delta\varphi v)|_2^2 \le \varepsilon_1|\nabla\varphi|_2^2 + c/\varepsilon_1|\Delta\varphi v|_{6/5}^2 \\ &\le \varepsilon_1|\nabla\varphi|_2^2 + c/\varepsilon_1|\Delta\varphi|_{3/2}^2|v|_6^2. \end{aligned} \tag{2.35}$$

Consider the third term on the r.h.s. of (2.32). Using that $F = \operatorname{div} F'$ we get

$$\left|\int_\Omega F\cdot\varphi dx\right| = \left|\int_\Omega F'\cdot\nabla\varphi\right| dx \le \varepsilon_2|\nabla\varphi|_2^2 + c/\varepsilon_2|F'|_2^2,$$

where
(2.36)

$$\begin{aligned} |F'|_2^2 &= |\Delta^{-1}[-\eta v_t - \eta v\cdot\nabla v + (p_\varrho(a) - p_\varrho(a+\eta))\nabla\eta + af_g + \eta f]|_2^2 \\ &\le c(|\eta v_t|_{6/5}^2 + |\eta v\cdot\nabla v|_{6/5}^2 + |\eta\nabla\eta|_{6/5}^2 + |f_g|_{6/5}^2 + |\eta f|_{6/5}^2) \\ &\le c(|\eta|_3^2|v_t|_2^2 + |\eta|_6^2|v|_6^2|\nabla v|_2^2 + |\eta|_6^2|\nabla\eta|_2^2 + |f_g|_{6/5}^2 + |\eta|_3^2|f|_2^2). \end{aligned}$$

We express the fourth term on the r.h.s. of (2.32) in the form

$$\int_\Omega \bar\eta\varphi dx = \int_\Omega \bar\eta\bar\varphi dx \equiv I_3.$$

Hence

$$|I_3| \le \varepsilon_3|\nabla\varphi|_2^2 + c/\varepsilon_3|\bar\eta|_2^2 \le \varepsilon_3|\nabla\varphi|_2^2 + c/\varepsilon_3|\eta|_2^2.$$

Finally, the last term on the r.h.s. of (2.32) is bounded by

$$\frac{|\fint \varphi_t dx|}{\varphi_*}|\varphi|_2^2.$$

Using the above estimates in (2.32) and assuming that $\varepsilon_1 - \varepsilon_3$ are sufficiently small we derive the inequality

$$\begin{aligned} a\frac{d}{dt}|\varphi|_2^2 + (\mu+\nu)|\nabla\varphi|_2^2 &\le \bigg[\frac{c|D_1|_{p/(p-2)}^{1/(1-\varkappa)}}{[(\mu+\nu)^\varkappa\varphi_*]^{1/(1-\varkappa)}} + \frac{|\fint \varphi_t dx|}{\varphi_*}\bigg]|\varphi|_2^2 \\ &\quad + \frac{c}{\mu+\nu}[|\Delta\varphi|_{3/2}^2|v|_6^2 + |F'|_2^2 + |\eta|_2^2] \\ &\equiv cd^2|\varphi|_2^2 + \frac{c}{\mu+\nu}[|\Delta\varphi|_{3/2}^2|v|_6^2 + |F'|_2^2 + |\eta|_2^2]. \end{aligned} \tag{2.37}$$

From (2.37) we have

$$\begin{aligned} &a\frac{d}{dt}\bigg(|\varphi|_2^2\exp\bigg[-c\int_0^t d^2(t')dt'\bigg]\bigg) + (\mu+\nu)|\nabla\varphi|_2^2\exp\bigg[-c\int_0^t d^2(t')dt'\bigg] \\ &\le \frac{c}{\mu+\nu}[|\Delta\varphi|_{3/2}^2|v|_6^2 + |F'|_2^2 + |\eta|_2^2]\exp\bigg[-c\int_0^t d^2(t')dt'\bigg]. \end{aligned} \tag{2.38}$$

Integrating (2.38) with respect to time implies

$$\begin{aligned} a|\varphi(t)|_2^2 + (\mu+\nu)|\nabla\varphi|_{2,\Omega^t}^2 &\le \exp\bigg[c\int_0^t d^2(t')dt'\bigg]\cdot \\ &\quad\cdot\bigg[\frac{c}{\mu+\nu}(|\Delta\varphi|_{3/2,\infty,k\Omega^t}^2 A_1^2 + |F'|_{2,\Omega^t}^2 + t|\eta|_{2,\infty,\Omega^t}^2) + a|\varphi(0)|_2^2\bigg]. \end{aligned} \tag{2.39}$$

This inequality implies (2.31) and concludes the proof. □

Now we obtain bounds of φ from below and from above. We follow considerations from [LSU, Ch. 2, Sections 5, 6].

Lemma 2.4. *Let φ be solution to (2.30). Let $\varphi(0) \in L_\infty(\Omega)$. Assume that $\hat{k} = \frac{c_1}{\nu^\varkappa}$ and $|\varphi(0)|_\infty < \frac{c_1}{\nu^\varkappa}$. Assume that $\eta \in L_\infty(\Omega^t)$, $\nabla\eta \in L_6(\Omega^t)$, $v_t \in L_{30/(22-9\varkappa)}(\Omega^t)$, $v \in L_{20/(4-3\varkappa)}(\Omega^t) \cap L_{60/(17-9\varkappa)}(\Omega^t)$, $f_g, f \in L_{30/(22-9\varkappa)}(\Omega^t)$,*

$\varkappa \in (0,4/3)$, $t \le T$. *Then*

$$
\begin{aligned}
&|\varphi|_{\infty,\Omega^t} \le 2\hat{k}[1 + 2^{2/\varkappa+1/\varkappa^2}(\beta\gamma)^{1+1/\varkappa} t^{(1+\varkappa)/r_0} \mathrm{meas}^{(1+\varkappa)/p_0}(\Omega)]\\
&\equiv \gamma_*, \quad t \le T,\\
&\frac{3}{p_0} + \frac{2}{r_0} = \frac{3}{2}, \quad \beta = \frac{c}{(\mu+\nu)^{1/q}}, \quad \frac{3}{p} + \frac{2}{q} = \frac{3}{2},\\
&\gamma = \frac{c}{(\mu+\nu)^{1/q}}(|v|^2_{20/(4-3\varkappa),\Omega^t} + G(\varkappa,t)) \equiv \frac{c}{(\mu+\nu)^{1/q}} G_0(\varkappa,t),
\end{aligned}
\tag{2.40}
$$

where G *is defined in (2.50).*

Proof. From (2.30) we have

$$
\begin{aligned}
&a\varphi_t - (\mu+\nu)\Delta\varphi\\
&= -a\Delta^{-1}\partial_{x_i}\partial_{x_j}(v_iv_j) + a\Delta^{-1}\partial_{x_i}(\Delta\varphi v_i) + F - \bar{\eta} + a \fint_\Omega \varphi_t dx
\end{aligned}
\tag{2.41}
$$

Let $\varphi^{(k)} = \max\{\varphi^{(k)}(x,t) - k, 0\}$. Multiply (2.41) by $\varphi^{(k)}$ and integrate over Ω. Then we have

$$
\begin{aligned}
&\frac{a}{2}\frac{d}{dt}|\varphi^{(k)}|_2^2 + (\mu+\nu)|\nabla\varphi^{(k)}|_2^2\\
&= -\int_\Omega \Delta^{-1}\partial_{x_i}\partial_{x_j}(v_iv_j)\varphi^{(k)} dx\\
&\quad + a\int_\Omega \Delta^{-1}\partial_{x_i}(\Delta\varphi v_i)\varphi^{(k)} dx + \int_\Omega F\varphi^{(k)} dx - \int_\Omega \bar{\eta}\varphi^{(k)} dx\\
&\quad + a\int_\Omega \fint_\Omega \varphi_t(x') dx' \varphi^{(k)}(x) dx.
\end{aligned}
\tag{2.42}
$$

Assume that

$$\hat{k} < k.$$

Integrating (2.42) with respect to time and using $k > \hat{k}$ gives

$$
\begin{aligned}
&\|\varphi^{(k)}\|^2_{V(\Omega^t)} \equiv a|\varphi^{(k)}(t)|_2^2 + (\mu+\nu)|\nabla\varphi^{(k)}|^2_{2,\Omega^t} \le c|vv|_{p',q',A_k^t(t)}|\varphi^{(k)}|_{p,q,\Omega^t}\\
&\quad + c(|\Delta\varphi v|_{p',q',A_k^t(t)} + c|F|_{p',q',A_k^t(t)})|\varphi^{(k)}|_{p,q,\Omega^t} + |\bar{\eta}|_{p',q',A_k^t(t)}|\varphi^{(k)}|_{p,q,\Omega^t}\\
&\quad + \sup_t \int_\Omega |\varphi_t(x,t)| dx \int_0^t \int_\Omega |\varphi^{(k)}(x,t)| dx dt,
\end{aligned}
\tag{2.43}
$$

where $\frac{3}{p}+\frac{2}{q}=\frac{3}{2}$ and $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$ and $A_k(t)=\{x\in\Omega:\bar\varphi(x,t)>k\}$. Then $\frac{3}{p'}+\frac{2}{q'}=7/2$.

Now we have to estimate $|\varphi^{(k)}|_{p,q,\Omega^t}$ by the norm on the l.h.s. of (2.43), where we have to take under account the coefficient $\mu+\nu$ which is assumed to be large. From [LSU, Ch. 2, Sect. 3] we have

$$|\varphi^{(k)}|_{p,q,\Omega^t}\le c|\varphi^{(k)}|^{1-2/q}_{2,\infty,\Omega^t}|\nabla\varphi^{(k)}|^{2/q}_{2,\Omega^t}. \tag{2.44}$$

Let β be a power function of $(\mu+\nu)$. Then we have

$$\beta|\varphi^{(k)}|^{1-2/q}_{2,\infty,\Omega^t}|\nabla\varphi^{(k)}|^{2/q}_{2,\Omega^t}\le c\beta^{q/2}|\varphi^{(k)}_x|_{2,\Omega^t}+c|\varphi^{(k)}|_{2,\infty,\Omega^t}.$$

Comparing this with the norm from the l.h.s. of (2.43) we have

$$\beta^{q/2}=(\mu+\nu)^{1/2}\quad\text{so}\quad\beta=(\mu+\nu)^{1/q}.$$

Then, (2.43) takes the form

(2.45)

$$\begin{aligned}&|\varphi^{(k)}(t)|_{2,\infty,\Omega^t}+(\mu+\nu)^{1/2}|\nabla\varphi^{(k)}|_{2,\Omega^t}\le\frac{c}{(\mu+\nu)^{1/q}}|vv|_{p',q',A_k^t(t)}\\&+\frac{1}{(\mu+\nu)^{1/q}}|F|_{p',q',A_k^t(t)}+\frac{1}{(\mu+\nu)^{1/q}}(|\Delta\varphi v|_{p',q',A_k^t(t)}+|\bar\eta|_{p',q',A_k^t(t)})\\&+\frac{1}{(\mu+\nu)^{1/q}}|\varphi_t|_{1,\infty,\Omega^t}\bigg(\int_0^t|A_k(t')|^{q'/p'}dt'\bigg)^{1/q'}.\end{aligned}$$

Now, we examine the terms from the r.h.s. of (2.45). Examine the first term. Let $h=v\cdot v$. Then we have

$$\begin{aligned}&\frac{c}{(\mu+\nu)^{1/q}}|h|_{p',q',A_k^t(t)}\equiv\frac{c}{(\mu+\nu)^{1/q}}\bigg(\int_0^t\bigg(\int_{A_k(t')}|h|^{p'}dx\bigg)^{q'/p'}dt'\bigg)^{1/q'}\\&\le\frac{c}{(\mu+\nu)^{1/q}}\bigg(\int_0^t\bigg[\bigg(\int_{A_k(t')}1dx\bigg)^{1/\lambda p'}\bigg(\int_{A_k(t')}|h|^{p'\lambda'}\bigg)^{1/p'\lambda'}\bigg]^{q'}dt'\bigg)^{1/q'}\\&=\frac{c}{(\mu+\nu)^{1/q}}\bigg(\int_0^t|A_k(t')|^{q'/\lambda p'}|h|^{q'}_{p'\lambda',\Omega}dt'\bigg)^{1/q'}\\&\le\frac{c}{(\mu+\nu)^{1/q}}\bigg(\int_0^t|A_k(t')|^{\gamma q'/\lambda p'}dt'\bigg)^{1/\gamma q'}\bigg(\int_0^t|h|^{\gamma'q'}_{p'\lambda',\Omega}dt'\bigg)^{1/\gamma'q'}\equiv I_1,\end{aligned}$$

where $1/\lambda+1/\lambda'=1$, $1/\gamma+1/\gamma'=1$, $|A_k(t)|=\text{meas}A_k(t)$, $\mu(k,t)=\int_0^t|A_k(t')|^{r_0/p_0}dt'$, $\frac{3}{p_0}+\frac{2}{r_0}=\frac{3}{2}$.
Let $\frac{r_0}{p_0}=\frac{\gamma q'}{\lambda p'}$, $1/\gamma q'=\frac{1+\varkappa}{r_0}$, where $\varkappa>0$. Then $\gamma q'=\frac{r_0}{1+\varkappa}$, $\lambda p'=\frac{p_0}{1+\varkappa}$.
Since $p'=\frac{p_0}{(1+\varkappa)\lambda}$, $q'=\frac{r_0}{(1+\varkappa)\gamma}$ we have the following two equations for p_0, r_0,

$$\frac{3}{p_0}+\frac{2}{r_0}=\frac{3}{2},\quad \frac{3(1+\varkappa)\lambda}{p_0}+\frac{2(1+\varkappa)\gamma}{r_0}=\frac{7}{2}. \tag{2.46}$$

Hence, we obtain

$$\begin{aligned}&\frac{2}{r_0}(1+\varkappa)(\gamma-\lambda)=\frac{7}{2}-\frac{3}{2}(1+\varkappa)\lambda,\\ &\frac{3}{p_0}(1+\varkappa)(\lambda-\gamma)=\frac{7}{2}-\frac{3}{2}(1+\varkappa)\gamma.\end{aligned} \tag{2.47}$$

Consider the case $\gamma=\lambda$. Then $\gamma'=\lambda'$, $\lambda=\frac{7}{3(1+\varkappa)}$ and $\lambda'=\frac{7}{4-3\varkappa}$. Therefore

$$I_1\le\frac{c}{(\mu+\nu)^{1/q}}\mu(k,t)^{\frac{1+\varkappa}{r_0}}\bigg(\int_0^t|h|_{p'\lambda'}^{q'\lambda'}dt'\bigg)^{1/q'\lambda'}\equiv I_2.$$

Since $3/p'+2/q'=7/2$ we have that $3/p'\lambda'+2/q'\lambda'=\frac{7}{2\lambda'}=\frac{4-3\varkappa}{2}$. Let $p_*=p'\lambda'$, $q_*=q'\lambda'$. Then

$$\frac{3}{p_*}+\frac{2}{q_*}=\frac{4-3\varkappa}{2},$$

where $0<\varkappa<4/3$. For $p_*=q_*$ we have that $p_*=\frac{10}{4-3\varkappa}$. But $h\sim v^2$ so to have I_2 bounded we need that

$$v\in L_{20/(4-3\varkappa)}(\Omega^t),\quad\text{so}\quad I_2\le\frac{c}{(\mu+\nu)^{1/q}}\mu(k,t)^{\frac{1+\varkappa}{r_0}}|v|^2_{20/(4-3\varkappa),\Omega^t} \tag{2.48}$$

Looking for solutions such that $v\in L_\infty(0,t;H^2(\Omega))$ we see that (2.48) may hold.
Looking for the second term on the r.h.s. of (2.45) and using the above considerations we have to find an estimate for

$$|F|_{10/(4-3\varkappa),\Omega^t}.$$

Using the form of F we calculate

$$\begin{aligned}|F|_{10/(4-3\varkappa),\Omega^t}=\bigg(\int_{\Omega^t}&|\Delta^{-1}\text{div}\,[-\eta v_t-\eta v\cdot\nabla v\\ &+(p_\varrho(a)-p_\varrho(a+\eta))\nabla\eta+af_g+\eta f]|^{10/(4-3\varkappa)}dxdt'\bigg)^{(4-3\varkappa)/10}.\end{aligned} \tag{2.49}$$

Now, we examine the particular terms from (2.49)

$$\begin{aligned}
&|\Delta^{-1}\mathrm{div}\,(\eta v_t)|_{10/(4-3\varkappa),\Omega^t}\le c|\eta v_t|_{30/(22-9\varkappa),10/(4-3\varkappa),\Omega^t}\\
&\le|\eta|_{\infty,\Omega^t}|v_t|_{30/(22-9\varkappa),10/(4-3\varkappa),\Omega^t},\\
&|\Delta^{-1}\mathrm{div}\,(\eta v\cdot\nabla v)|_{10/(4-3\varkappa),\Omega^t}\le|\Delta^{-1}\partial_{x_i}\partial_{x_j}(\eta v_iv_j)|_{10/(4-3\varkappa),\Omega^t}\\
&\quad+|\Delta^{-1}\partial_{x_j}(\eta\Delta\varphi v_j)|_{10/(4-3\varkappa),\Omega^t}\\
&\quad+|\Delta^{-1}\partial_{x_j}(\partial_{x_i}\eta v_iv_j)|_{10/(4-3\varkappa),\Omega^t}\\
&\le c|\eta|_{\infty,\Omega^t}|v^2|_{10/(4-3\varkappa),\Omega^t}+c|\eta|_{\infty,\Omega^t}|\Delta\varphi v|_{10/(4-3\varkappa),\Omega^t}\\
&\quad+c|\nabla\eta v^2|_{30/(22-9\varkappa),10/(4-3\varkappa),\Omega^t}\le c|\eta|_{\infty,\Omega^t}|v|^2_{20/(4-3\varkappa),\Omega^t}\\
&\quad+c|\eta|_{\infty,\Omega^t}|\Delta\varphi|_{20/(4-3\varkappa),\Omega^t}|v|_{20/(4-3\varkappa),\Omega^t}\\
&\quad+c|\nabla\eta|_{30\lambda_1/(22-9\varkappa),\Omega^t}|v^2|_{30\lambda_2/(22-9\varkappa),\Omega^t}\equiv J_1+J_2,
\end{aligned}$$

where $1/\lambda_1+1/\lambda_2=1$.
Since

$$\frac{30\lambda_1}{22-9\varkappa}=6\quad\text{we have that}\quad\lambda_1=\frac{22-9\varkappa}{5}$$

so $\lambda_2=\frac{22-9\varkappa}{17-9\varkappa}$.
Therefore

$$\begin{aligned}
J_2&\le c|\nabla\eta|_{6,\infty,\Omega^t}|v^2|_{30/(17-9\varkappa),10/(4-3\varkappa),\Omega^t}\\
&\le c|\nabla\eta|_{6,\infty,\Omega^t}|v|^2_{60/(17-9\varkappa),20/(4-3\varkappa),\Omega^t}.
\end{aligned}$$

Continuing,

$$\begin{aligned}
&|\Delta^{-1}\mathrm{div}\,(p_\varrho(a)-p_\varrho(a+\eta))\nabla\eta|_{10/(4-3\varkappa),\Omega^t}\le c|\eta\nabla\eta|_{30/(22-9\varkappa),10/(4-3\varkappa),\Omega^t}\\
&\le c|\eta|_{\infty,\Omega^t}|\nabla\eta|_{30/(22-9\varkappa),10/(4-3\varkappa),\Omega^t},
\end{aligned}$$

where $\frac{30}{22-9\varkappa}\le 6$ so $5\le 22-9\varkappa$. Hence, $\varkappa\le\frac{17}{9}$. Finally, we have

$$|\Delta^{-1}\mathrm{div}\,f_g|_{10/(4-3\varkappa),\Omega^t}\le c|f_g|_{30/(22-9\varkappa),10/(4-3\varkappa),\Omega^t}$$

and

$$|\Delta^{-1}\mathrm{div}\,(\eta f)|_{10/(4-3\varkappa),\Omega^t}\le c|\eta|_{\infty,\Omega^t}|f|_{30/(22-9\varkappa),10/(4-3\varkappa),\Omega^t}.$$

Using the above estimates in (2.49) yields

$$\begin{aligned}
&|F|_{10/(4-3\varkappa),\Omega^t}\le c[|\eta|_{\infty,\Omega^t}|v_t|_{30/(22-9\varkappa),10/(4-3\varkappa),\Omega^t}\\
&\quad+|\eta|_{\infty,\Omega^t}|v|^2_{20/(4-3\varkappa),\Omega^t}\\
&\quad+|\eta|_{\infty,\Omega^t}|\Delta\varphi|_{20/(4-3\varkappa),\Omega^t}|v|_{20/(4-3\varkappa),\Omega^t}\\
&\quad+|\nabla\eta|_{6,\infty,\Omega^t}|v|^2_{60/(17-9\varkappa),20/(4-3\varkappa),\Omega^t}\\
&\quad+|\eta|_{\infty,\Omega^t}|\nabla\eta|_{30/(22-9\varkappa),10/(4-3\varkappa),\Omega^t}+|f_g|_{30/(22-9\varkappa),10/(4-3\varkappa),\Omega^t}\\
&\quad+|\eta|_{\infty,\Omega^t}|f|_{30/(22-9\varkappa),10/(4-3\varkappa),\Omega^t}]\\
&\equiv G_1(\varkappa,t)
\end{aligned}\tag{2.50}$$

In view of (2.50) the second term on the r.h.s. of (2.45) is bounded by

$$
(2.51)\qquad
\begin{aligned}
&\frac{1}{(\mu+\nu)^{1/q}}|F|_{p',q',\Omega^t}\le\frac{c}{(\mu+\nu)^{1/q}}\mu(k,t)^{\frac{1+\varkappa}{r_0}}G_1(\varkappa,t),\\
&|\Delta\varphi v|_{10/(4-3\varkappa),\Omega^t}\le|\Delta\varphi|_{20/(4-3\varkappa),\Omega^t}|v|_{20/(4-3\varkappa),\Omega^t}\equiv G_2,\\
&|\bar\eta|_{10/(4-3\varkappa),\Omega^t}\equiv G_3,\\
&G=G_1+G_2+G_3.
\end{aligned}
$$

Considering the last term on the r.h.s. of (2.45) we have $3/p'+2/q'=7/2$, $q'/p'=r_0/p_0$, $1/q'=(1+\varkappa)/r_0$ so $q'=r_0/(1+\varkappa)$, $p'=p_0/(1+\varkappa)$. Then equation $3/p_0+2/q_0=3/2$, $(3/p_0+2/q_0)(1+\varkappa)=7/2$ imply that $\varkappa=4/3$. Hence the last term on the r.h.s. of (2.45) is bounded by $\frac{c}{(\mu+\nu)^{1/q}}|\varphi_t|_{1,\infty,\Omega^t}\mu(k,t)\frac{1+\varkappa_0}{r_0}$, where $\varkappa_0=4/3$.
Employing estimates (2.48) and (2.51) in (2.45) implies the inequality

$$
(2.52)\qquad
\begin{aligned}
&|\varphi^{(k)}(t)|_{2,\infty,\Omega^t}+(\mu+\nu)^{1/2}|\nabla\varphi^{(k)}(t)|_{2,\Omega^t}\\
&\le\frac{c}{(\mu+\nu)^{1/q}}\mu(k,t)^{\frac{1+\varkappa}{r_0}}(|v|^2_{20/(4-3\varkappa),\Omega^t}+G(\varkappa,t)+|\varphi_t|_{1,\infty,\Omega^t}),
\end{aligned}
$$

where we used that $\mu(k,t)$ is finite, q and r_0 follow from the relations

$$
\frac{3}{p}+\frac{2}{q}=\frac{3}{2},\quad\frac{3}{p_0}+\frac{2}{r_0}=\frac{3}{2}\quad\text{and}\quad0<\varkappa<4/3.
$$

We apply Lemma 6.1 from [LSU, Ch. 2, Sect. 6]. Then for $k\ge\hat k$ we obtain from (2.52) the inequality

$$
(2.53)\qquad\|\varphi^{(k)}\|_{V(\Omega^t)}\le\gamma k\mu^{\frac{1+\varkappa}{r_0}}(k),
$$

where the norm of $V(\Omega^t)$ is determined by the l.h.s. of (2.52) and

$$
\gamma=\frac{c}{(\mu+\nu)^{1/q}}(|v|^2_{20/(4-3\varkappa),\Omega^t}+G(\varkappa,t)+|\varphi_t|_{1,\infty,\Omega^t}),\quad t\le T.
$$

Moreover, (2.44) is used in the form

$$
(2.54)\qquad|\varphi^{(k)}|_{p,q,\Omega^t}\le\beta(\mu+\nu)^{1/q}|\varphi^{(k)}|^{1-2/q}_{2,\infty,\Omega^t}|\nabla\varphi^{(k)}|^{2/q}_{2,\Omega^t}\le\beta|\varphi^{(k)}\|_{V(\Omega^t)},
$$

where $\beta=c/(\mu+\nu)^{1/q}$ appears in formula (3.2) from [LSU, Ch. 2, Sect. 3]. The β appears also in Theorem 6.1 from [LSU, Ch. 2, Sect. 6]. Then Theorem 6.1 yields the estimate for $\varphi>0$

$$
(2.55)\qquad\operatorname*{ess\,sup}_{\Omega^t}\varphi\le2\hat k[1+2^{2/\varkappa+1/\varkappa^2}(\beta\gamma)^{1+1/\varkappa}t^{\frac{1+\varkappa}{r_0}}\operatorname{meas}^{\frac{1+\varkappa}{p_0}}\Omega].
$$

Next we consider (2.41) in the form

$$a(-\varphi)_t - (\mu+\nu)\Delta(-\varphi) = a\Delta^{-1}\partial_{x_i}\partial_{x_j}(v_iv_j)$$
$$- a\Delta^{-1}\partial_{x_i}(\Delta\varphi v_i) - F + \bar\eta - a\fint_\Omega \varphi_t dx.$$

Repeating the considerations leading to (2.55) we derive the estimate for $\varphi < 0$

$$\operatorname*{ess\,sup}_{\Omega^t}(-\varphi) \le \gamma_* \tag{2.56}$$

From (2.55) and (2.56) we obtain (2.40) and conclude the proof. □

Remark 2.2. Since (2.41) is invariant with respect to the translation $\varphi \to \varphi + L$, $L = \text{const}$, we can consider the function

$$L + \varphi = L - (-\varphi) \ge L - \gamma_* = \varphi_*.$$

We assume that $L = \frac{c_0}{\nu^\varkappa}$ and we have that $\gamma_* = \frac{c_2}{\nu^\varkappa}$. So we take such c_0 that

$$\varphi_* = L - \gamma_* = \frac{c_0 - c_2}{\nu^\varkappa} = \frac{c_*}{\nu^\varkappa}, \quad c_* > 0.$$

Hence φ_* is used in the proof of Lemma 2.3.

The fact that φ is defined up to an arbitrary constant, say L, is connected with the considered periodic boundary conditions. Therefore, we have some freedom with determining the magnitude of φ.

Remark 2.3. To estimate the second term on the r.h.s. of (2.25) we need (2.28). Then we examine

$$\nu|\Delta\varphi|_{18/7,6,\Omega^t} \le c\nu^{2/3}|\nabla\Delta\varphi|^{2/3}_{2,\infty,\Omega^t}\nu^{1/3}|\nabla\varphi|^{1/3}_{2,\Omega^t} \equiv I. \tag{2.57}$$

Our aim is the following estimate for I

$$I \le c\frac{\Psi^\alpha}{\nu^\beta}, \tag{2.58}$$

where α, β are positive numbers and $\Psi = \nu|\nabla\varphi|_{3,1,2,\Omega^t}$.

Hence

$$I \le c\Psi^{2/3}\nu^{1/3}|\nabla\varphi|^{1/3}_{2,\Omega^t} \equiv I_1. \tag{2.59}$$

To derive the bound (2.58) for I_1 in the case of large ν we need to estimate $|\nabla\varphi|_{2,\Omega^t}$. For this purpose we use (2.31). To derive bound (2.58) from (2.31) we need to know that the coefficient with exponent is independent of ν.
For this purpose we assume that there exist positive constants c_1, c_2, $c_1 < c_2$ such that

(2.60) $$c_1 \le (\mu+\nu)^{\varkappa}\varphi_* \le c_2,$$

where $\varkappa = 3/2 - 3/p$, $2 < p < 6$.
The second term under the exponent in the r.h.s. of (2.31) equals is bounded by

$$\begin{aligned} I_2 &= \frac{1}{\varphi_*}\int_0^t \bigg|\,\fint_\Omega \varphi_t dx\bigg| dt' \\ &\le \frac{1}{\varphi_*}\int_0^t |\varphi_{t'}(t')|_1 dt' \le \frac{\sqrt{t}}{\varphi_*}|\varphi_t|_{1,2,\Omega^t} \le \frac{\sqrt{t}}{\varphi_*}\frac{\Psi}{\nu}. \end{aligned}$$

Then (2.39), which is a simpler version of (2.31), takes the form
(2.61)
$$\begin{aligned} a|\varphi(t)|_2^2 + (\mu+\nu)|\nabla\varphi|_{2,\Omega^t}^2 &\le c\exp(c|v|_{2p/(p-2),2/(1-\varkappa),\Omega^t}^{2/(1-\varkappa)} + c\sqrt{t}\Psi/\varphi_*\nu)\cdot \\ &\cdot\bigg[\frac{c}{\mu+\nu}(A_1^2|\Delta\varphi|_{3/2,\infty,\Omega^t}^2 + |F'|_{2,\Omega^t}^2 + t|\eta|_{2,\infty,\Omega^t}^2) \\ &+ a|\varphi(0)|_2^2\bigg]. \end{aligned}$$

From the problem for η

$$\eta_t + v\cdot\nabla\eta = -a\Delta\varphi - \eta\Delta\varphi, \quad \eta|_{t=0} = \eta(0),$$

we have

(2.62) $$\begin{aligned} |\eta(t)|_r &\le \exp(c|\Delta\varphi|_{\infty,1,\Omega^t})\bigg(\int_0^t |\Delta\varphi|_r dt' + |\eta(0)|_r\bigg) \\ &\le \exp\bigg(ct^{1/2}\frac{\Psi}{\nu}\bigg)\bigg(t^{1/2}\frac{\Psi}{\nu} + |\eta(0)|_r\bigg), \quad r \le \infty. \end{aligned}$$

Assuming that

(2.63) $$|\eta(0)|_r \le \frac{c_3}{\nu}$$

we obtain that

$$|\eta(t)|_r \le \frac{c(t)}{\nu}(\Psi + c_3). \tag{2.64}$$

In view of the assumptions of Lemma 2.4 we have

$$|F'|_{2,\Omega^t} \le \frac{c(t)}{\nu}(\Psi + c_3). \tag{2.65}$$

Then (2.61) implies

$$|\nabla\varphi|_{2,\Omega^t} \le \frac{c(t)(1+t)^{1/2}}{(\mu+\nu)\nu}(\Psi + c_3) + \frac{c}{(\mu+\nu)^{1/2}}|\varphi(0)|_2. \tag{2.66}$$

Inserting estimate (2.66) in (2.57) yields

$$\begin{aligned} I &\le c\Psi^{2/3}\bigg[\nu^{1/3}\bigg[(1+t)^{1/2}\bigg(\frac{\Psi}{\nu^2} + \frac{c_3}{\nu^2}\bigg)\bigg]^{1/3} \\ &\quad + \nu^{1/3}\frac{1}{\nu^{1/6}}|\varphi(0)|_2^{1/3}\bigg] \\ &\le c(1+t)^{1/6}\bigg[\frac{\Psi}{\nu^{1/3}} + \frac{\Psi^{2/3}c_3^{1/3}}{\nu^{1/3}}\bigg] + c\Psi^{2/3}\nu^{1/6}|\varphi(0)|_2^{1/3}. \end{aligned} \tag{2.67}$$

We see that (2.67) does not have from (2.58).
In view of restriction (2.60) we can assume that

$$\nu^{\varkappa}|\varphi(0)|_2 \le c_4. \tag{2.68}$$

The restriction is compatible with (2.60).
In view of (2.68) the last element on the r.h.s. of (2.67) is estimated in the following way

$$c\Psi^{2/3}\nu^{1/6}|\varphi(0)|_2^{1/3} \le c\Psi^{2/3}\nu^{1/6-\varkappa/3}(\nu^{\varkappa}|\varphi(0)|_2)^{1/3} = \frac{c\Psi^{2/3}c_5^{1/3}}{\nu^{\varkappa/3-1/6}}. \tag{2.69}$$

To have estimate (2.58) we need

$$\frac{\varkappa}{3} - \frac{1}{6} = \frac{1}{3}(\varkappa - 1/2) > 0.$$

Hence

$$\varkappa > 1/2 \ \text{ so } \ \frac{3}{2} - \frac{3}{p} > \frac{1}{2} \ \text{ implies that } \ p > 3. \tag{2.70}$$

Therefore, we can formulate the corollary.

Corollary 2.5. *Assume that there exist positive constants $c_1 - c_4$ such that*

$$\begin{aligned} &c_1 \le (\mu+\nu)^{\varkappa}\varphi_* \le c_2,\\ &|\eta(0)|_r \le \frac{c_3}{\nu}, \quad |\varphi(0)|_2 \le \frac{c_4}{\nu^{\varkappa}}, \end{aligned} \tag{2.71}$$

where $\varkappa = 3/2 - 3/p > 1/2$, $3 < p < 6$.
Assume that

$$\begin{aligned} &v \in L_{2p/(p-2),2/(1-\varkappa)}(\Omega^t), \quad F' \in L_2(\Omega^t),\\ &|F'|_{2,\Omega^t} \le \frac{c}{\nu}(\Psi + c_3), \qquad |\eta(t)|_r \le \frac{c}{\nu}(\Psi + c_3). \end{aligned}$$

Then

$$\nu|\Delta\varphi|_{18/7,6,\Omega^t} \le c\bigg(\frac{\Psi}{\nu^{1/3}} + \frac{\Psi^{2/3}}{\nu^{1/3}} + \frac{\Psi^{2/3}}{\nu^{\varkappa/3-1/6}}\bigg). \tag{2.72}$$

Lemma 2.6. *Let η be a solution to (1.4). Assume also that $\eta(0) \in L_r(\Omega)$, $\operatorname{div} v \in L_r(\Omega) \cap L_\infty(\Omega)$, $r \in [1,\infty]$. Then*

$$|\eta(t)|_r \le \exp\bigg[\bigg(1 - \frac{1}{r}\bigg)\int_0^t |\operatorname{div} v(t')|_\infty dt'\bigg]\bigg[\int_0^t a|\operatorname{div} v|_r dt' + |\eta(0)|_r\bigg]. \tag{2.73}$$

Proof. Multiplying $(1.4)_1$ by $\eta|\eta|^{r-2}$ and integrating over Ω yields

$$\frac{1}{r}\frac{d}{dt}|\eta|_r^r + \frac{1}{r}\int_\Omega v\cdot\nabla|\eta|^r dx + \int_\Omega |\eta|^r \operatorname{div} v dx + \int_\Omega a\eta|\eta|^{r-2}\operatorname{div} v dx = 0.$$

Integrating by parts we have

$$\frac{1}{r}\frac{d}{dt}|\eta|_r^r + \bigg(1 - \frac{1}{r}\bigg)\int_\Omega \operatorname{div} v|\eta|_r^r dx + a\int_\Omega \operatorname{div} v\eta|\eta|^{r-2} dx = 0.$$

Continuing,

$$\frac{1}{r}\frac{d}{dt}|\eta|_r^r \le \bigg(1 - \frac{1}{r}\bigg)|\operatorname{div} v|_\infty|\eta|_r^r + a|\operatorname{div} v|_r|\eta|_r^{r-1}.$$

Simplifying,

$$\frac{d}{dt}|\eta|_r \le (1 - 1/r)|\operatorname{div} v|_\infty|\eta|_r + a|\operatorname{div} v|_r.$$

Integrating with respect to time yields (2.73). This concludes the proof. □

Next, we obtain estimates for derivatives of η.

Lemma 2.7. *Let η be a solution to (1.4). Let $\nabla\varphi, \operatorname{rot}\psi \in L_1(0,t;H^3(\Omega))$, $\eta(0) \in H^2(\Omega)$. Then*
(2.74)
$$\|\eta(t)\|_2 \le \exp\left[c\int_0^t(\|\nabla\varphi(t')\|_3 + \|\operatorname{rot}\psi(t')\|_3)dt'\right]\left[c\int_0^t\|\nabla\varphi(t')\|_3dt' + \|\eta(0)\|_2\right].$$

Proof. Multiplying $(1.5)_1$ by η, integrating over Ω and by parts we get

$$\frac{1}{2}\frac{d}{dt}|\eta|_2^2 + \frac{1}{2}\int_\Omega\Delta\varphi\eta^2dx + a\int_\Omega\Delta\varphi\eta dx = 0.$$

By the Hölder inequality we have

$$\frac{d}{dt}|\eta|_2 \le \frac{1}{2}|\Delta\varphi|_\infty|\eta|_2 + a|\Delta\varphi|_2. \tag{2.75}$$

Differentiating $(1.4)_1$ with respect to x, multiplying by $\eta_{,x}$ and integrating over Ω implies

$$\frac{1}{2}\frac{d}{dt}|\eta_{,x}|_2^2 + a\int_\Omega\Delta\varphi_{,x}\eta_{,x}dx + \frac{1}{2}\int_\Omega\Delta\varphi\eta_{,x}^2dx + \int_\Omega v_{,x}\cdot\nabla\eta\eta_{,x}dx + \int_\Omega\eta\Delta\varphi_{,x}\eta_{,x}dx.$$

By the Hölder inequality we get

$$\frac{d}{dt}|\eta_{,x}|_2 \le c(|\operatorname{rot}\psi_{,x}|_\infty + |\nabla^2\varphi|_\infty)|\eta_{,x}|_2 + |\Delta\varphi_{,x}|_2 + c\|\eta\|_1\|\Delta\varphi_{,x}\|_1. \tag{2.76}$$

Finally, we differentiate $(1.4)_1$ twice with respect to x, multiply by $\eta_{,xx}$ and integrate over Ω. Then we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\eta_{,xx}|_2^2 &\le \frac{1}{2}|\Delta\varphi|_\infty|\eta_{,xx}|_2^2 + |v_{,x}|_\infty|\eta_{,xx}|_2^2 + |v_{,xx}|_4|\eta_{,x}|_4|\eta_{,xx}|_2\\ &\quad + a|\Delta\varphi_{,xx}|_2|\eta_{,xx}|_2 + |\Delta\varphi_{,x}|_4|\eta_{,x}|_4|\eta_{,xx}|_2 + |\eta|_\infty|\Delta\varphi_{,xx}|_2|\eta_{,xx}|_2.\end{aligned}$$

Simplifying we get

$$\frac{d}{dt}|\eta_{,xx}|_2 \le c(\|\nabla\varphi\|_3 + \|\operatorname{rot}\psi\|_3)\|\eta\|_2 + |\nabla\varphi_{,xxx}|_2. \tag{2.77}$$

Adding (2.75), (2.76) and (2.77) yields

$$\frac{d}{dt}\|\eta\|_2 \le c(\|\nabla\varphi\|_3 + \|\operatorname{rot}\psi\|_3)\|\eta\|_2 + c\|\nabla\varphi\|_3. \tag{2.78}$$

From (2.78) we have

$$\begin{aligned}&\frac{d}{dt}\bigg[\|\eta\|_2 \exp\bigg(-c\int_0^t(\|\nabla\varphi\|_3 + \|\operatorname{rot}\psi\|_3)dt'\bigg)\bigg]\\ &\le c\|\nabla\varphi\|_3 \exp\bigg(-c\int_0^t(\|\nabla\varphi\|_3 + \|\operatorname{rot}\psi\|_3)dt'\bigg).\end{aligned} \tag{2.79}$$

Integrating (2.79) with respect to time yields (2.74). This concludes the proof. □

Lemma 2.8. *Assume that* $\operatorname{rot}\psi, \nabla\varphi \in L_1(0,t;\Gamma_1^3(\Omega))$, $\eta(0) \in \Gamma_1^2(\Omega)$, $t \le T$. *Then*

$$\begin{aligned}|\eta(t)|_{2,1} \le \exp&\bigg[c\int_0^t(|\nabla\varphi(t')|_{3,1} + |\operatorname{rot}\psi(t')|_{3,1})dt'\bigg]\\ &\cdot\bigg[c\int_0^t|\nabla\varphi(t')|_{3,1}dt' + \|\eta(0)\|_2 + \|\eta_t(0)\|_1\bigg].\end{aligned} \tag{2.80}$$

Proof. We consider the equation

$$\eta_t = -v\cdot\nabla\eta - a\Delta\varphi - \eta\Delta\varphi \tag{2.81}$$

From (2.81) we have

$$\begin{aligned}\int_\Omega \eta_{xtt}\eta_{xt}dx = &-\int_\Omega(v_{xt}\cdot\nabla\eta + v_x\nabla\eta_t + v_t\nabla\eta_x + v\cdot\nabla\eta_{xt})\eta_{xt}dx\\ &- a\int_\Omega \Delta\varphi_{xt}\eta_{xt}dx - \int_\Omega(\eta_{xt}\Delta\varphi + \eta_x\Delta\varphi_t + \eta_t\Delta\varphi_x + \eta\Delta\varphi_{xt})\eta_{xt}dx.\end{aligned}$$

Hence

$$\frac{d}{dt}|\eta_{xt}|_2 \le c(|v|_{3,1} + |\nabla\varphi|_{3,1})(\|\eta\|_2 + \|\eta_t\|_1) + a|\Delta\varphi_{xt}|_2^2$$

and

$$\frac{d}{dt}|\eta|_2 \le c(|v_t|_\infty + |\nabla\varphi|_{3,1})(|\eta_t|_2 + \|\eta\|_2) + c|\Delta\varphi_t|_2.$$

Using (2.78) yields

$$\frac{d}{dt}(\|\eta\|_2+\|\eta_t\|_1)\le c|v|_{3,1}(\|\eta\|_2+\|\eta_t\|_1)+c|\nabla\varphi|_{3,1}(\|\eta\|_2+\|\eta_t\|_1)+|\nabla\varphi|_{3,1}.$$

This implies (2.80) and concludes the proof. □

Remark 2.4. From (2.25) for $r=6$ and from (2.72) we have

$$\tag{2.82} \begin{aligned} |v|_6 &\le c|\eta|_{18/7,6,\Omega^t}+\phi\bigg(\Psi^{2/3}/\nu^{1/3}+\Psi/\nu^{1/3}+\frac{\Psi^{2/3}}{\nu^{\varkappa/3-1/6}}\bigg)\\ &\quad+c|f|_{18/7,6,\Omega^t}+c|\varrho_0|_\infty^{1/6}|v_0|_6+A_1\equiv D_1, \end{aligned}$$

where $\phi=\phi(|\eta|_{6,\infty,\Omega^t},|v_t|_{2,\Omega^t},|v|_{6,\infty,\Omega^t},|f_g|_{2,\Omega^t},|f|_{2,\Omega^t})$, $\varkappa>1/2$ appears in (2.71) and Ψ is introduced in (2.58).

Differentiating (2.13) with respect to t yields

$$\tag{2.83} \begin{aligned} &\varrho v_{tt}+\varrho_tv_t+\varrho v\cdot\nabla v_t+\varrho_tv\cdot\nabla v+\varrho v_t\cdot\nabla v-\mu\Delta v_t\\ &\quad-\nu_1\nabla\mathrm{div}\,v_t+\nabla(p_t-\nu_2\mathrm{div}\,v_t)=\varrho f_t+\varrho_tf, \end{aligned}$$

where $\nu=\nu_1+\nu_2$, $\nu_i>0$, $i=1,2$.
Next we derive the result

Lemma 2.9. *Assume* $\eta\in L_\infty(0,T;\Gamma_1^2(\Omega))$, $v\in L_\infty(0,T;L_6(\Omega))\cap$ $\cap\ L_\infty(0,T;L_2(\Omega))$, $\Delta\varphi\in L_2(0,T;L_6(\Omega))$, $\Delta\varphi_t\in L_2(0,T;L_3(\Omega))$, $f_t\in L_2(0,T;L_{6/5}(\Omega))$, $f\in L_2(0,T;L_{3/2}(\Omega))$.
Recall the estimates

$$|v(t)|_2\le A_1\quad(\text{see (2.1)}),\qquad |v(t)|_6\le D_1\quad(\text{see (2.82)})$$

Then

$$\tag{2.84} \begin{aligned} &|v_t(t)|_2^2+\mu|\nabla v_t|_{2,2,\Omega^t}^2+\nu|\Delta\varphi_t|_{2,2,\Omega^t}^2\\ &\le c(a)\exp[cB_1(t)][B_2(t)+|\varrho_0|_\infty|v_t(0)|_2^2]\equiv D_2^2, \end{aligned}$$

$$\begin{aligned} &|v_t(t)|_2^2+\mu\|v_t\|_{1,2,\Omega^t}^2+\nu|\Delta\varphi_t|_{2,\Omega^t}^2\\ &\le c(a)\exp(c\bar B_1(t))[\bar B_2(t)+|\varrho_0|_\infty|v_t(0)|_2^2]\equiv\bar D_2^2, \end{aligned}$$

where

$$\begin{aligned} B_1(t)&=\int_0^t|\eta_t|_3^2dt'+\sup_t\|\eta(t)\|_2^2A_1^2+D_1^2A_1^2+A_1^2,\\ \bar B_1&=B_1+|\eta|_{2,\Omega^t}^2 \end{aligned}$$

and

$$B_2(t) = |\eta_t|^2_{2,2,\Omega^t} + \|\eta_t\|^2_{1,\infty,\Omega^t}|\Delta\varphi|^2_{6,2,\Omega^t}A_1^2$$
$$+ \|\eta_t\|^2_{1,\infty,\Omega^t}D_1^2A_1^2 + |\Delta\varphi_t|^2_{3,2,\Omega^t}A_1^2 + |f_t|^2_{6/5,2,\Omega^t} + \|\eta_t\|^2_{1,\infty,\Omega^t}|f|^2_{3/2,2,\Omega^t},$$
$$\bar{B}_2 = B_2 + (1 + |\eta|^2_{2,\infty,\Omega^t})|f|^2_{2,\Omega^t}.$$

Proof. Multiplying (2.83) by v_t, integrating over Ω, using $(1.1)_2$ and boundary conditions we get

$$\begin{aligned}(2.85)\qquad &\frac{1}{2}\frac{d}{dt}\int\limits_\Omega \varrho|v_t|^2dx + \mu|\nabla v_t|_2^2 + \nu_1|\mathrm{div}\, v_t|_2^2 = \int\limits_\Omega (p_t - \nu_2\mathrm{div}\, v_t)\mathrm{div}\, v_tdx\\ &- \int\limits_\Omega \varrho_tv_t^2dx - \int\limits_\Omega \varrho_tv\cdot\nabla v\cdot v_tdx - \int\limits_\Omega \varrho v_t\cdot\nabla v\cdot v_tdx\\ &+ \int\limits_\Omega (\varrho f_t + \varrho_tf)\cdot v_tdx.\end{aligned}$$

From (2.85) we have

$$\begin{aligned}(2.86)\qquad &\frac{1}{2}\frac{d}{dt}\int\limits_\Omega \varrho|v_t|^2dx + \mu|\nabla v_t|_2^2 + \nu|\mathrm{div}\, v_t|_2^2 \le \frac{\varepsilon}{2}|\mathrm{div}\, v_t|_2^2\\ &+ \frac{1}{2\varepsilon}|\eta_t|_2^2 - \int\limits_\Omega \varrho_tv_t^2dx - \int\limits_\Omega \varrho_tv\cdot\nabla v\cdot v_tdx - \int\limits_\Omega \varrho v_t\cdot\nabla v\cdot v_tdx\\ &+ \int\limits_\Omega (\varrho f_t + \varrho_tf)\cdot v_tdx.\end{aligned}$$

Now we estimate non-positive terms from the r.h.s. of (2.86). We estimate the third term by

$$\bigg|\int\limits_\Omega \eta_tv_t^2dx\bigg| \le |\eta_t|_3|v_t|_2|v_t|_6 \le \varepsilon/2|v_t|_6^2 + c/\varepsilon|\eta_t|_3^2|v_t|_2^2$$

We intergate by parts in the fourth term on the r.h.s. of (2.86). Then we have

$$\begin{aligned}&-\int\limits_\Omega \varrho_tv\cdot\nabla vv_tdx = \int\limits_\Omega v\cdot\nabla\varrho_tv\cdot v_tdx + \int\limits_\Omega \varrho_t\mathrm{div}\, vv\cdot v_tdx\\ &+ \int\limits_\Omega \varrho_tv\cdot\nabla v_t\cdot vdx \equiv \sum_{i=1}^3 K_i,\end{aligned}$$

where

$$|K_1| \le \varepsilon |v_t|_6^2 + c/\varepsilon |v|_6^4 |\eta_{xt}|_2^2,$$
$$|K_2| \le \varepsilon |v_t|_6^2 + c/\varepsilon |\eta_t|_6^2 |\Delta\varphi|_6^2 |v|_2^2,$$
$$|K_3| \le \varepsilon |\nabla v_t|_2^2 + c/\varepsilon |\eta_t|_6^2 |v|_6^4.$$

Integrating by parts in the fifth term on the r.h.s. of (2.86) yields

$$-\int_\Omega \varrho v_t \cdot \nabla v \cdot v_t dx = \int_\Omega v_t \cdot \nabla \varrho v \cdot v_t dx + \int_\Omega \varrho \Delta\varphi_t v \cdot v_t dx$$
$$+ \int_\Omega \varrho v_t \cdot \nabla v_t \cdot v dx \equiv \sum_{i=1}^3 L_i,$$

where

$$|L_1| \le \varepsilon |v_t|_6^2 + c/\varepsilon |\eta_x|_6^2 |v|_6^2 |v_t|_2^2,$$
$$|L_2| \le \varepsilon |v_t|_6^2 + c/\varepsilon |\varrho|_\infty^2 |\Delta\varphi_t|_3^2 |v|_2^2,$$
$$|L_3| \le \varepsilon |\nabla v_t|_2^2 + c/\varepsilon |\varrho|_\infty^2 |v_t|_3^2 |v|_6^2 \equiv L_3^1 + L_3^2.$$

To estimate L_3^2 we use the interpolation

$$|v_t|_3 \le c |\nabla v_t|_2^{1/2} |v_t|_2^{1/2} + |v_t|_2.$$

Hence

$$L_3^2 \le \varepsilon |\nabla v_t|_2^2 + c/\varepsilon |\varrho|_\infty^4 |v|_6^4 |v_t|_2^2 + c/\varepsilon |\varrho|_\infty^2 |v|_6^2 |v_t|_2^2.$$

Finally, we estimate the last term on the r.h.s. of (2.86) by

$$\left| \int_\Omega (\varrho f_t + \varrho_t f) \cdot v_t dx \right| \le \varepsilon |v_t|_6^2 + c/\varepsilon |\varrho|_\infty^2 |f_t|_{6/5}^2 + c/\varepsilon |\varrho_t|_6^2 |f|_{3/2}^2.$$

Employing the above estimates in (2.86) and assuming that ε is sufficiently small we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \varrho |v_t|^2 dx + \frac{\mu}{2} |\nabla v_t|_2^2 + \frac{\nu}{2} |\operatorname{div} v_t|_2^2 \\ &\le c[|\eta_t|_2^2 + |\eta_t|_3^2 |v_t|_2^2 + |v|_6^4 |\eta_{xt}|_2^2 + |\eta_t|_6^2 |\Delta\varphi|_6^2 |v|_2^2 + |\eta_t|_6^2 |v|_6^4 \\ &\quad + |\eta_x|_6^2 |v|_6^2 |v_t|_2^2 + |\varrho|_\infty^2 |\Delta\varphi_t|_3^2 |v|_2^2 + |\varrho|_\infty^4 |v|_6^4 |v_t|_2^2 + |\varrho|_\infty^2 |v|_6^2 |v_t|_2^2 \\ &\quad + |\varrho|_\infty^2 |f_t|_{6/5}^2 + |\varrho_t|_6^2 |f|_{3/2}^2]. \end{aligned} \tag{2.87}$$

Using Lemma 2.1 and that $a/2 \le \varrho \le 3a/2$ we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_\Omega \varrho|v_t|^2dx + \mu|\nabla v_t|_2^2 + \nu|\operatorname{div} v_t|_2^2 \\
&\le c(|\eta_t|_3^2 + |\eta_x|_6^2|v|_6^2 + |v|_6^4 + |v|_6^2)\int\limits_\Omega \varrho|v_t|^2dx \\
&\quad + c(|\eta_t|_2^2 + \|\eta_t\|_1^2|v|_6^4 + \|\eta_t\|_1^2|\Delta\varphi|_6^2A_1^2 + \|\eta_t\|_1^2|v|_6^4 \\
&\quad + |\Delta\varphi_t|_3^2A_1^2 + |f_t|_{6/5}^2 + \|\eta_t\|_1^2|f|_{3/2}^2).
\end{aligned}
\tag{2.88}
$$

Introduce the quantities

$$
\begin{aligned}
&B_1'(t) = |\eta_t(t)|_3^2 + |\eta_x(t)|_6^2|v(t)|_6^2 + |v(t)|_6^4 + |v(t)|_6^2, \\
&B_2'(t) = |\eta_t(t)|_2^2 + \|\eta_t(t)\|_1^2|v(t)|_6^4 + \|\eta_t(t)\|_1^2|\Delta\varphi(t)|_6^2A_1^2 \\
&\quad + |\Delta\varphi_t|_3^2A_1^2 + |f_t(t)|_{6/5}^2 + \|\eta_t(t)\|_1^2|f(t)|_{3/2}^2.
\end{aligned}
$$

Using the quantities in (2.88) it takes the form

$$
\frac{d}{dt}\int\limits_\Omega \varrho|v_t|^2dx + \mu|\nabla v_t|_2^2 + \nu|\operatorname{div} v_t|_2^2 \le cB_1'\int\limits_\Omega \varrho|v_t|^2dx + cB_2'. \tag{2.89}
$$

Integrating (2.89) with respect to time yields

$$
\begin{aligned}
&\int\limits_\Omega \varrho|v_t|^2dx + \mu\int\limits_0^t|\nabla v_t|_2^2dt' + \nu\int\limits_0^t|\operatorname{div} v_t|^2dt' \\
&\le \exp\left[c\int\limits_0^t B_1'(t')dt'\right]\left[c\int\limits_0^t B_2'(t')dt' + |\varrho_0|_\infty|v_t(0)|_2^2\right].
\end{aligned}
\tag{2.90}
$$

In view of Lemma 2.1 and (2.82) we have

(2.91)

$$
\begin{aligned}
&B_1 = \int\limits_0^t B_1'dt' \le |\eta_t|_{3,2,\Omega^t}^2 + \|\eta\|_{2,\infty,\Omega^t}^2A_1^2 + D_1^2A_1^2 + A_1^2, \\
&B_2 = \int\limits_0^t B_2'dt' \le |\eta_t|_{2,2,\Omega^t}^2 + \|\eta_t\|_{1,\infty,\Omega^t}^2D_1^2A_1^2 + \|\eta_t\|_{1,\infty,\Omega^t}^2|\Delta\varphi|_{6,2,\Omega^t}^2A_1^2 \\
&\quad + |\Delta\varphi_t|_{3,2,\Omega^t}^2A_1^2 + |f_t|_{6/5,2,\Omega^t}^2 \\
&\quad + \|\eta_t\|_{1,\infty,\Omega^t}^2|f|_{3/2,2,\Omega^t}^2.
\end{aligned}
$$

Exployting the estimates in (2.90) implies $(2.84)_1$. To obtain $(2.84)_2$ we use the continuity equation

$$a\int\limits_\Omega v_t dx + \int\limits_\Omega (\eta v)_t dx = \int\limits_\Omega (a+\eta) f dx.$$

Hence

$$\bigg|\int\limits_\Omega v_t dx\bigg| \le c[|\eta_t|_2|v|_2 + |\eta|_2|v_t|_2 + (1+|\eta|_2)|f|_2].$$

Then

$$\begin{aligned}|v_t|^2_{2,\Omega^t} \le \bigg|v_t - {\int\!\!\!\!\!-}_\Omega v_t dx\bigg|^2_{2,\Omega^t} + \bigg|{\int\!\!\!\!\!-}_\Omega v_t\bigg|^2_{2,\Omega^t} \le c(|\nabla v_t|^2_{2,\Omega^t}\\ + |\eta_t|^2_{2,\infty,\Omega^t} A_1^2 + |\eta|^2_{2,\infty,\Omega^t}|v_t|^2_{2,\Omega^t} + (1+|\eta|^2_{2,\infty,\Omega^t})|f|^2_{2,\Omega^t})\end{aligned}$$

Considering (2.88) and (2.89) we introduce

$$\begin{aligned}\bar B_1'(t) &= B_1'(t) + |\eta|_2^2, & \bar B_1(t) &= \int\limits_0^t \bar B_1'(t')dt',\\ \bar B_2'(t) &= B_2'(t) + (1+|\eta|_2^2)|f|_2^2, & \bar B_2(t) &= \int\limits_0^t \bar B_2'(t')dt',\end{aligned}$$

Therefore we derive $(2.84)_2$. This concludes the proof. □

To formulate Section 3 more explicitly we introduce the notation

Notation 2.10. We introduce the quantities

$$\begin{aligned}
&\chi_1^2(t) = \nu|\nabla\varphi|^2_{2,1,\infty,\Omega^t} + |\mathrm{rot}\,\psi|^2_{2,1,\infty,\Omega^t},\\
&\chi_2^2(t) = \nu|\nabla\varphi|^2_{3,1,2,\Omega^t} + |\mathrm{rot}\,\psi|^2_{3,1,2,\Omega^t},\\
&\Psi^2(t) = \nu^2|\nabla\varphi|^2_{3,1,2,\Omega^t},\\
&\Phi_1^2(t) = \nu(U_1^2+V_1^2) \equiv \nu(|\nabla\varphi|^2_{2,1,\infty,\Omega^t} + |\nabla\varphi|^2_{3,1,2,\Omega^t}) \equiv \Phi_{11}^2(t) + \Phi_{12}^2(t),\\
&\Phi_2^2(t) = \nu(U_2^2+V_2^2) \equiv |\mathrm{rot}\,\psi|^2_{2,1,\infty,\Omega^t} + |\mathrm{rot}\,\psi|^2_{3,1,2,\Omega^t} \equiv \Phi_{21}^2(t) + \Phi_{22}^2(t),\\
&\chi_0^2 = \nu|\nabla\varphi|^2_{2,1,\infty,\Omega^t} \equiv \Phi_{11}^2(t),\\
&\Phi_0^2(t) = \nu|\nabla\varphi(t)|^2_{2,1} + |\mathrm{rot}\,\psi(t)|^2_{2,1},\\
&\Phi_*^2(t) = |\nabla\varphi|^2_{3,1,2,\Omega^t} + |\mathrm{rot}\,\psi|^2_{3,1,2,\Omega^t}.
\end{aligned}$$

From (2.71) we have

$$(2.92)\qquad |\eta(t)|_r \le \exp\bigg[(1-1/r)\int_0^t |\Delta\varphi(t')|_\infty dt'\bigg]\bigg[\int_0^t |\Delta\varphi|_r dt' + |\eta(0)|_r\bigg].$$

From (2.92) we have

$$(2.93)\qquad |\eta(t)|_r \le \exp\bigg(t^{1/2}\frac{\Psi}{\nu}\bigg)\bigg[t^{1/2}\frac{\Psi}{\nu} + |\eta(0)|_r\bigg]$$

From (2.78) we derive the estimate
(2.94)

$$\begin{aligned}
|\eta(t)|_{2,1} &\le \exp\bigg[ct^{1/2}\bigg(\bigg(\int_0^t |\nabla\varphi(t')|_{3,1}^2 dt'\bigg)^{1/2} + \bigg(\int_0^t |\mathrm{rot}\,\psi(t')|_{3,1}^2 dt'\bigg)^{1/2}\bigg)\bigg]\\
&\quad\cdot\bigg[t^{1/2}\bigg(\int_0^t |\nabla\varphi(t')|_{3,1}^2 dt'\bigg)^{1/2} + |\eta(0)|_{2,1}\bigg]\\
&\le \exp\bigg[ct^{1/2}\bigg(\frac{\Psi}{\nu} + \Phi_2\bigg)\bigg]\bigg[t^{1/2}\frac{\Psi}{\nu} + |\eta(0)|_{2,1}\bigg].
\end{aligned}$$

From (2.82) and (2.93) we have

$$(2.95)\qquad \begin{aligned}
D_1^2 = c\bigg(&\exp\bigg(t^{1/2}\frac{\Psi}{\nu}\bigg)\bigg(t^{2/3}\frac{\Psi}{\nu} + t^{1/6}|\eta(0)|_3^2\bigg)\\
&+\phi\bigg(\frac{\Psi^{4/3}}{\nu^{2/3}} + \frac{\Psi^2}{\nu^{2/3}} + \frac{\Psi^{4/3}}{\nu^{2\varkappa/3-1/3}}\bigg)\bigg) + A_2^2 + A_1^2,
\end{aligned}$$

where $A_2 = |f|_{3,6,\Omega^t} + |\varrho_0|_\infty^{1/6}|v_0|_6$ (see also (2.82)), and from (2.83) we obtain

$$(2.96)\qquad \begin{aligned}
D_2^2 \le\ &\|\eta_t\|_{1,2,\Omega^t}^2\bigg(1 + D_1^2A_1^2 + \frac{\Psi^2}{\nu^2}A_1^2 + |f|_{3/2,2,\Omega^t}^2\bigg)\\
&+\frac{\Psi^2}{\nu^2}A_1^2 + |f_t|_{6/5,2,\Omega^t}^2,
\end{aligned}$$

where $A_1^2 = e^{\mu' T}\big(B(T)/(1-e^{-\mu' T/2}) + \frac{3}{2}\int_\Omega\big(\frac{1}{2}\varrho_0 v_0^2 + \frac{A}{\varkappa-1}\varrho_0^\varkappa\big)dx\big)$ (see (2.1)),

$$\begin{aligned}
B(T) = c\sup_{k\in\mathbb{N}_0}\exp\bigg(c\int_{kT}^{(k+1)T}\eta^2(t)dt\bigg)&\cdot\\
\cdot\,[|\varrho_0|_1^2|f|_{\infty,2,\Omega\times(kT,(k+1)T)}^2 + \mu' a^\varkappa + |\varrho_0|_1^2|f|_{\infty,1,\Omega^t}^2 + |\varrho_0|_2^2|v_0|_2^2].&
\end{aligned}$$

3 Differential inequality

We recall the considered equations

$$\eta_t + v\cdot\nabla\eta + a\Delta\varphi + \eta\Delta\varphi = 0, \quad \eta|_{t=0} = \eta(0), \tag{3.1}$$

and

$$\begin{aligned}&(a+\eta)v_t - \mu\Delta v - \nu\nabla\Delta\varphi + a_0\nabla\eta + (a+\eta)v\cdot\nabla v\\ &= [p_\varrho(a) - p_\varrho(a+\eta)]\nabla\eta + (a+\eta)f, \quad v|_{t=0} = v(0),\end{aligned} \tag{3.2}$$

where $a_0 = p_\varrho(a)$ and

$$v = \nabla\varphi + \operatorname{rot}\psi + G, \tag{3.3}$$

where G is defined below (1.6).

Lemma 3.1. *Assume that $|v|_2 \le A_1$ (see Lemma 2.1), $|v|_6 \le D_1$ (see (2.82)), $|v_t| \le D_2$ (see (2.84)). Then for sufficiently regular solutions to (3.1)–(3.3) we have*

$$\begin{aligned}&\frac{a}{2}\frac{d}{dt}\bigg(|\nabla\varphi|_2^2 + \frac{1}{\nu}|\operatorname{rot}\psi|_2^2\bigg) + \mu|\nabla^2\varphi|_2^2 + \nu|\Delta\varphi|_2^2 + \frac{\mu}{\nu}|\nabla\operatorname{rot}\psi|_2^2\\ &\le \frac{c}{\nu}[|\eta|_2^2 + |\eta|_3^2|v_t|_2^2 + (1+|\eta|_\infty^2)|\Delta\varphi|_3^2A_1^2 + (1+|\eta|_\infty^2)D_1^2|v|_3^2\\ &\quad + |\eta|_3^2|\nabla\eta|_2^2 + (1+|\eta|_3^2)|f|_2^2]\\ &\le \frac{c}{\nu}\phi(|\eta|_\infty, D_1)[|\eta|_2^2 + |v_t|_2^2 + |\Delta\varphi|_3^2 + |v|_3^2 + |\nabla\eta|_2^2 + |f|_2^2]\end{aligned} \tag{3.4}$$

and integrating the above inequality with respect to time yields

$$\begin{aligned}&\frac{a}{2}\bigg(|\nabla\varphi(t)|_2^2 + \frac{1}{\nu}|\operatorname{rot}\psi(t)|_2^2\bigg) + \mu|\nabla^2\varphi|_{2,2,\Omega^t}^2 + \nu|\Delta\varphi|_{2,2,\Omega^t}^2 + \frac{\mu}{\nu}|\nabla\operatorname{rot}\psi|_{2,2,\Omega^t}^2\\ &\le \frac{c}{\nu}[|\eta|_{2,2,\Omega^t}^2 + |\eta|_{3,\infty,\Omega^t}^2D_2^2 + (1+|\eta|_{\infty,\infty,\Omega^t}^2)|\Delta\varphi|_{3,2,\Omega^t}^2A_1^2\\ &\quad + (1+\|\eta\|_{2,\infty,\Omega^t}^2)D_1^2A_1^2 + |\eta|_{3,\infty,\Omega^t}^2 + |\nabla\eta|_{2,2,\Omega^t}^2 + (1+|\eta|_{3,\infty,\Omega^t}^2)|f|_{2,2,\Omega^t}^2]\\ &\quad + \frac{a}{2}\bigg(|\nabla\varphi(0)|_2^2 + \frac{1}{\nu}|\operatorname{rot}\psi(0)|_2^2\bigg).\end{aligned}$$

Proof. Multiplying (3.2) by $\nabla\varphi$ and integrating over Ω yields

$$\begin{aligned}&\frac{a}{2}\frac{d}{dt}|\nabla\varphi|_2^2 + \mu|\nabla^2\varphi|_2^2 + \nu|\Delta\varphi|_2^2 = -a_0\int_\Omega\nabla\eta\cdot\nabla\varphi dx\\ &\quad - \int_\Omega\eta v_t\cdot\nabla\varphi dx - \int_\Omega(a+\eta)v\cdot\nabla v\cdot\nabla\varphi dx\\ &\quad + \int_\Omega[p_\varphi(a) - p_\varphi(a+\eta)]\nabla\eta\cdot\nabla\varphi dx + \int_\Omega(a+\eta)f\cdot\nabla\varphi dx.\end{aligned} \tag{3.5}$$

Integrating by parts in the first term on the r.h.s. of (3.5) we estimate it by

$$\varepsilon|\Delta\varphi|_2^2 + c/\varepsilon|\eta|_2^2.$$

The second term on the r.h.s. of (3.5) is bounded by

$$\varepsilon|\nabla\varphi|_6^2 + c/\varepsilon|\eta|_3^2|v_{,t}|_2^2 \le \varepsilon|\nabla\varphi|_6^2 + c/\varepsilon|\eta|_3^2 D_2^2,$$

where (2.67) is used.
The third term on the r.h.s. of (3.5) is expressed in the form

$$-a\int\limits_\Omega v\cdot\nabla v\cdot\nabla\varphi dx - \int\limits_\Omega \eta v\cdot\nabla v\cdot\nabla\varphi dx \equiv I_1 + I_2.$$

We drop the factor a in I_1 for simplicity. Integrating by parts we express it in the form

$$I_1 = \int\limits_\Omega \operatorname{div} vv\cdot\nabla\varphi dx + \int\limits_\Omega v\cdot v\cdot\nabla\nabla\varphi dx \equiv I_{11} + I_{12}.$$

Then

$$|I_{11}| \le \varepsilon|\nabla\varphi|_6^2 + c/\varepsilon|\Delta\varphi|_3^2|v|_2^2 \le \varepsilon|\nabla\varphi|_6^2 + c/\varepsilon|\Delta\varphi|_3^2 A_1^2.$$

and

$$|I_{12}| \le \varepsilon|\nabla^2\varphi|_2^2 + c/\varepsilon|v|_6^2|v|_3^2 \le \varepsilon|\nabla^2\varphi|_2^2 + c/\varepsilon D_1^2|v|_3^2.$$

Next we estimate I_2. Integration by parts yields

$$\begin{aligned} I_2 &= \int\limits_\Omega \nabla\eta\cdot vv\cdot\nabla\varphi dx + \int\limits_\Omega \eta\Delta\varphi v\cdot\nabla\varphi dx + \int\limits_\Omega \eta vv\cdot\nabla^2\varphi dx \\ &\equiv I_2^1 + I_2^2 + I_2^3, \end{aligned}$$

where

$$\begin{aligned} |I_2^1| &\le \varepsilon|\nabla\varphi|_6^2 + c/\varepsilon|\nabla\eta|_6^2|v|_6^2|v|_2^2 \le \varepsilon|\nabla\varphi|_6^2 + c/\varepsilon\|\eta\|_2^2 D_1^2 A_1^2, \\ |I_2^2| &\le \varepsilon|\nabla\varphi|_6^2 + c/\varepsilon|\eta|_\infty^2|\Delta\varphi|_3^2|v|_2^2 \le \varepsilon|\nabla\varphi|_6^2 + c/\varepsilon|\eta|_\infty^2|\Delta\varphi|_3^2 A_1^2, \\ |I_2^3| &\le \varepsilon|\nabla^2\varphi|_2^2 + c/\varepsilon|\eta|_\infty^2 D_1^2|v|_3^2. \end{aligned}$$

The fourth term on the r.h.s. of (3.5) is bounded by

$$c\int\limits_\Omega |\eta|\,|\nabla\eta|\,|\nabla\varphi| dx \le \varepsilon|\nabla\varphi|_6^2 + c/\varepsilon|\eta|_3^2|\nabla\eta|_2^2.$$

Finally, the last term on the r.h.s. of (3.5) equals

$$I_3 = a\int\limits_\Omega f_g\cdot\nabla\varphi dx + \int\limits_\Omega \eta f\cdot\nabla\varphi dx.$$

Hence, it is bounded by

$$|I_3| \le \varepsilon|\nabla\varphi|_2^2 + c/\varepsilon|f_g|_2^2 + \varepsilon|\nabla\varphi|_6^2 + c/\varepsilon|\eta|_3^2|f|_2^2.$$

Employing the above estimates in (3.5) and assuming that ε is sufficiently small yields

$$\begin{aligned}(3.6)\qquad & a\frac{d}{dt}|\nabla\varphi|_2^2 + \mu|\nabla^2\varphi|_2^2 + \nu|\Delta\varphi|_2^2 \le \frac{c}{\nu}[|\eta|_2^2 + |\eta|_3^2|v_t|_2^2\\ &\quad + (1+|\eta|_\infty^2)|\Delta\varphi|_3^2A_1^2 + (1+|\eta|_\infty^2)D_1^2|v|_3^2 + \|\eta\|_2^2D_1^2|v|_2^2\\ &\quad + |\eta|_3^2|\nabla\eta|_2^2 + (1+|\eta|_3^2)|f|_2^2].\end{aligned}$$

Multiplying (3.2) by $\operatorname{rot}\psi$ and integrating the result over Ω implies

$$\begin{aligned}(3.7)\qquad & \frac{a}{2}\frac{d}{dt}|\operatorname{rot}\psi|_2^2 + \mu|\nabla\operatorname{rot}\psi|_2^2 = -\int\limits_\Omega \eta v_t\cdot\operatorname{rot}\psi dx\\ &\quad -\int\limits_\Omega (a+\eta)v\cdot\nabla v\cdot\operatorname{rot}\psi dx + \int\limits_\Omega (a+\eta)f\cdot\operatorname{rot}\psi dx.\end{aligned}$$

We estimate the first term on the r.h.s. of (3.7) by

$$\varepsilon|\operatorname{rot}\psi|_6^2 + c/\varepsilon|\eta|_3^2|v_t|_2^2.$$

The second term on the r.h.s. of (3.7) is expressed in the form

$$-a\int\limits_\Omega v\cdot\nabla v\cdot\operatorname{rot}\psi dx - \int\limits_\Omega \eta v\cdot\nabla v\cdot\operatorname{rot}\psi dx \equiv J_1 + J_2.$$

Integrating by parts in J_1 yields

$$J_1 = \int\limits_\Omega \Delta\varphi v\cdot\operatorname{rot}\psi dx + \int\limits_\Omega v\cdot v\cdot\nabla\operatorname{rot}\psi dx \equiv J_1^1 + J_1^2,$$

where

$$\begin{aligned}|J_1^1| &\le \varepsilon|\operatorname{rot}\psi|_6^2 + c/\varepsilon|\Delta\varphi|_3^2|v|_2^2 \le \varepsilon|\operatorname{rot}\psi|_6^2 + c/\varepsilon|\Delta\varphi|_3^2A_1^2,\\ |J_1^2| &\le \varepsilon|\nabla\operatorname{rot}\psi|_2^2 + c/\varepsilon|v|_6^2|v|_3^2 \le \varepsilon|\nabla\operatorname{rot}\psi|_2^2 + c/\varepsilon D_1^2|v|_3^2.\end{aligned}$$

Next we examine J_2. Integration by parts gives

$$
\begin{aligned}
J_2 &= \int_\Omega \nabla\eta\cdot vv\cdot \operatorname{rot}\psi dx + \int_\Omega \eta\Delta\varphi v\cdot\operatorname{rot}\psi dx + \int_\Omega \eta v\cdot v\cdot\nabla\operatorname{rot}\psi dx\\
&\equiv J_2^1 + J_2^2 + J_2^3,
\end{aligned}
$$

where

$$
\begin{aligned}
&|J_2^1| \le \varepsilon|\operatorname{rot}\psi|_6^2 + c/\varepsilon|\nabla\eta|_6^2|v|_6^2|v|_2^2 \le \varepsilon|\operatorname{rot}\psi|_6^2 + c/\varepsilon\|\eta\|_2^2 D_1^2|v|_2^2,\\
&|J_2^2| \le \varepsilon|\operatorname{rot}\psi|_6^2 + c/\varepsilon|\eta|_\infty^2|\Delta\varphi|_3^2|v|_2^2 \le \varepsilon|\operatorname{rot}\psi|_6^2 + c/\varepsilon|\eta|_\infty^2|\Delta\varphi|_3^2 A_1^2,\\
&|J_2^3| \le \varepsilon|\nabla\operatorname{rot}\psi|_2^2 + c/\varepsilon|\eta|_\infty^2|v|_6^2|v|_3^2 \le \varepsilon|\nabla\operatorname{rot}\psi|_2^2 + c/\varepsilon|\eta|_\infty^2 D_1^2|v|_3^2.
\end{aligned}
$$

Finally, the last term on the r.h.s. of (3.7) has the form

$$
a\int_\Omega f_r\cdot\operatorname{rot}\psi dx + \int_\Omega \eta f\cdot\operatorname{rot}\psi dx \equiv J_3.
$$

Hence

$$
|J_3| \le \varepsilon|\operatorname{rot}\psi|_2^2 + c/\varepsilon|f_r|_2^2 + \varepsilon|\operatorname{rot}\psi|_6^2 + c/\varepsilon|\eta|_3^2|f|_2^2.
$$

Employing the above estimates in (3.7) and using that ε is sufficiently small we derive the inequality

$$
\begin{aligned}
&\frac{a}{2}\frac{d}{dt}|\operatorname{rot}\psi|_2^2 + \mu|\nabla\operatorname{rot}\psi|_2^2 \le c[|\eta|_3^2|v_t|_2^2 + (1+|\eta|_\infty^2)|\Delta\varphi|_3^2 A_1^2\\
&\quad + (1+|\eta|_\infty^2)D_1^2|v|_3^2 + (1+|\eta|_3^2)|f|_2^2].
\end{aligned}
\tag{3.8}
$$

Multiplying (3.8) by $1/\nu$ and adding to (3.6) yields

$$
\begin{aligned}
&\frac{a}{2}\frac{d}{dt}\left(|\nabla\varphi|_2^2 + \frac{1}{\nu}|\operatorname{rot}\psi|_2^2\right) + \mu|\nabla^2\varphi|_2^2 + \nu|\Delta\varphi|_2^2 + \frac{\mu}{\nu}|\nabla\operatorname{rot}|_2^2\\
&\le \frac{c}{\nu}[|\eta|_2^2 + |\eta|_3^2|v_t|_2^2 + (1+|\eta|_\infty^2)|\Delta\varphi|_3^2 A_1^2 + (1+|\eta|_\infty^2)D_1^2|v|_3^2\\
&\quad + |\eta|_3^2|\nabla\eta|_2^2 + (1+|\eta|_3^2)|f|_2^2].
\end{aligned}
\tag{3.9}
$$

Using that

$$
(1+|\eta|_\infty^2)D_1^2|v|_3^2 \le \varepsilon|\nabla v|_2^2 + c/\varepsilon(1+|\eta|_\infty^2)^2 D_1^4 A_1^2
$$

we obtain (3.4).

Integrating (3.9) with respect to time and using (2.1), (2.67) gives

$$
\begin{aligned}
&\frac{a}{2}\bigg(|\nabla\varphi(t)|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi(t)|_2^2\bigg)+\mu|\nabla^2\varphi|_{2,2,\Omega^t}^2+\nu|\Delta\varphi|_{2,2,\Omega^t}^2\\
&+\frac{\mu}{\nu}|\nabla\mathrm{rot}\,\psi|_{2,2,\Omega^t}^2\le\frac{c}{\nu}[|\eta|_{2,2,\Omega^t}^2+|\eta|_{3,\infty,\Omega^t}^2D_2^2\\
&+(1+|\eta|_{\infty,\infty,\Omega^t}^2)|\Delta\varphi|_{3,2,\Omega^t}^2A_1^2+(1+\|\eta\|_{2,\infty,\Omega^t}^2)D_1^2A_1^2\\
&+|\eta|_{3,\infty,\Omega^t}^2|\nabla\eta|_{2,2,\Omega^t}^2+(1+|\eta|_{3,\infty,\Omega^t}^2)|f|_{2,2,\Omega^t}^2]\\
&+\frac{a}{2}\bigg(|\nabla\varphi(0)|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi(0)|_2^2\bigg).
\end{aligned}
\tag{3.10}
$$

The above inequalities imply (3.4) and conclude the proof. □

Remark 3.1. To simplify considerations we introduce the quantity

$$
\phi_1=(1+\|\eta\|_{2,\infty,\Omega^t}^2)(1+A_1^2+D_1^2)
\tag{3.11}
$$

Then (3.10) and $(3.4)_2$ take the form

$$
\begin{aligned}
&a\bigg(|\nabla\varphi(t)|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi(t)|_2^2\bigg)+\mu\bigg(|\nabla^2\varphi|_{2,\Omega^t}^2+\frac{1}{\nu}|\nabla\mathrm{rot}\,\psi|_{2,\Omega^t}^2\bigg)\\
&+\nu|\Delta\varphi|_{2,\Omega^t}^2\le\frac{c}{\nu}\phi_1\bigg[\|\eta\|_{1,2,\Omega^t}^2+A_1^2+D_2^2+\frac{\Psi^2}{\nu^2}+|f|_{2,\Omega^t}^2\bigg]\\
&+a\bigg(|\nabla\varphi(0)|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi(0)|_2^2\bigg).
\end{aligned}
\tag{3.12}
$$

To obtain estimates for time derivatives we express (3.2) in the form

$$
\begin{aligned}
&\nabla\varphi_t+\mathrm{rot}\,\psi_t+G_t-\frac{\mu}{a}\Delta v-\frac{\nu}{a}\nabla\Delta\varphi+\frac{a_0}{a+\eta}\nabla\eta\\
&=-\frac{\mu}{a}\frac{\eta}{a+\eta}\Delta v-\frac{\nu}{a}\frac{\eta}{a+\eta}\nabla\Delta\varphi-v\cdot\nabla v\\
&\quad+\frac{1}{a+\eta}[p_\varrho(a)-p_\varrho(a+\eta)]\nabla\eta+f.
\end{aligned}
\tag{3.13}
$$

Lemma 3.2. *Let the assumptions of Lemma 3.1 hold. Let ϕ_2 be defined by (3.21). Then*

(3.14)

$$
\begin{aligned}
&\frac{d}{dt}\bigg(|\nabla\varphi_t|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_t|_2^2\bigg)+\frac{\mu}{a}\|\nabla\varphi_t\|_1^2+\frac{\mu}{\nu a}\|\mathrm{rot}\,\psi_t\|_1^2+\frac{\nu}{a}|\Delta\varphi_t|_2^2\\
&\le\frac{c}{\nu}\phi_2[|\eta|_{2,1}^2+|\eta|_{2,1}^2|\nabla\mathrm{rot}\,\psi|_3^2+|\nabla^2\varphi|_3^2+|\eta|_{2,1}^2|\nabla\mathrm{rot}\,\psi_t|_2^2+\|\nabla\varphi_{,t}\|_1^2\\
&\quad+|\Delta\varphi|_3^2(|\nabla\mathrm{rot}\,\psi_t|_2^2+\|\nabla\varphi_{,t}\|_1^2+D_2^2)+\|\nabla\varphi_{,t}\|_1^2(|\nabla v|_2^2+D_1^2)\\
&\quad+|v_t|_3^2D_1^2+|f_t|_2^2]\\
&\quad+c\nu[|\eta|_{2,1}^2(1+|\eta|_{2,1}^2)|\Delta\varphi|_3^2+\|\eta\|_2^2\|\nabla\varphi_t\|_1^2].
\end{aligned}
$$

(3.15)

$$
\begin{aligned}
&|\nabla\varphi_t(t)|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_t(t)|_2^2+\frac{\mu}{a}\|\nabla\varphi_t\|_{1,2,\Omega^t}^2+\frac{\nu}{a}|\Delta\varphi_t|_{2,2,\Omega^t}^2+\frac{\mu}{\nu a}\|\mathrm{rot}\,\psi_t\|_{1,2,\Omega^t}^2\\
&\le\frac{c}{\nu}\phi_2(|\eta|_{2,1,\infty,\Omega^t})\bigg[|\eta|_{2,1,2,\Omega^t}^2+|\eta|_{2,1,\infty,\Omega^t}^2(|\nabla\mathrm{rot}\,\psi|_{3,2,\Omega^t}^2\\
&\quad+|\nabla\mathrm{rot}\,\psi_t|_{2,2,\Omega^t}^2)+\frac{\Psi^2}{\nu^2}+\frac{\chi_0^2}{\nu}\bigg(|\nabla\mathrm{rot}\,\psi_t|_{2,2,\Omega^t}^2+D_2^2+A_1^2+D_1^2\\
&\quad+\frac{\psi^2}{\nu^2}D_2^2D_1^2\bigg)+|f_t|_{2,2,\Omega^t}^2\bigg]+c\nu|\eta|_{2,1,\infty,\Omega^t}^2(1+|\eta|_{2,1,\infty,\Omega^t}^2)\frac{\Psi^2}{\nu^2}+|\nabla\varphi_t(0)|_2^2\\
&\quad+\frac{1}{\nu}|\mathrm{rot}\,\psi_t(0)|_2^2.
\end{aligned}
$$

Proof. Differentiate (3.13) with respect to t, multiply by $\nabla\varphi_t$ and integrate over Ω. Then we have

(3.16)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla\varphi_t|_2^2+\frac{\mu}{a}|\nabla^2\varphi_t|_2^2+\frac{\nu}{a}|\Delta\varphi_t|_2^2=-\int\limits_\Omega\left(\frac{a_0}{a+\eta}\nabla\eta\right)_t\cdot\nabla\varphi_t dx\\
&\quad-\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\Delta v\right)_{,t}\cdot\nabla\varphi_t dx-\frac{\nu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\nabla\Delta\varphi\right)_{,t}\cdot\nabla\varphi_t dx\\
&\quad-\int\limits_\Omega(v\cdot\nabla v)_{,t}\cdot\nabla\varphi_t dx+\int\limits_\Omega\left[\frac{1}{a+\eta}(p_\varrho(a)-p_\varrho(a+\eta))\nabla\eta\right]_{,t}\cdot\nabla\varphi_t dx\\
&\quad+\int\limits_\Omega f_{gt}\cdot\nabla\varphi_t dx.
\end{aligned}
$$

Now, we estimate the terms from the r.h.s. of (3.16). The first term on the r.h.s. of (3.16) is bounded by

$$
\begin{aligned}
c\int\limits_\Omega(|\nabla\eta_t|\,|\nabla\varphi_t|+|\eta_t|\,|\nabla\eta|\,|\nabla\varphi_t|)dx&\le\varepsilon|\nabla\varphi_t|_2^2+c/\varepsilon|\nabla\eta_t|_2^2\\
&\quad+\varepsilon|\nabla\varphi_t|_6^2+c/\varepsilon|\eta_t|_3^2|\nabla\eta|_2^2.
\end{aligned}
$$

The second term on the r.h.s. of (3.16) equals

$$
-\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\Delta\mathrm{rot}\,\psi\right)_{,t}\cdot\nabla\varphi_t dx-\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\Delta\nabla\varphi\right)_{,t}\cdot\nabla\varphi_t dx\equiv I_1+I_2,
$$

where

$$
\begin{aligned}
I_1&=-\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_t\Delta\mathrm{rot}\,\psi\cdot\nabla\varphi_t dx-\frac{\mu}{a}\int\limits_\Omega\frac{\eta}{a+\eta}\Delta\mathrm{rot}\,\psi_t\cdot\nabla\varphi_t dx\\
&\equiv I_{11}+I_{12}.
\end{aligned}
$$

Integrating by parts in I_{11} yields

$$I_{11} = \frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta\eta} \nabla\eta\eta_t\nabla\mathrm{rot}\,\psi\cdot\nabla\varphi_t dx$$
$$+\frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta} \nabla\eta_t\cdot\nabla\mathrm{rot}\,\psi\cdot\nabla\varphi_t dx$$
$$+\frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta} \eta_t\nabla\mathrm{rot}\,\psi\cdot\nabla\nabla\varphi_t dx \equiv I_{11}^1 + I_{11}^2 + I_{11}^3.$$

Continuing, we have

$$|I_{11}^1| \le \varepsilon|\nabla\varphi_t|_6^2 + c/\varepsilon|\nabla\eta|_6^2|\eta_t|_6^2|\nabla\mathrm{rot}\,\psi|_2^2,$$
$$|I_{11}^2| \le \varepsilon|\nabla\varphi_t|_6^2 + c/\varepsilon|\nabla\eta_t|_2^2|\nabla\mathrm{rot}\,\psi|_3^2,$$
$$|I_{11}^3| \le \varepsilon|\nabla^2\varphi_t|_2^2 + c/\varepsilon|\eta_t|_6^2|\nabla\mathrm{rot}\,\psi|_3^2.$$

Next we examine I_{12}. Integration by parts yields

$$I_{12} = \frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta} \nabla\eta\cdot\nabla\mathrm{rot}\,\psi_t\cdot\nabla\varphi_t dx$$
$$+\frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}\nabla\mathrm{rot}\,\psi_t\cdot\nabla^2\varphi_t dx \equiv I_{12}^1 + I_{12}^2.$$

Continuing, we have

$$|I_{12}^1| \le \varepsilon|\nabla\varphi_t|_6^2 + c/\varepsilon|\nabla\eta|_3^2|\nabla\mathrm{rot}\,\psi_t|_2^2,$$
$$|I_{12}^2| \le \varepsilon|\nabla^2\varphi_t|_2^2 + c/\varepsilon|\eta|_\infty^2|\nabla\mathrm{rot}\,\psi_t|_2^2.$$

Summarizing the estimates yields

$$|I_1| \le \varepsilon\|\nabla\varphi_t\|_1^2 + c/\varepsilon|\eta|_{2,1}^4|\nabla\mathrm{rot}\,\psi|_2^2 + c/\varepsilon|\eta|_{2,1}^2|\nabla\mathrm{rot}\,\psi|_3^2$$
$$+ c/\varepsilon\|\eta\|_2^2|\nabla\mathrm{rot}\,\psi_{,t}|_2^2.$$

Next, we estimate I_2. Performing differentiation with respect to time implies

$$I_2 = -\frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta} \eta_t\Delta\nabla\varphi\cdot\nabla\varphi_t dx - \frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}\Delta\nabla\varphi_t\nabla\varphi_t dx \equiv I_{21} + I_{22}.$$

Consider I_{21}. Integration by parts gives

$$I_{21} = \frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta\eta} \nabla\eta\eta_t\Delta\varphi\cdot\nabla\varphi_t dx + \frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta} \nabla\eta_t\Delta\varphi\cdot\nabla\varphi_t dx$$
$$+\frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta} \eta_t\Delta\varphi\cdot\Delta\varphi_t dx \equiv I_{21}^1 + I_{21}^2 + I_{21}^3.$$

Continuing, we have

$$\begin{aligned}|I_{21}^1| &\le \varepsilon|\nabla\varphi_t|_6^2 + c/\varepsilon|\nabla\eta|_6^2|\eta_t|_6^2|\Delta\varphi|_2^2,\\ |I_{21}^2| &\le \varepsilon|\nabla\varphi_t|_6^2 + c/\varepsilon|\nabla\eta_t|_2^2|\Delta\varphi|_3^2,\\ |I_{21}^3| &\le \varepsilon|\Delta\varphi_t|_2^2 + c/\varepsilon|\eta_t|_6^2|\Delta\varphi|_3^2.\end{aligned}$$

Integration by parts in I_{22} implies

$$I_{22} = \frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\nabla\eta\nabla^2\varphi_t\cdot\nabla\varphi_t dx + \frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}|\nabla^2\varphi_t|^2 dx \equiv I_{22}^1 + I_{22}^2,$$

where I_{22}^2 is qabsorbed by the second term on the l.h.s. of (3.16) in such way that

$$\begin{aligned}&\frac{\mu}{a}\int\limits_\Omega |\nabla^2\varphi_t|^2 dx - \frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}|\nabla^2\varphi_t|^2 dx\\ &= \frac{\mu}{a}\int\limits_\Omega \frac{a}{a+\eta}|\nabla^2\varphi_t|^2 dx \ge \frac{2}{3}\frac{\mu}{a}\int\limits_\Omega |\nabla^2\varphi_t|^2 dx,\end{aligned}$$

where we used that $|\eta|_\infty \le a/2$. Finally,

$$|I_{22}^1| \le \varepsilon|\nabla^2\varphi_t|_2^2 + c/\varepsilon|\nabla\eta|_6^2|\nabla\varphi_t|_3^2.$$

Summarizing the above estimates implies

$$|I_2 - I_{22}^2| \le \varepsilon\|\nabla\varphi_t\|_1^2 + c/\varepsilon(|\eta|_{2,1}^4|\nabla^2\varphi|_2^2 + |\eta|_{2,1}^2|\nabla^2\varphi|_3^2 + \|\eta\|_2^2|\nabla\varphi_t|_3^2).$$

Next, we examine the third term on the r.h.s. of (3.16). First, we write it in the form

$$I_3 = -\frac{\nu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_t\nabla\Delta\varphi\cdot\nabla\varphi_t dx - \frac{\nu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}\nabla\Delta\varphi_t\cdot\nabla\varphi_t dx \equiv I_{31} + I_{32}.$$

Integration by parts in I_{31} yields

$$\begin{aligned}I_{31} &= \frac{\nu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta\eta}\nabla\eta\eta_t\Delta\varphi\cdot\nabla\varphi_t dx + \frac{\nu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\nabla\eta_t\Delta\varphi\nabla\varphi_t dx\\ &\quad + \frac{\nu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_t\Delta\varphi\Delta\varphi_t dx \equiv I_{31}^1 + I_{31}^2 + I_{31}^3,\end{aligned}$$

where

$$\begin{aligned}|I_{31}^1| &\le \nu\varepsilon|\nabla\varphi_t|_6^2 + \frac{\nu c}{\varepsilon}|\nabla\eta|_6^2|\eta_t|_6^2|\Delta\varphi|_2^2,\\ |I_{31}^2| &\le \nu\varepsilon|\nabla\varphi_t|_6^2 + \frac{\nu c}{\varepsilon}|\nabla\eta_t|_2^2|\Delta\varphi|_3^2,\\ |I_{31}^3| &\le \nu\varepsilon|\Delta\varphi_t|_2^2 + \frac{\nu c}{\varepsilon}|\eta_t|_6^2|\Delta\varphi|_3^2.\end{aligned}$$

Finally, we examine I_{32}. Integration by parts yields

$$I_{32}=\frac{\nu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\right)_{,\eta}\nabla\eta\Delta\varphi_t\cdot\nabla\varphi_tdx+\frac{\nu}{a}\int\limits_\Omega\frac{\eta}{a+\eta}|\Delta\varphi_t|^2dx\equiv I_{32}^1+I_{32}^2,$$

where I_{32}^2 is absorbed by the third term on the l.h.s. of (3.16) in the following way

$$\begin{aligned}&\frac{\nu}{a}\int\limits_\Omega|\Delta\varphi_t|^2dx-\frac{\nu}{a}\int\limits_\Omega\frac{\eta}{a+\eta}|\Delta\varphi_t|^2dx\\&=\frac{\nu}{a}\int\limits_\Omega\frac{a}{a+\eta}|\Delta\varphi_t|^2dx\ge\frac{2}{3}\frac{\nu}{a}\int\limits_\Omega|\Delta\varphi_t|^2dx\end{aligned}$$

which holds for $|\eta|\le a/2$ and

$$|I_{32}^1|\le\nu\varepsilon|\Delta\varphi_t|_2^2+\frac{\nu c}{\varepsilon}|\nabla\eta|_6^2|\nabla\varphi_t|_3^2.$$

Summarizing, we have

$$|I_3-I_{32}^2|\le\nu\varepsilon\|\nabla\varphi_t\|_1^2+\frac{\nu c}{\varepsilon}(|\eta|_{2,1}^4|\Delta\varphi|_2^2+|\eta|_{2,1}^2|\Delta\varphi|_3^2+\|\eta\|_2^2|\nabla\varphi_t|_3^2).$$

We write the fourth term on the r.h.s. of (3.16) in the form

$$\begin{aligned}I_4=&\int\limits_\Omega v_t\cdot\nabla v\cdot\nabla\varphi_tdx+\int\limits_\Omega v\cdot\nabla v_t\cdot\nabla\varphi_tdx=-\int\limits_\Omega\Delta\varphi_tv\cdot\nabla\varphi_tdx\\&-\int\limits_\Omega v_tv\cdot\nabla^2\varphi_tdx-\int\limits_\Omega\Delta\varphi v_t\cdot\nabla\varphi_tdx-\int\limits_\Omega vv_t\cdot\nabla^2\varphi_tdx.\end{aligned}$$

Hence, we have

$$\begin{aligned}|I_4|\le&\ \varepsilon|\nabla\varphi_t|_6^2+c/\varepsilon(|\Delta\varphi_t|_2^2|v|_3^2+|\Delta\varphi|_3^2|v_t|_2^2)\\&+\varepsilon|\nabla^2\varphi_t|_2^2+c/\varepsilon|v_t|_3^2|v|_6^2\le\varepsilon|\nabla\varphi_t|_6^2+c/\varepsilon(|\Delta\varphi_t|_2^2D_1^2+|\Delta\varphi|_3^2D_2^2)\\&+\varepsilon|\nabla^2\varphi_t|_2^2+c/\varepsilon|v_t|_3^2D_1^2.\end{aligned}$$

We estimate the fifth term on the r.h.s. of (3.16) by

$$\begin{aligned}|I_5|&\le c\int\limits_\Omega(|\eta_|\,|\eta|\,|\nabla\eta|+|\eta_t|\,|\nabla\eta|+|\eta|\nabla\eta_t|)|\nabla\varphi_t|dx\\&\le\varepsilon|\nabla\varphi_t|_6^2+c/\varepsilon|(|\eta|_{2,1}^6+\eta|_{2,1}^4).\end{aligned}$$

Finally, we estimate the last term on the r.h.s. of (3.16) by

$$\varepsilon|\nabla\varphi_t|_6^2+c/\varepsilon|f_{gt}|_2^2.$$

Employing the above estimates in (3.16) and assuming that ε is sufficiently small we have

$$
\begin{aligned}
&\frac{d}{dt}|\nabla\varphi_t|_2^2+\frac{\mu}{a}\|\nabla\varphi_t\|_1^2+\frac{\nu}{a}|\Delta\varphi_t|_2^2\\
&\le\frac{c}{\nu}[|\eta|_{2,1}^2+|\eta|_{2,1}^4+|\eta|_{2,1}^6+|\eta|_{2,1}^4(|\nabla\mathrm{rot}\,\psi|_2^2+|\nabla^2\varphi|_2^2)\\
&\quad+|\eta|_{2,1}^2(|\nabla\mathrm{rot}\,\psi|_3^2+|\nabla^2\varphi|_3^2)+\|\eta\|_2^2(|\nabla\mathrm{rot}\,\psi_t|_2^2+|\nabla\varphi_t|_3^2)\\
&\quad+|\Delta\varphi_t|_2^2D_1^2+|\Delta\varphi|_3^2D_2^2+|v_t|_3^2D_1^2+|f_{gt}|_2^2]\\
&\quad+c\nu[|\eta|_{2,1}^4|\Delta\varphi|_2^2+|\eta|_{2,1}^2|\Delta\varphi|_3^2+\|\eta\|_2^2|\nabla\varphi_t|_3^2].
\end{aligned}\tag{3.17}
$$

Differentiate (3.13) with respect to t, multiply by $\mathrm{rot}\,\psi_t$ and integrate over Ω. Then we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\mathrm{rot}\,\psi_t|_2^2+\frac{\mu}{a}|\nabla\mathrm{rot}\,\psi_t|_2^2=-\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\Delta v\right)_{,t}\cdot\mathrm{rot}\,\psi_t dx\\
&-\frac{\nu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\Delta\nabla\varphi\right)_{,t}\cdot\mathrm{rot}\,\psi_t dx-\int\limits_\Omega(v\cdot\nabla v)_{,t}\cdot\mathrm{rot}\,\psi_t dx\\
&+\int\limits_\Omega f_{r,t}\cdot\mathrm{rot}\,\psi_t dx.
\end{aligned}\tag{3.18}
$$

Now we examine the particular terms from the r.h.s. of (3.18). The first term is written in the form

$$
-\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\Delta\mathrm{rot}\,\psi\right)_{,t}\cdot\mathrm{rot}\,\psi_t dx-\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\Delta\nabla\varphi\right)_{,t}\cdot\mathrm{rot}\,\psi_t dx\equiv I_1+I_2.
$$

Performing differentiation in I_1 yields

$$
\begin{aligned}
I_1&=-\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_t\Delta\mathrm{rot}\,\psi\cdot\mathrm{rot}\,\psi_t dx-\frac{\mu}{a}\int\limits_\Omega\frac{\eta}{a+\eta}\Delta\mathrm{rot}\,\psi_t\cdot\mathrm{rot}\,\psi_t dx\\
&\equiv I_{11}+I_{12}.
\end{aligned}
$$

Integration by parts in I_{11} implies

$$
\begin{aligned}
I_{11}&=\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\right)_{,\eta\eta}\nabla\eta\eta_t\nabla\mathrm{rot}\,\psi\cdot\mathrm{rot}\,\psi_t dx\\
&\quad+\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\right)_{,\eta}\nabla\eta_t\nabla\mathrm{rot}\,\psi\cdot\mathrm{rot}\,\psi_t dx\\
&\quad+\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_t\nabla\mathrm{rot}\,\psi\cdot\nabla\mathrm{rot}\,\psi_t dx\equiv I_{11}^1+I_{11}^2+I_{11}^3.
\end{aligned}
$$

Continuing, we have the estimates

$$\begin{aligned}
&|I_{11}^1|\le\varepsilon|\mathrm{rot}\,\psi_t|_6^2+c/\varepsilon|\nabla\eta|_6^2|\eta_t|_6^2|\nabla\mathrm{rot}\,\psi|_2^2,\\
&|I_{11}^2|\le\varepsilon|\mathrm{rot}\,\psi_t|_6^2+c/\varepsilon|\nabla\eta_t|_2^2|\nabla\mathrm{rot}\,\psi|_3^2,\\
&|I_{11}^3|\le\varepsilon|\nabla\mathrm{rot}\,\psi_t|_2^2+c/\varepsilon|\eta_t|_6^2|\nabla\mathrm{rot}\,\psi|_3^2.
\end{aligned}$$

Integrating by parts in I_{12} gives

$$\begin{aligned}
I_{12}&=\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\right)_{,\eta}\nabla\eta\nabla\mathrm{rot}\,\psi_t\cdot\mathrm{rot}\,\psi_t dx+\frac{\mu}{a}\int\limits_\Omega\frac{\eta}{a+\eta}|\nabla\mathrm{rot}\,\psi_t|^2dx\\
&\equiv I_{12}^1+I_{12}^2,
\end{aligned}$$

where

$$|I_{12}^1|\le\varepsilon|\nabla\mathrm{rot}\,\psi_t|_2^2+c/\varepsilon|\nabla\eta|_6^2|\mathrm{rot}\,\psi_t|_3^2$$

and I_{12}^2 is absorbed by the second term on the l.h.s. of (3.18) in such way that

$$\begin{aligned}
&\frac{\mu}{a}\int\limits_\Omega|\nabla\mathrm{rot}\,\psi_t|^2dx-\frac{\mu}{a}\int\limits_\Omega\frac{\eta}{a+\eta}|\nabla\mathrm{rot}\,\psi_t|^2dx\\
&=\frac{\mu}{a}\int\limits_\Omega\frac{a}{a+\eta}|\nabla\mathrm{rot}\,\psi_t|^2dx\ge\frac{2}{3}\frac{\mu}{a}|\nabla\mathrm{rot}\,\psi_t|_2^2\quad\text{for } |\eta|\le a/2.
\end{aligned}$$

Next, we examine I_2. Performing differentiation with respect to time gives

$$\begin{aligned}
I_2=&-\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_t\Delta\nabla\varphi\cdot\mathrm{rot}\,\psi_t dx\\
&-\frac{\mu}{a}\int\limits_\Omega\frac{\eta}{a+\eta}\Delta\nabla\varphi_t\cdot\mathrm{rot}\,\psi_t dx\equiv I_{21}+I_{22}.
\end{aligned}$$

Integrating by parts in I_{21} implies

$$\begin{aligned}
I_{21}=&\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\right)_{,\eta\eta}\nabla\eta\eta_t\Delta\varphi\cdot\mathrm{rot}\,\psi_t dx\\
&+\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\right)_{,\eta}\nabla\eta_t\Delta\varphi\cdot\mathrm{rot}\,\psi_t dx\equiv I_{21}^1+I_{21}^2,
\end{aligned}$$

where

$$\begin{aligned}
&|I_{21}^1|\le\varepsilon|\mathrm{rot}\,\psi_t|_6^2+c/\varepsilon|\nabla\eta|_6^2|\eta_t|_6^2|\Delta\varphi|_2^2,\\
&|I_{21}^2|\le\varepsilon|\mathrm{rot}\,\psi_t|_6^2+c/\varepsilon|\nabla\eta_t|_2^2|\Delta\varphi|_3^2.
\end{aligned}$$

Integration by parts in I_{22} gives

$$I_{22} = \frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta} \nabla\eta\Delta\varphi_t \cdot \operatorname{rot}\psi_t dx.$$

Hence,

$$|I_{22}| \le c\int\limits_\Omega |\nabla\eta|\,|\Delta\varphi_t|\,|\operatorname{rot}\psi_t|dx \le \varepsilon|\operatorname{rot}\psi_t|_6^2 + c/\varepsilon|\nabla\eta|_3^2|\Delta\varphi_t|_2^2.$$

Collecting the above estimates yields

$$\begin{aligned}
&|I_1| \le \varepsilon\|\operatorname{rot}\psi_t\|_1^2 + c/\varepsilon(|\eta|_{2,1}^4|\nabla\operatorname{rot}\psi|_2^2 + |\eta|_{2,1}^2|\nabla\operatorname{rot}\psi|_3^2 + \|\eta\|_2^2|\operatorname{rot}\psi_t|_3^2),\\
&|I_2| \le \varepsilon|\operatorname{rot}\psi_t|_6^2 + c/\varepsilon(|\eta|_{2,1}^4|\Delta\varphi|_2^2 + |\nabla\eta|_3^2|\Delta\varphi_t|_2^2 + |\eta|_{2,1}^2|\Delta\varphi|_3^2).
\end{aligned}$$

The second term on the r.h.s. of (3.18) equals

$$I_3 = -\frac{\nu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_t\Delta\nabla\varphi\cdot\operatorname{rot}\psi_t dx - \frac{\nu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}\Delta\nabla\varphi_t\cdot\operatorname{rot}\psi_t dx.$$

Integrating by parts in the above integrals yields

$$\begin{aligned}
I_3 &= \frac{\nu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta\eta}\nabla\eta\eta_t\Delta\varphi\cdot\operatorname{rot}\psi_t dx\\
&\quad + \frac{\nu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\nabla\eta_t\Delta\varphi\cdot\operatorname{rot}\psi_t dx\\
&\quad + \frac{\nu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\nabla\eta\Delta\varphi_t\cdot\operatorname{rot}\psi_t dx.
\end{aligned}$$

Hence, we have

$$|I_3| \le \varepsilon|\operatorname{rot}\psi_t|_6^2 + \frac{c\nu^2}{\varepsilon}(|\nabla\eta|_6^2|\eta_t|_6^2|\Delta\varphi|_2^2 + |\nabla\eta_t|_2^2|\Delta\varphi|_3^2 + |\nabla\eta|_6^2|\Delta\varphi_t|_{3/2}^2).$$

The third term on the r.h.s. of (3.18) equals

$$I_4 = -\int\limits_\Omega (v\cdot\nabla v_t + v_t\cdot\nabla v)\cdot\operatorname{rot}\psi_t dx \equiv I_{41} + I_{42}.$$

First we examine I_{41}. We write it in the form

$$I_{41} = -\int\limits_\Omega v\cdot\nabla\operatorname{rot}\psi_t\cdot\operatorname{rot}\psi_t dx - \int\limits_\Omega v\cdot\nabla\nabla\varphi_t\cdot\operatorname{rot}\psi_t dx \equiv I_{41}^1 + I_{41}^2,$$

where

$$I_{41}^1 = -\frac{1}{2}\int\limits_\Omega v\cdot\nabla|\mathrm{rot}\,\psi_t|^2dx = \frac{1}{2}\int\limits_\Omega \Delta\varphi|\mathrm{rot}\,\psi_t|^2dx$$

so

$$|I_{41}^1| \le \varepsilon|\mathrm{rot}\,\psi_t|_6^2 + c/\varepsilon|\Delta\varphi|_3^2|\mathrm{rot}\,\psi_t|_2^2.$$

Integrating by parts in I_{41}^2 yields

$$I_{41}^2 = \int\limits_\Omega \nabla v\cdot\nabla\varphi_t\cdot\mathrm{rot}\,\psi_t dx.$$

Hence,we have

$$|I_{41}^2| \le \varepsilon|\mathrm{rot}\,\psi_t|_6^2 + c/\varepsilon|\nabla v|_2^2|\nabla\varphi_t|_3^2.$$

Next, we examine I_{42}. Integration by parts implies

$$I_{42} = \int\limits_\Omega \Delta\varphi_t v\cdot\mathrm{rot}\,\psi_t dx + \int\limits_\Omega v_t\cdot v\cdot\nabla\mathrm{rot}\,\psi_t dx.$$

Hence, we obtain

$$|I_{42}| \le \varepsilon\|\mathrm{rot}\,\psi_t\|_1^2 + c/\varepsilon(|\Delta\varphi_t|_2^2|v|_3^2 + |v_t|_3^2|v|_6^2).$$

Finally, the last term on the r.h.s. of (3.18) is bounded by

$$\varepsilon|\mathrm{rot}\,\psi_t|_2^2 + c/\varepsilon|f_{rt}|_2^2.$$

Employing the above estimates in (3.18) and using that ε is sufficiently small we derive the inequality
(3.19)

$$\begin{aligned}&\frac{d}{dt}|\mathrm{rot}\,\psi_t|_2^2 + \frac{\mu}{a}|\nabla\mathrm{rot}\,\psi_t|_2^2 \le c[|\eta|_{2,1}^2(|\eta|_{2,1}^2+1)|\nabla\mathrm{rot}\,\psi|_3^2\\&\quad + (\|\eta\|_2^2 + |\Delta\varphi|_3^2)|\mathrm{rot}\,\psi_t|_3^2 + (|\eta|_{2,1}^4 + |\eta|_{2,1}^2)|\Delta\varphi|_3^2 + |\nabla\eta|_3^2|\Delta\varphi_t|_2^2\\&\quad + |\nabla v|_2^2|\nabla\varphi_t|_3^2 + (|\Delta\varphi_t|_2^2 + |v_t|_3^2)D_1^2 + |f_{rt}|_2^2] + c\nu^2[|\eta|_{2,1}^2(|\eta|_{2,1}^2+1)|\Delta\varphi|_3^2\\&\quad + \|\eta\|_2^2|\Delta\varphi_t|_2^2].\end{aligned}$$

Multiplying (3.19) by $1/\nu$ and adding to (3.17) gives the inequality

$$\begin{aligned}&\frac{d}{dt}\left(|\nabla\varphi_t|_2^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi_t|_2^2\right) + \frac{\mu}{a}\|\nabla\varphi_t\|_1^2 + \frac{\mu}{\nu a}\|\mathrm{rot}\,\psi_t\|_1^2 + \frac{\nu}{a}|\Delta\varphi_t|_2^2\\&\le \frac{c}{\nu}[|\eta|_{2,1}^2 + |\eta|_{2,1}^4 + |\eta|_{2,1}^6 + |\eta|_{2,1}^2(1+|\eta|_{2,1}^2)(|\nabla\mathrm{rot}\,\psi|_3^2 + |\nabla^2\varphi|_3^2)\\&\quad + (\|\eta\|_2^2 + |\Delta\varphi|_3^2)(|\nabla\mathrm{rot}\,\psi_t|_2^2 + \|\nabla\varphi_t\|_1^2) + |\nabla v|_2^2|\nabla\varphi_t|_3^2\\&\quad + |\Delta\varphi_t|_2^2D_1^2 + |\Delta\varphi|_3^2D_2^2 + |v_t|_3^2D_1^2 + |f_{rt}|_2^2 + |f_{gt}|_2^2]\\&\quad + c\nu[|\eta|_{2,1}^2(1+|\eta|_{2,1}^2)|\Delta\varphi|_3^2 + \|\eta\|_2^2\|\nabla\varphi_t\|_1^2].\end{aligned} \tag{3.20}$$

Introduce the quantity

$$\phi_2 = \phi_2(|\eta|_{2,1}), \quad \phi_2(0) \neq 0. \tag{3.21}$$

Then (3.20) takes the form

$$\begin{aligned}
&\frac{d}{dt}\bigg(|\nabla\varphi_t|_2^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi_t|_2^2\bigg) + \frac{\mu}{a}\|\nabla\varphi_t\|_1^2 + \frac{\mu}{\nu a}\|\mathrm{rot}\,\psi_t\|_1^2 + \frac{\nu}{a}|\Delta\varphi_t|_2^2 \\
&\le \frac{c}{\nu}\phi_2[|\eta|_{2,1}^2 + |\eta|_{2,1}^2(|\nabla\mathrm{rot}\,\psi|_3^2 + |\nabla^2\varphi|_3^2 + |\nabla\mathrm{rot}\,\psi_t|_2^2 + \|\nabla\varphi_{,t}\|_1^2) \\
&\quad + |\Delta\varphi|_3^2(|\nabla\mathrm{rot}\,\psi_t|_2^2 + \|\nabla\varphi_t\|_1^2 + D_2^2) + \|\nabla\varphi_t\|_1^2(|\nabla v|_2^2 + D_1^2) \\
&\quad + |v_t|_3^2 D_1^2 + |f_t|_2^2] + c\nu[|\eta|_{2,1}^2(1 + |\eta|_{2,1}^2)|\Delta\varphi|_3^2 + \|\eta\|_2^2\|\nabla\varphi_t\|_1^2].
\end{aligned} \tag{3.22}$$

Hence (3.22) implies (3.14). Integrating (3.22) with respect to time and using Notation 2.10 and the fact that $|v|_6 \le D_1$, $|v_t|_2 \le D_2$, $|v|_{6,2,\Omega^t} \le A_1$, $|v_t|_{2,\Omega^t} \le D_2$, $|\nabla v|_{2,\Omega^t} \le A_1$ implies (3.15). This concludes the proof of Lemma 3.2. □

Lemma 3.3. *Let the assumptions of Lemma 3.1 hold. Then*

$$\begin{aligned}
&a\frac{d}{dt}\bigg(|\nabla\varphi_x|_2^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi_x|_2^2\bigg) + \mu|\nabla^2\varphi_x|_2^2 + \nu|\Delta\varphi_x|_2^2 + \frac{\mu}{\nu}|\nabla\mathrm{rot}\,\psi_x|_2^2 \\
&\le \frac{c}{\nu}[|\eta|_\infty^2|v_t|_2^2 + (1 + |\eta|_\infty^4)|\nabla v|_2^2 D_1^4 + |\nabla\eta|_2^2 + |\eta|_\infty^2|\nabla\eta|_2^2 \\
&\quad + (1 + |\eta|_\infty^2)|f|_2^2],
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
&|\nabla\varphi_x(t)|_2^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi_x(t)|_2^2 + \frac{\mu}{a}|\nabla^2\varphi_x|_{2,2,\Omega^t}^2 + \frac{\nu}{a}|\Delta\varphi_x|_{2,2,\Omega^t}^2 \\
&\quad + \frac{\mu}{\nu a}|\nabla\mathrm{rot}\,\psi_x|_{2,2,\Omega^t}^2 \le \frac{c}{\nu}[|\eta|_{\infty,\Omega^t}^2 D_2^2 + (1 + |\eta|_{\infty,\Omega^t}^2)A_1^2 D_1^4 \\
&\quad + |\nabla\eta|_{2,\Omega^t}^2 + |\eta|_{\infty,\Omega^t}^2|\nabla\eta|_{2,\Omega^t}^2 + (1 + |\eta|_{\infty,\Omega^t}^2)|f|_{2,2,\Omega^t}^2] \\
&\quad + |\nabla\varphi_x(0)|_2^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi_x(0)|_2^2.
\end{aligned} \tag{3.24}$$

Proof. Differentiate (3.2) with respect to x, multiply by $\nabla\varphi_x$ and integrate over Ω. Then we get

$$
\begin{aligned}
&\frac{a}{2}\frac{d}{dt}|\nabla\varphi_{,x}|_2^2+\mu|\nabla^2\varphi_x|_2^2+\mu|\Delta\varphi_x|_2^2=-\int\limits_\Omega(\eta v_t)_{,x}\nabla\varphi_x dx\\
&-a_0\int\limits_\Omega\nabla\eta_x\nabla\varphi_x dx-\int\limits_\Omega[(a+\eta)v\cdot\nabla v]_{,x}\cdot\nabla\varphi_x dx\\
&+\int\limits_\Omega[(p_\varrho(a)-p_\varrho(a+\eta))\nabla\eta]_{,x}\cdot\nabla\varphi_x dx\\
&+\int\limits_\Omega[(a+\eta)f]_{,x}\cdot\nabla\varphi_x dx.
\end{aligned}
\tag{3.25}
$$

After integration by parts in the first term on the r.h.s. of (3.25) we bound it by

$$\varepsilon|\nabla\varphi_{xx}|_2^2+c/\varepsilon|\eta|_\infty^2|v_t|_2^2.$$

The second term on the r.h.s. of (3.25) is bounded by

$$\varepsilon|\nabla\varphi_{xx}|_2^2+c/\varepsilon|\nabla\eta|_2^2.$$

After integration by parts the third term on the r.h.s. of (3.25) is estimated by

$$\varepsilon|\nabla\varphi_{xx}|_2^2+c/\varepsilon(1+|\eta|_\infty^2)|v|_6^2|\nabla v|_3^2\le\varepsilon|\nabla\varphi_{xx}|_2^2+c/\varepsilon|\nabla v|_3^2D_1^2(1+|\eta|_\infty^2)$$

and the fourth term by

$$\varepsilon|\nabla\varphi_{xx}|_2^2+c/\varepsilon|\eta|_\infty^2|\nabla\eta|_2^2.$$

Finally, the last term on the r.h.s. of (3.25) equals

$$I_1=-a\int\limits_\Omega f_g\cdot\nabla\varphi_{xx}dx-\int\limits_\Omega\eta f\cdot\nabla\varphi_{xx}dx,$$

so it is estimated by

$$|I_1|\le\varepsilon|\nabla\varphi_{xx}|_2^2+c/\varepsilon(|f_g|_2^2+|\eta|_\infty^2|f|_2^2).$$

Employing the above estimates in (3.25) and assuming that ε is sufficiently small we obtain the inequality

$$
\begin{aligned}
&a\frac{d}{dt}|\nabla\varphi_x|_2^2+\mu|\nabla^2\varphi_x|_2^2+\nu|\Delta\varphi_x|_2^2\\
&\le\frac{c}{\nu}[|\eta|_\infty^2|v_t|_2^2+|\nabla\eta|_2^2+|\nabla v|_3^2D_1^2(1+|\eta|_\infty^2)+|\eta|_\infty^2|\nabla\eta|_2^2\\
&\quad+|f|_2^2(1+|\eta|_\infty^2)].
\end{aligned}
\tag{3.26}
$$

Differentiate (3.2) with respect to x, multiply by $\operatorname{rot}\psi_x$ and integrate over Ω. Then we derive

$$
\begin{aligned}
&\frac{a}{2}\frac{d}{dt}|\operatorname{rot}\psi_x|_2^2+\mu|\nabla\operatorname{rot}\psi_x|_2^2=-\int\limits_\Omega(\eta v_t)_{,x}\operatorname{rot}\psi_x dx\\
&-\int\limits_\Omega[(a+\eta)v\cdot\nabla v]_{,x}\cdot\operatorname{rot}\psi_x dx+\int\limits_\Omega[(a+\eta)f]_{,x}\cdot\operatorname{rot}\psi_x dx.
\end{aligned}\tag{3.27}
$$

Integrating by parts in the terms from the r.h.s. of the above equality we obtain

$$
\begin{aligned}
&a\frac{d}{dt}|\operatorname{rot}\psi_x|_2^2+\mu|\nabla\operatorname{rot}\psi_x|_2^2\\
&\le c[|\eta|_\infty^2|v_t|_2^2+(1+|\eta|_\infty^2)|v|_6^2|\nabla v|_3^2+(1+|\eta|_\infty^2)|f|_2^2].
\end{aligned}\tag{3.28}
$$

Multiplying (3.28) by $1/\nu$ and adding to (3.26) yields

$$
\begin{aligned}
&a\frac{d}{dt}\bigg(|\nabla\varphi_x|_2^2+\frac{1}{\nu}|\operatorname{rot}\psi_x|_2^2\bigg)+\mu|\nabla^2\varphi_x|_2^2+\nu|\Delta\varphi_x|_2^2+\frac{\mu}{\nu}|\nabla\operatorname{rot}\psi_x|_2^2\\
&\le\frac{c}{\nu}[|\eta|_\infty^2|v_t|_2^2+(1+|\eta|_\infty^2)|\nabla v|_3^2D_1^2+|\nabla\eta|_2^2+|\eta|_\infty^2|\nabla\eta|_2^2\\
&\quad+(1+|\eta|_\infty^2)|f|_2^2].
\end{aligned}\tag{3.29}
$$

Using the interpolation

$$
|\nabla v|_3\le c|\nabla^2v|_2^{1/2}|\nabla v|_2^{1/2}
$$

We obtain from (3.29) the inequality

$$
\begin{aligned}
&a\frac{d}{dt}\bigg(|\nabla\varphi_x|_2^2+\frac{1}{\nu}|\operatorname{rot}\psi_x|_2^2\bigg)+\mu|\nabla^2\varphi_x|_2^2+\nu|\Delta\varphi|_2^2+\frac{\mu}{\nu}|\nabla\operatorname{rot}\psi_x|_2^2\\
&\frac{c}{\nu}[|\eta|_\infty^2|v_t|_2^2+(1+|\eta|_\infty^4)|\nabla v|_2^2D_1^4+|\nabla\eta|_2^2+|\eta|_\infty^2|\nabla\eta|_2^2\\
&\quad+(1+|\eta|_\infty^2)|f|_2^2]
\end{aligned}\tag{3.30}
$$

Integrating (3.30) with respect to time yields

$$
\begin{aligned}
&a\bigg(|\nabla\varphi_x(t)|_2^2+\frac{1}{\nu}|\operatorname{rot}\psi_x(t)|_2^2\bigg)+\mu|\nabla^2\varphi_x|_{2,2,\Omega^t}^2+\nu|\Delta\varphi_x|_{2,2,\Omega^t}^2\\
&\quad+\frac{\mu}{\nu}|\nabla\operatorname{rot}\psi_x|_{2,2,\Omega^t}^2\le\frac{c}{\nu}[|\eta|_{\infty,2,\Omega^t}^2D_2^2+(1+|\eta|_{\infty,2,\Omega^t}^4)A_1^2D_1^4\\
&\quad+|\nabla\eta|_{2,2,\Omega^t}^2+|\eta|_{\infty,\infty,\Omega^t}^2|\nabla\eta|_{2,2,\Omega^t}^2+(1+|\eta|_{\infty,\infty,\Omega^t}^2)|f|_{2,2,\Omega^t}^2]\\
&\quad+a\bigg(|\nabla\varphi_x(0)|_2^2+\frac{1}{\nu}|\operatorname{rot}\psi_x(0)|_2^2\bigg).
\end{aligned}\tag{3.31}
$$

Since $|\eta(t)|_{2,1}$ is bounded by a constant dependent of time the r.h.s. of (3.31) is an increasing function of time. To obtain such estimate that the r.h.s. of (3.31) does not depend on time needs another approach. However, we shall show such result that $\nu \to \infty$ implies that $t \to \infty$. This concludes the proof. □

Remark 3.2. We simplify (3.29). Introduce the quantities

$$\phi_3 = \phi_3(|\eta|_\infty, D_1). \tag{3.32}$$

Then (3.29) takes the form

$$\begin{aligned} &a\frac{d}{dt}\left(|\nabla\varphi_x|_2^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi_x|_2^2\right) + \mu|\nabla^2\varphi_x|_2^2 + \nu|\Delta\varphi_x|_2^2 + \frac{\mu}{\nu}|\nabla\mathrm{rot}\,\psi_x|_2^2 \\ &\le \frac{c}{\nu}\phi_3[|v_t|_2^2 + |\nabla v|_3^2 + |f|_2^2]. \end{aligned} \tag{3.33}$$

Using the interpolation

$$\alpha|\nabla v|_3^2 \le \varepsilon^{4/3}(|\nabla^2\varphi_x|_2^2 + |\nabla\mathrm{rot}\,\psi_x|_2^2) + \frac{c}{\varepsilon^4}\alpha^4(|\nabla\varphi|_2^2 + |\mathrm{rot}\,\psi|_2^2) \tag{3.34}$$

we obtain from (3.33) the inequality

$$\begin{aligned} &a\frac{d}{dt}\left(|\nabla\varphi_x|_2^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi_x|_2^2\right) + \mu|\nabla^2\varphi_x|_2^2 + \nu|\Delta\varphi_x|_2^2 + \frac{\mu}{\nu}|\nabla\mathrm{rot}\,\psi_x|_2^2 \\ &\le \frac{c}{\nu}\phi_3[|v_t|_2^2 + |v|_2^2 + |f|_2^2]. \end{aligned} \tag{3.35}$$

Adding $(3.4)_1$ and (3.35) we have

$$\begin{aligned} &a\frac{d}{dt}\left(\|\nabla\varphi\|_1^2 + \frac{1}{\nu}\|\mathrm{rot}\,\psi\|_1^2\right) + \mu\|\nabla\varphi\|_2^2 + \nu\|\nabla\varphi\|_2^2 + \frac{\mu}{\nu}\|\mathrm{rot}\,\psi\|_2^2 \\ &\le \frac{c}{\nu}\phi_4(|\eta|_\infty, D_1)[\|\eta\|_1^2 + |v_t|_2^2 + |v|_3^2 + |\Delta\varphi|_3^2 + |f|_2^2]. \end{aligned} \tag{3.36}$$

Using interpolation (3.34) in the r.h.s. of (3.22) to terms $|\nabla\mathrm{rot}\,\psi|_3^2$, $|\nabla^2\varphi|_3^2$, $|\Delta\varphi|_3^2$ we obtain from (3.36) and (3.22) the inequality

(3.37)

$$\begin{aligned} &a\frac{d}{dt}\left(|\nabla\varphi|_{1,1}^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi|_{1,1}^2\right) + \mu|\nabla\varphi|_{2,1}^2 + \nu|\nabla\varphi|_{2,1}^2 + \frac{\mu}{\nu}|\mathrm{rot}\,\psi|_{2,1}^2 \\ &\le \frac{c}{\nu}\phi(|\eta|_{2,1}, A_1, D_1, D_2)[|\eta|_{2,1}^2 + |v_t|_2^2 + |\nabla v_t|_2^2 + |\Delta\varphi_t|_3^2 \\ &\quad + |\Delta\varphi|_3^2(|\nabla\mathrm{rot}\,\psi_t|_2^2 + \|\nabla\varphi_t\|_1^2) + |\nabla v|_2^2\|\nabla\varphi_t\|_1^2 + \|\nabla\varphi_t\|_1^2 D_1^2 \\ &\quad + |v_t|_3^2 D_1^2 + |f|_2^2 + |f_t|_2^2] + c\nu[|\eta|_{2,1}^2(1 + |\eta|_{2,1}^2)|\Delta\varphi|_3^2 + \|\eta\|_2^2\|\nabla\varphi_t\|_1^2]. \end{aligned}$$

Lemma 3.4. *Let the assumptions of Lemma 3.1 hold. Let Notation 2.10 be applied. Then*

(3.38)
$$
\begin{aligned}
&a\frac{d}{dt}\bigg(|\nabla\varphi_{xt}|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_{xt}|_2^2\bigg)+\mu|\nabla^2\varphi_{xt}|_2^2+\nu|\Delta\varphi_{xt}|_2^2+\frac{\mu}{\nu}|\nabla\mathrm{rot}\,\psi_{xt}|_2^2\\
&\le\frac{\varepsilon}{\nu}|v_{xxx}|_2^2+\nu\varepsilon_1|\nabla\varphi_{,xxx}|_2^2\\
&\frac{c}{\nu\varepsilon}\phi_6[|v|_2^2+|v_t|_2^2+|\nabla\varphi_{xt}|_2^2+|\nabla\varphi_{xx}|_2^2|\mathrm{rot}\,\psi_{xt}|_2+|\nabla v|_3^2|\nabla\varphi_{,xt}|_2^2\\
&\quad+|\nabla v|_3^2|\nabla v_{,t}|_2^2+|f_t|_2^2]+\frac{c\nu}{\varepsilon_1}|\eta|_{2,1}^4(|\nabla\varphi_{,xx}|_2^2+|\nabla\varphi_{,xt}|_2^2),
\end{aligned}
$$

where ϕ_6 is introduced below (3.47). Next (3.49) implies

(3.39)
$$
\begin{aligned}
&a\bigg(|\nabla\varphi_{,xt}(t)|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_{,xt}(t)|_2^2\bigg)+\mu|\nabla^2\varphi_{,xt}|_{2,2,\Omega^t}^2+\nu|\Delta\varphi_{,xt}|_{2,2,\Omega^t}^2\\
&\quad+\frac{\mu}{\nu}|\nabla\mathrm{rot}\,\psi_{,xt}|_{2,2,\Omega^t}^2\le\frac{\varepsilon}{\nu}|v_{,xxx}|_{2,2,\Omega^t}^2+\nu\varepsilon_1|\nabla\varphi_{,xxx}|_{2,2,\Omega^t}^2+\varepsilon_2|v_{xx}|_{2,\infty,\Omega^t}^2\\
&\quad+\varepsilon_3|v_{xx}|_{2,\Omega^t}^2+\frac{c}{\nu}\phi_8+c\nu|\eta|_{2,1,\infty,\Omega^t}^2(1+|\eta|_{2,1,\infty,\Omega^t}^2)\frac{\Psi^2}{\nu^2}\\
&\quad+|\nabla\varphi_{,xt}(0)|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_{,xt}(0)|_2^2,
\end{aligned}
$$

where ϕ_8 is describes the last but one term on the r.h.s. of (3.50).

Proof. Differentiate (3.13) with respect to x and t, multiply by $\nabla\varphi_{xt}$ and integrate over Ω. Then we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla\varphi_{xt}|_2^2+\frac{\mu}{a}|\nabla^2\varphi_{xt}|_2^2+\frac{\nu}{a}|\Delta\varphi_{xt}|_2^2\\
&=-\int_\Omega\bigg(\frac{a_0}{a+\eta}\nabla\eta\bigg)_{,xt}\cdot\nabla\varphi_{,xt}dx-\frac{\mu}{a}\int_\Omega\bigg(\frac{\eta}{a+\eta}\Delta v\bigg)_{,xt}\cdot\nabla\varphi_{xt}dx\\
&\quad-\frac{\nu}{a}\int_\Omega\bigg(\frac{\eta}{a+\eta}\nabla\Delta\varphi\bigg)_{,xt}\cdot\nabla\varphi_{xt}dx-\int_\Omega(v\cdot\nabla v)_{,xt}\cdot\nabla\varphi_{xt}dx\\
&\quad+\int_\Omega\bigg[\frac{1}{a+\eta}(p_\varrho(a)-p_\varrho(a+\eta))\nabla\eta\bigg]_{,xt}\cdot\nabla\varphi_{xt}dx\\
&\quad+\int_\Omega f_{,xt}\cdot\nabla\varphi_{,xt}dx.
\end{aligned}\tag{3.40}
$$

Now, we examine the particular terms from the r.h.s. of (3.40). Integrating by parts in the first term we get

$$I_1 = \int\limits_\Omega \left[\left(\frac{a_0}{a+\eta}\right)_{,\eta} \eta_t \nabla\eta + \frac{a_0}{a+\eta}\nabla\eta_t\right] \cdot \nabla\varphi_{,xxt} dx.$$

Hence, we have

$$|I_1| \le \varepsilon|\nabla\varphi_{,xxt}|_2^2 + c/\varepsilon(|\eta_t|_4^2|\nabla\eta|_4^2 + |\nabla\eta_t|_2^2).$$

We express the second term on the r.h.s. of (3.40) in the form

$$\begin{aligned} I_2 = &-\frac{\mu}{a}\int\limits_\Omega \left[\left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_t\Delta v\right]_{,x} \cdot \nabla\varphi_{,xt}dx \\ &-\frac{\mu}{a}\int\limits_\Omega \left[\frac{\eta}{a+\eta}\Delta v_{,t}\right]_{.x} \cdot \nabla\varphi_{,xt}dx \equiv I_{21} + I_{22}. \end{aligned}$$

Integrating by parts in I_{21} yields

$$I_{21} = \frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_t\Delta v \cdot \nabla\varphi_{,xxt}dx$$

and

$$|I_{21}| \le \varepsilon|\nabla\varphi_{,xxt}|_2^2 + c/\varepsilon|\eta_t|_6^2|\Delta v|_3^2.$$

Performing differentiation in I_{22} gives

$$\begin{aligned} I_{22} = &-\frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_x\Delta v_{,t}\nabla\varphi_{,xt}dx \\ &-\frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}\Delta v_{tx} \cdot \nabla\varphi_{,xt}dx \equiv I_{221} + I_{222}. \end{aligned}$$

Integrating by parts in I_{221} implies

$$\begin{aligned} I_{221} = &\frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta\eta}\nabla\eta\eta_x \cdot \nabla v_t \cdot \nabla\varphi_{,xt}dx \\ &+\frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\nabla\eta_x\nabla v_t \cdot \nabla\varphi_{,xt}dx \\ &+\frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_{,x}\nabla v_t \cdot \nabla^2\varphi_{,xt}dx \equiv I_{221}^1 + I_{221}^2 + I_{221}^3, \end{aligned}$$

where

$$|I_{221}^1| \le \varepsilon |\nabla\varphi_{,xt}|_6^2 + c/\varepsilon |\eta_x|_6^4 |v_{,xt}|_2^2,$$
$$|I_{221}^2| \le \varepsilon |\nabla\varphi_{,xt}|_6^2 + c/\varepsilon |\eta_{xx}|_2^2 |v_{,xt}|_3^2,$$
$$|I_{221}^3| \le \varepsilon |\nabla^2\varphi_{,xt}|_2^2 + c/\varepsilon |\eta_x|_6^2 |v_{,xt}|_3^2.$$

Next we examine I_{222}. We express it in the form

$$\begin{aligned} I_{222} &= -\frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}\Delta \mathrm{rot}\,\psi_{,xt}\cdot\nabla\varphi_{,xt}dx \\ &\quad -\frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}\Delta\nabla\varphi_{,xt}\cdot\nabla\varphi_{,xt}dx \equiv J_1 + J_2, \end{aligned}$$

where

$$\begin{aligned} J_1 &= \frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\nabla\eta\nabla \mathrm{rot}\,\psi_{,xt}\cdot\nabla\varphi_{,xt}dx \\ &\quad +\frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}\nabla \mathrm{rot}\,\psi_{,xt}\nabla^2\varphi_{,xt}dx \\ &\equiv J_{11} + J_{12}. \end{aligned}$$

and

$$\begin{aligned} J_{12} &= \frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}\nabla_j(\mathrm{rot}\,\psi)_{i,xt}\nabla_j\nabla_i\varphi_{,xt}dx \\ &= -\frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\nabla_i\eta\nabla_j(\mathrm{rot}\,\psi)_{i,xt}\nabla_j\varphi_{,xt}dx. \end{aligned}$$

Hence,

$$\begin{aligned} |J_{11}| &\le \varepsilon_1 |\nabla \mathrm{rot}\,\psi_{,xt}|_2^2 + c/\varepsilon_1 |\nabla\eta|_6^2 |\nabla\varphi_{,xt}|_3^2 \\ &\le \varepsilon_1 |\nabla \mathrm{rot}\,\psi_{xt}|_2^2 + c/\varepsilon_1 |\nabla\eta|_6^2 |\varphi_{,xxxt}|_2 |\varphi_{,xt}|_2 \\ &\le \varepsilon_1 |\nabla \mathrm{rot}\,\psi_{xt}|_2^2 + \varepsilon |\varphi_{,xxxt}|_2^2 + c/\varepsilon_1^2\varepsilon |\nabla\eta|_6^4 |\varphi_{,xt}|_2^2 \end{aligned}$$

and

$$\begin{aligned} |J_{12}| &\le \varepsilon_1 |\nabla \mathrm{rot}\,\psi_{,xt}|_2^2 + c/\varepsilon_1 |\nabla\eta|_6^2 |\nabla\varphi_{,xt}|_3^2 \\ &\le \varepsilon_1 |\nabla \mathrm{rot}\,\psi_{,xt}|_2^2 + c/\varepsilon_1 |\nabla\eta|_6^2 |\nabla\varphi_{,xxt}|_2 |\nabla\varphi_{,xt}|_2 \\ &\le \varepsilon_1 |\nabla \mathrm{rot}\,\psi_{,xt}|_2^2 + \varepsilon |\nabla\varphi_{,xxt}|_2^2 + c/\varepsilon_1^2\varepsilon |\nabla\eta|_6^4 |\nabla\varphi_{,xt}|_2^2. \end{aligned}$$

Summarizing,

$$|J_1| \le \varepsilon_1 |\nabla \mathrm{rot}\,\psi_{,xt}|_2^2 + \varepsilon |\nabla\varphi_{,xxt}|_2^2 + c/\varepsilon_1^2\varepsilon \|\eta\|_2^4 |\nabla\varphi_{,xt}|_2^2.$$

Finally, we examine J_2. Integrating by parts yields

$$
\begin{aligned}
J_2 &= \frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta} \nabla\eta\nabla\nabla\varphi_{,xt}\cdot\nabla\varphi_{xt}dx \\
&\quad + \frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}|\nabla^2\varphi_{,xt}|^2 dx \equiv J_{21}+J_{22},
\end{aligned}
$$

where

$$
\begin{aligned}
|J_{21}| &\le \varepsilon|\nabla^2\varphi_{,xt}|_2^2 + c/\varepsilon|\nabla\eta|_6^2|\nabla\varphi_{,xt}|_3^2 \\
&\le \varepsilon|\nabla^2\varphi_{,xt}|_2^2 + c/\varepsilon|\nabla\eta|_6^2|\nabla\varphi_{,xxt}|_2|\nabla\varphi_{,xt}|_2 \\
&\le \varepsilon|\nabla^2\varphi_{,xt}|_2^2 + c/\varepsilon^3|\nabla\eta|_6^4|\nabla\varphi_{xt}|_2^2.
\end{aligned}
$$

Summarizing the estimates we have

$$
\begin{aligned}
|I_2| &\le \varepsilon_1(|\nabla\mathrm{rot}\,\psi_{,xt}|_2^2 + \varepsilon\|\nabla\varphi_{,xt}\|_1^2) + c/\varepsilon[\|\eta_{,t}\|_1^2|\Delta v|_3^2 \\
&\quad + \|\eta\|_2^4|v_{,xt}|_2^2 + \|\eta\|_2^2|v_{,xt}|_3^2] + (1/\varepsilon_1^2\varepsilon + 1/\varepsilon^3)\|\eta\|_2^4|\nabla\varphi_{,xt}|_2^2 \\
&\quad + \frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}|\nabla^2\varphi_{,xt}|^2 dx.
\end{aligned}
$$

The term J_{22} is absorbed by the second term on the l.h.s. of (3.40) in the following way

$$
\begin{aligned}
&\frac{\mu}{a}\int\limits_\Omega |\nabla^2\varphi_{,xt}|^2dx - \frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}|\nabla^2\varphi_{,xt}|^2dx = \frac{\mu}{a}\int\limits_\Omega \frac{a}{a+\eta}|\nabla^2\varphi_{,xt}|^2dx \\
&\ge \frac{2}{3}\frac{\mu}{a}|\nabla^2\varphi_{,xt}|_2^2,
\end{aligned}
$$

where $|\eta| \le a/2$. Now we estimate the third term on the r.h.s. of (3.40). Performing differentiation with respect to time yields

$$
\begin{aligned}
I_3 &= -\frac{\nu}{a}\int\limits_\Omega \left[\left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_t\nabla\Delta\varphi\right]_{,x}\cdot\nabla\varphi_{,xt}dx \\
&\quad - \frac{\nu}{a}\int\limits_\Omega \left[\frac{\eta}{a+\eta}\nabla\Delta\varphi_{,t}\right]_{,x}\nabla\varphi_{,xt}dx \equiv I_{31}+I_{32}.
\end{aligned}
$$

Integrating by parts in I_{31} yields

$$
I_{31} = \frac{\nu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_t\Delta\nabla\varphi\cdot\nabla\varphi_{,xxt}dx
$$

so

$$
\begin{aligned}
|I_{31}| &\le \nu\varepsilon|\nabla\varphi_{,xxt}|_2^2 + \frac{c\nu}{\varepsilon}|\eta_t|_6^2|\Delta\nabla\varphi|_3^2 \\
&\le \nu\varepsilon|\nabla\varphi_{,xxt}|_2^2 + \frac{c\nu}{\varepsilon}|\eta_t|_6^2|\Delta\nabla\varphi_x|_2|\Delta\nabla\varphi|_2 \\
&\le \nu\varepsilon|\nabla\varphi_{,xxt}|_2^2 + \nu\varepsilon_2|\nabla\varphi_{,xxx}|_2^2 + \frac{c\nu}{\varepsilon^2\varepsilon_2}|\eta_t|_6^4|\Delta\nabla\varphi|_2^2.
\end{aligned}
$$

Performing differentiation with respect to x in I_{32} we have

$$
\begin{aligned}
I_{32} = &-\frac{\nu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_x\nabla\Delta\varphi_{,t}\cdot\nabla\varphi_{,xt}dx \\
&-\frac{\nu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}\nabla\Delta\varphi_{xt}\cdot\nabla\varphi_{,xt}dx \equiv L_1 + L_2,
\end{aligned}
$$

where

$$
\begin{aligned}
|L_1| &\le \nu\varepsilon|\Delta\nabla\varphi_{,t}|_2^2 + \frac{c\nu}{\varepsilon}|\eta_{,x}|_6^2|\nabla\varphi_{,xt}|_3^2 \\
&\le \nu\varepsilon|\nabla\varphi_{,xxt}|_2^2 + \frac{c\nu}{\varepsilon}|\eta_{,x}|_6^2|\nabla\varphi_{,xxt}|_2|\nabla\varphi_{,xt}|_2 \\
&\le \nu\varepsilon|\nabla\varphi_{,xxt}|_2^2 + \frac{c\nu}{\varepsilon^3}|\eta_{,x}|_6^4|\nabla\varphi_{,xt}|_2^2.
\end{aligned}
$$

Integrating by parts in L_2 yields

$$
\begin{aligned}
L_2 = &\frac{\nu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\nabla\eta\Delta\varphi_{,xt}\cdot\nabla\varphi_{,xt}dx \\
&+\frac{\nu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}|\Delta\varphi_{,xt}|^2dx \equiv L_2^1 + L_2^2,
\end{aligned}
$$

where

$$
\begin{aligned}
|L_2^1| &\le \nu\varepsilon|\nabla\varphi_{,xxt}|_2^2 + \frac{\nu c}{\varepsilon}|\nabla\eta|_6^2|\nabla\varphi_{,xt}|_3^2 \\
&\le \nu\varepsilon|\nabla\varphi_{,xxt}|_2^2 + \frac{\nu c}{\varepsilon}|\nabla\eta|_6^2|\nabla\varphi_{,xxt}|_2|\nabla\varphi_{,xt}|_2 \\
&\le \nu\varepsilon|\nabla\varphi_{,xxt}|_2^2 + \frac{\nu c}{\varepsilon^3}|\nabla\eta|_6^4|\nabla\varphi_{,xt}|_2^2
\end{aligned}
$$

and L_2^2 is absorbed by the third term on the r.h.s. of (3.40) in the following way

$$
\begin{aligned}
&\frac{\nu}{a}\int\limits_\Omega |\Delta\varphi_{,xt}|^2dx - \frac{\nu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}|\Delta\varphi_{,xt}|^2dx = \frac{\nu}{a}\int\limits_\Omega |\Delta\varphi_{,xt}|^2dx \\
&\ge \frac{2}{3}\frac{\nu}{a}|\Delta\varphi_{,xt}|_2^2 \quad \text{for} \quad |\eta|_\infty \le a/2.
\end{aligned}
$$

Summarizing, the third term on the r.h.s. of (3.40) is estimated in the following way

$$|I_3| \le \nu\varepsilon|\nabla\varphi_{,xxt}|_2^2 + \nu\varepsilon_2|\nabla\varphi_{,xxx}|_2^2 + \frac{c\nu}{\varepsilon^2\varepsilon_2}\|\eta_t\|_1^4|\nabla\varphi_{,xx}|_2^2 + \frac{c\nu}{\varepsilon^3}\|\eta\|_2^4|\nabla\varphi_{xt}|_2^2$$
$$+ \frac{\nu}{a}\int_\Omega \frac{\eta}{a+\eta}|\Delta\varphi_{,xt}|^2 dx.$$

Consider the fourth term on the r.h.s. of (3.40). Performing differentiations it takes the form

$$I_4 = \int_\Omega v_{,xt}\cdot\nabla v\cdot\nabla\varphi_{,xt}dx + \int_\Omega v_{,t}\cdot\nabla v_{,x}\cdot\nabla\varphi_{,xt}dx + \int_\Omega v_{,x}\cdot\nabla v_{,t}\cdot\nabla\varphi_{,xt}dx$$
$$+ \int_\Omega v\cdot\nabla v_{,xt}\cdot\nabla\varphi_{,xt}dx \equiv \sum_{i=1}^4 I_{4i}.$$

First we consider

$$I_{44} = \int_\Omega v\cdot\nabla v_{xt}\cdot\nabla\varphi_{xt}dx = \int_\Omega v\cdot\nabla\nabla\varphi_{xt}\cdot\nabla\varphi_{xt}dx + \int_\Omega v\cdot\nabla\mathrm{rot}\,\psi_{xt}\cdot\nabla\varphi_{xt}dx$$
$$\equiv I_{44}^1 + I_{44}^2,$$

where

$$I_{44}^1 = \frac{1}{2}\int_\Omega v\cdot\nabla|\nabla\varphi_{xt}|^2dx = -\frac{1}{2}\int_\Omega \Delta\varphi|\nabla\varphi_{xt}|^2dx,$$

so

$$|I_{44}^1| \le \varepsilon|\nabla\varphi_{xt}|_6^2 + c/\varepsilon|\Delta\varphi|_3^2|\nabla\varphi_{xt}|_2^2.$$

We integrate by parts in I_{44}^2. Then we have

$$I_{44}^2 = \int_\Omega v_i\nabla_i(\mathrm{rot}\,\psi)_{jxt}\nabla_j\varphi_{xt}dx = -\int_\Omega \nabla_j v_i\nabla_i(\mathrm{rot}\,\psi)_{jxt}\varphi_{xt}dx$$
$$= \int_\Omega \nabla_j\Delta\varphi(\mathrm{rot}\,\psi)_{jxt}\varphi_{xt}dx + \int_\Omega \nabla_j v_i(\mathrm{rot}\,\psi)_{jxt}\nabla_i\varphi_{xt}dx$$
$$\equiv K_1 + K_2,$$

where

$$|K_1| \le \varepsilon|\varphi_{xt}|_\infty^2 + c/\varepsilon|\Delta\nabla\varphi|_2^2|\mathrm{rot}\,\psi_{xt}|_2^2$$

and

$$|K_2| \le \varepsilon|\nabla\varphi_{xt}|_6^2 + c/\varepsilon|\nabla v|_3^2|\mathrm{rot}\,\psi_{xt}|_2^2$$
$$\le \varepsilon|\nabla\varphi_{xt}|_6^2 + c/\varepsilon|\nabla v|_3^2(|v_{xt}|_2^2 + |\nabla\varphi_{xt}|_2^2).$$

Next, we examine

$$I_{41} = -\int\limits_\Omega \Delta\varphi_{xt} v\cdot\nabla\varphi_{,xt}dx - \int\limits_\Omega v_{xt}v\cdot\nabla^2\varphi_{xt}dx \equiv I_{41}^1 + I_{41}^2,$$

where

$$\begin{aligned}&|I_{41}^1| \le \varepsilon|\Delta\varphi_{,xt}|_2^2 + c/\varepsilon|\nabla\varphi_{,xt}|_3^2|v|_6^2 \le \varepsilon|\nabla\varphi_{,xxt}|_2^2 + c/\varepsilon^3|v|_6^4|\nabla\varphi_{,xt}|_2^2,\\&|I_{41}^2| \le \varepsilon|\nabla^2\varphi_{,xt}|_2^2 + c/\varepsilon|v_{,xt}|_3^2|v|_6^2.\end{aligned}$$

The second term in I_4 is bounded by

$$|I_{42}| \le \varepsilon|\nabla\varphi_{,xt}|_6^2 + c/\varepsilon|v_t|_2^2|v_{xx}|_3^2.$$

Finally, we have the estimate

$$|I_{43}| \le \varepsilon|\nabla\varphi_{,xt}|_6^2 + c/\varepsilon|v_x|_3^2|\nabla v_t|_2^2.$$

Summarizing, we have the estimate

$$\begin{aligned}|I_4| \le{}& \varepsilon\|\nabla\varphi_{,t}\|_2^2 + c/\varepsilon[|\Delta\varphi|_3^2|\nabla\varphi_{,xt}|_2^2 + |v|_6^4|\nabla\varphi_{,xt}|_2^2 + |v|_6^2|v_{,xt}|_3^2\\&+|v_{,t}|_2^2|v_{,xx}|_3^2 + |\nabla v|_3^2(|v_{,xt}|_2^2 + |\nabla\varphi_{,xt}|_2^2) + |\nabla\varphi_{,xx}|_2^2|\operatorname{rot}\psi_{,xt}|_2^2].\end{aligned}$$

The fifth term on the r.h.s. of (3.40) is expressed in the form

$$\begin{aligned}I_5 ={}& -\int\limits_\Omega \left(\frac{1}{a+\eta}\right)_{,\eta}\eta_t(p_\varrho(a) - p_\varrho(a+\eta))\nabla\eta\cdot\nabla\varphi_{,xxt}dx\\&+\int\limits_\Omega \frac{1}{a+\eta}p_{\varrho\varrho}\eta_t\nabla\eta\cdot\nabla\varphi_{,xxt}dx\\&+\int\limits_\Omega \frac{1}{a+\eta}(p_\varrho(a) - p_\varrho(a+\eta))\nabla\eta_t\cdot\nabla\varphi_{,xxt}dx \equiv I_{51}+I_{52}+I_{53}.\end{aligned}$$

Hence, we have

$$\begin{aligned}&|I_{51}| \le \varepsilon|\nabla\varphi_{,xxt}|_2^2 + c/\varepsilon|\eta_t|_6^2|\eta|_6^2|\nabla\eta|_6^2,\\&|I_{52}| \le \varepsilon|\nabla\varphi_{,xxt}|_2^2 + c/\varepsilon|\eta_t|_6^2|\nabla\eta|_3^2,\\&|I_{53}| \le \varepsilon|\nabla\varphi_{,xxt}|_2^2 + c/\varepsilon|\eta|_\infty^2|\nabla\eta_t|_2^2.\end{aligned}$$

Finally, the last term on the r.h.s. of (3.40) is bounded by

$$\varepsilon|\nabla\varphi_{,xxt}|_2^2 + c/\varepsilon|f_t|_2^2.$$

Employing the above estimates in (3.40) yields

(3.41)

$$
\begin{aligned}
&\frac{d}{dt}|\nabla\varphi_{,xt}|_2^2+\frac{\mu}{a}|\nabla^2\varphi_{,xt}|_2^2+\frac{\nu}{a}|\Delta\varphi_{,xt}|_2^2\le\frac{\varepsilon_1}{\nu}|\nabla\mathrm{rot}\,\psi_{,xt}|_2^2\\
&\quad+\nu\varepsilon_2|\nabla\varphi_{,xxx}|_2^2+\frac{c}{\nu}[\|\eta_{,t}\|_1^2\|\eta\|_2^4+\|\eta_{,t}\|_1^2\|\eta\|_2^2+\|\eta_{,t}\|_1^2+\|\eta_t\|_1^2|\Delta\varphi|_3^2\\
&\quad+\|\eta\|_2^4|v_{,xt}|_2^2+(|\Delta\varphi|_3^2+|v|_6^4+\|\eta\|_2^4+|v_{,x}|_3^2)|\nabla\varphi_{,xt}|_2^2\\
&\quad+|v|_6^2|v_{,xt}|_3^2+|v_{,t}|_2^2|v_{,xx}|_3^2+|\nabla\varphi_{,xx}|_2^2|\mathrm{rot}\,\psi_{,xt}|_2^2\\
&\quad+|v_{,x}|_3^2|\nabla v_{,t}|_2^2+|f_{,t}|_2^2]+\nu c(1+1/\varepsilon_1^2)\|\eta\|_2^4|\nabla\varphi_{,xt}|_2^2\\
&\quad+\frac{\nu c}{\varepsilon_2}\|\eta_{,t}\|_1^4|\nabla\varphi_{,xx}|_2^2.
\end{aligned}
$$

Differentiate (3.13) with respect to x and t, multiply by $\mathrm{rot}\,\psi_{,xt}$ and integrate over Ω. Then we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\mathrm{rot}\,\psi_{,xt}|_2^2+\frac{\mu}{a}|\nabla\mathrm{rot}\,\psi_{,xt}|_2^2=-\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\Delta v\right)_{,xt}\cdot\mathrm{rot}\,\psi_{xt}dx\\
&\quad-\frac{\nu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\Delta\nabla\varphi\right)_{,xt}\cdot\mathrm{rot}\,\psi_{xt}dx-\int\limits_\Omega(v\cdot\nabla v)_{,xt}\cdot\mathrm{rot}\,\psi_{xt}dx\\
&\quad+\int\limits_\Omega f_{,xt}\cdot\mathrm{rot}\,\psi_{,xt}dx.
\end{aligned}\tag{3.42}
$$

Now we estimate the terms from the r.h.s. of (3.42). We express the first term in the form

$$
\begin{aligned}
I_1&=-\frac{\mu}{a}\int\limits_\Omega\left[\left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_t\Delta v\right]_{,x}\mathrm{rot}\,\psi_{,xt}dx\\
&\quad-\frac{\mu}{a}\int\limits_\Omega\left[\frac{\eta}{a+\eta}\Delta v_{,t}\right]_{,x}\cdot\mathrm{rot}\,\psi_{,xt}dx\equiv I_{11}+I_{12}.
\end{aligned}
$$

Integrating by parts in I_{11} yields

$$
I_{11}=\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_t\Delta v\cdot\mathrm{rot}\,\psi_{,xxt}dx
$$

and

$$
\begin{aligned}
|I_{11}|&\le\varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2+c/\varepsilon|\eta_t|_6^2|\Delta v|_3^2\le\varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2+c/\varepsilon|\eta_t|_6^2|v_{,xxx}|_2^{5/3}|v|_2^{1/3}\\
&\le\varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2+\varepsilon_1|v_{,xxx}|_2^2+c/(\varepsilon\varepsilon_1)|\eta_t|_6^{12}A_1^2.
\end{aligned}
$$

Performing differentiation with respect to x in I_{12} implies

$$I_{12} = -\frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta} \eta_x \Delta v_t \cdot \operatorname{rot}\psi_{,xt} dx - \frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}\Delta v_{,xt}\cdot \operatorname{rot}\psi_{,xt}dx \equiv I_{12}^1 + I_{12}^2,$$

where

$$\begin{aligned}|I_{12}^1| &\le \varepsilon|v_{,xxt}|_2^2 + c/\varepsilon|\eta_{,x}|_6^2|\operatorname{rot}\psi_{,xt}|_3^2\\ &\le \varepsilon|v_{,xxt}|_2^2 + c/\varepsilon\|\eta\|_2^2|\operatorname{rot}\psi_{,xxt}|_2|\operatorname{rot}\psi_{,xt}|_2\\ &\le \varepsilon(v_{,xxt}|_2^2 + |\operatorname{rot}\psi_{,xxt}|_2^2) + c/\varepsilon\|\eta\|_2^4|\operatorname{rot}\psi_{xt}|_2^2.\end{aligned}$$

To examine I_{12}^2 we write it in the form

$$I_{12}^2 = -\frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}\Delta\operatorname{rot}\psi_{,xt}\cdot\operatorname{rot}\psi_{,xt}dx - \frac{\mu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}\Delta\nabla\varphi_{,xt}\cdot\operatorname{rot}\psi_{,xt}dx \equiv J_1 + J_2.$$

Integrating by parts in J_1 yields

$$J_1 = \frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\nabla\eta\nabla\operatorname{rot}\psi_{,xt}\cdot\operatorname{rot}\psi_{,xt}dx + \frac{\mu}{a}\int\limits_\Omega\frac{\eta}{a+\eta}|\nabla\operatorname{rot}\psi_{,xt}|^2dx \equiv J_{11} + J_{12},$$

where

$$\begin{aligned}|J_{11}| &\le \varepsilon|\nabla\operatorname{rot}\psi_{,xt}|_2^2 + c/\varepsilon|\nabla\eta|_6^2|\operatorname{rot}\psi_{,xt}|_3^2\\ &\le \varepsilon|\nabla\operatorname{rot}\psi_{,xt}|_2^2 + c/\varepsilon\|\eta\|_2^2|\operatorname{rot}\psi_{,xxt}|_2|\operatorname{rot}\psi_{,xt}|_2\\ &\le \varepsilon|\nabla\operatorname{rot}\psi_{,xt}|_2^2 + c/\varepsilon\|\eta\|_2^4|\operatorname{rot}\psi_{,xt}|_2^2\end{aligned}$$

and J_{12} is absorbed by the second term on the l.h.s. of (3.42) in the following way

$$\begin{aligned}&\frac{\mu}{a}\int\limits_\Omega|\nabla\operatorname{rot}\psi_{,xt}|^2dx - \frac{\mu}{a}\int\limits_\Omega\frac{\eta}{a+\eta}|\nabla\operatorname{rot}\psi_{,xt}|^2dx\\ &= \frac{\mu}{a}\int\limits_\Omega\frac{a}{a+\eta}|\nabla\operatorname{rot}\psi_{,xt}|^2dx \ge \frac{2}{3}\frac{\mu}{a}|\nabla\operatorname{rot}\psi_{,xt}|_2^2,\end{aligned}$$

where the last inequality holds for $|\eta|_\infty \le a/2$.

Integrating by parts in J_2 yields

$$J_2 = \frac{\mu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta} \nabla\eta\cdot \operatorname{rot}\psi_{,xt}\Delta\varphi_{,xt}dx.$$

Hence,

$$\begin{aligned}|J_2| &\le \varepsilon|\Delta\varphi_{,xt}|_2^2 + c/\varepsilon|\nabla\eta|_6^2|\operatorname{rot}\psi_{,xt}|_3^2\\ &\le \varepsilon(|\Delta\varphi_{,xt}|_2^2 + |\nabla \operatorname{rot}\psi_{,xt}|_2^2) + c/\varepsilon\|\eta\|_2^4|\operatorname{rot}\psi_{,xt}|_2^2.\end{aligned}$$

Next, we examine the second term on the r.h.s. of (3.42). We express it in the form

$$\begin{aligned}I_2 = &-\frac{\nu}{a}\int\limits_\Omega \left[\left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_t\nabla\Delta\varphi\right]_{,x}\cdot \operatorname{rot}\psi_{,xt}dx\\ &-\frac{\nu}{a}\int\limits_\Omega \left[\frac{\eta}{a+\eta}\nabla\Delta\varphi_{,t}\right]_{,x}\operatorname{rot}\psi_{,xt}dx \equiv I_{21}+I_{22}.\end{aligned}$$

Integrating by parts in I_{21} implies the estimate

$$\begin{aligned}|I_{21}| &\le \varepsilon|\operatorname{rot}\psi_{,xxt}|_2^2 + c\nu^2/\varepsilon|\eta_t|_6^2|\Delta\nabla\varphi|_3^2\\ &\le \varepsilon|\operatorname{rot}\psi_{,xxt}|_2^2 + \frac{c\nu^2}{\varepsilon}\varepsilon_1|\nabla\varphi_{,xxx}|_2^2 + \frac{c\nu^2}{\varepsilon\varepsilon_1}\|\eta_t\|_1^4|\nabla\varphi_{xx}|_2^2.\end{aligned}$$

Performing differentiation in I_{22} yields

$$\begin{aligned}I_{22} = &-\frac{\nu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta}\eta_x\Delta\nabla\varphi_{,t}\cdot \operatorname{rot}\psi_{,xt}dx\\ &-\frac{\nu}{a}\int\limits_\Omega \frac{\eta}{a+\eta}\Delta\nabla\varphi_{,tx}\cdot \operatorname{rot}\psi_{,xt}dx \equiv L_1+L_2,\end{aligned}$$

where

$$\begin{aligned}|L_1| &\le \nu^2\varepsilon_1|\Delta\nabla\varphi_{,t}|_2^2 + \frac{c}{\varepsilon_1}|\eta_{,x}|_6^2|\operatorname{rot}\psi_{,xt}|_3^2\\ &\le \nu^2\varepsilon_1|\Delta\nabla\varphi_{,t}|_2^2 + c/\varepsilon_1\|\eta\|_2^2|\operatorname{rot}\psi_{,xxt}|_2|\operatorname{rot}\psi_{,xt}|_2\\ &\le \nu^2\varepsilon_1|\Delta\nabla\varphi_{,t}|_2^2 + \varepsilon|\operatorname{rot}\psi_{,xxt}|_2^2 + c/(\varepsilon\varepsilon_1)\|\eta\|_2^4|\operatorname{rot}\psi_{,xt}|_2^2.\end{aligned}$$

Integrating by parts in L_2 gives

$$L_2 = \frac{\nu}{a}\int\limits_\Omega \left(\frac{\eta}{a+\eta}\right)_{,\eta} \nabla\eta\cdot \operatorname{rot}\psi_{,xt}\Delta\varphi_{,xt}dx.$$

Hence,

$$
\begin{aligned}
|L_2| &\le \nu^2\varepsilon_1|\Delta\varphi_{,xt}|_2^2 + c/\varepsilon_1|\nabla\eta|_6^2|\mathrm{rot}\,\psi_{,xt}|_3^2\\
&\le \nu^2\varepsilon_1|\Delta\varphi_{,xt}|_2^2 + \varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + c/(\varepsilon\varepsilon_1)\|\eta\|_2^4|\mathrm{rot}\,\psi_{,xt}|_2^2.
\end{aligned}
$$

Now we consider the third term on the r.h.s. of (3.42). Performing differentiations we write it in the form

$$
\begin{aligned}
I_3 &= \int_\Omega v_{,xt}\cdot\nabla v\cdot\mathrm{rot}\,\psi_{,xt}dx + \int_\Omega v_{,x}\cdot\nabla v_{,t}\cdot\mathrm{rot}\,\psi_{,xt}dx\\
&\quad + \int_\Omega v_{,t}\cdot\nabla v_{,x}\cdot\mathrm{rot}\,\psi_{,xt}dx + \int_\Omega v\cdot\nabla v_{,xt}\cdot\mathrm{rot}\,\psi_{,xt}dx\\
&\equiv I_{31}+I_{32}+I_{33}+I_{34}.
\end{aligned}
$$

Consider the sum

$$
\begin{aligned}
I_{31}+I_{33} &= -\int_\Omega v_{,t}\cdot\nabla v_{,x}\cdot\mathrm{rot}\,\psi_{,xt}dx - \int_\Omega v_{,t}\cdot\nabla v\cdot\mathrm{rot}\,\psi_{,xxt}dx\\
&\quad + \int_\Omega v_{,t}\cdot\nabla v_{,x}\cdot\mathrm{rot}\,\psi_{,xt}dx = -\int_\Omega v_{,t}\cdot\nabla v\cdot\mathrm{rot}\,\psi_{,xxt}dx \equiv I_0,
\end{aligned}
$$

where

$$
\begin{aligned}
|I_0| &\le \varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + c/\varepsilon|v_{,x}|_\infty^2|v_{,t}|_2^2 \le \varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + c/\varepsilon|v_{,x}|_\infty^2 D_2^2\\
&\le \varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + c/\varepsilon|v_{,xxx}|_2^{5/3}|v|_2^{1/3}D_2^2\\
&\le \varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + \varepsilon_1|v_{,xxx}|_2^2 + c/(\varepsilon\varepsilon_1)|v|_2^2 D_2^{12}\\
&\le \varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + \varepsilon_1|v_{,xxx}|_2^2 + c/(\varepsilon\varepsilon_1)A_1^2D_2^{12}.
\end{aligned}
$$

Next, we consider

$$
\begin{aligned}
I_{32}+I_{34} &= -\int_\Omega v\cdot\nabla v_{,xt}\cdot\mathrm{rot}\,\psi_{,xt}dx - \int_\Omega v\cdot\nabla v_{,t}\cdot\mathrm{rot}\,\psi_{,xxt}dx\\
&\quad + \int_\Omega v\cdot\nabla v_{,xt}\cdot\mathrm{rot}\,\psi_{,xt}dx = -\int_\Omega v\cdot\nabla v_{,t}\cdot\mathrm{rot}\,\psi_{,xxt}dx \equiv I_\infty.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
|I_\infty| &\le \varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + c/\varepsilon|v_{,xt}|_3^2|v|_6^2 \le \varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + c/\varepsilon|v_{,xt}|_3^2D_1^2\\
&\le \varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + c/\varepsilon|v_{xxt}|_2^{3/2}|v_t|_2^{1/2}D_1^2 \le \varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + \varepsilon_1|v_{,xxt}|_2^2\\
&\quad + c/(\varepsilon_1\varepsilon)D_1^8D_2^2.
\end{aligned}
$$

Summarizing, we have

$$|I_3| \le \varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + \varepsilon_1(|v_{,xxx}|_2^2 + |v_{,xxt}|_2^2) + c/(\varepsilon\varepsilon_1)(A_1^2D_2^{12} + D_1^8D_2^2).$$

Finally, the last term on the r.h.s. of (3.42) is bounded by

$$\varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + c/\varepsilon|f_t|_2^2.$$

Employing the above estimates in (3.42) and assuming that ε is sufficiently small implies

(3.43)

$$\begin{aligned}
&\frac{d}{dt}|\mathrm{rot}\,\psi_{,xt}|_2^2 + \frac{\mu}{a}|\nabla\mathrm{rot}\,\psi_{,xt}|_2^2 \le \varepsilon(|v_{,xxx}|_2^2 + |v_{,xxt}|_2^2)\\
&\quad + \varepsilon_1\nu^2(|\nabla\varphi_{,xxt}|_2^2 + |\nabla\varphi_{,xxx}|_2^2)\\
&\quad + c/\varepsilon[|\eta_t|_6^{12}|v|_2^2 + \|\eta\|_2^4|\mathrm{rot}\,\psi_{,xt}|_2^2\\
&\quad + D_2^{12}|v|_2^2 + D_1^8|v_t|_2^2 + |f_t|_2^2] + \frac{c}{\varepsilon_1}(\nu^2\|\eta_t\|_1^4|\nabla\varphi_{,xx}|_2^2 + \|\eta\|_2^4|\mathrm{rot}\,\psi_{xt}|_2^2).
\end{aligned}$$

Multiplying (3.43) by $1/\nu$ and adding to (3.41) we obtain

(3.44)

$$\begin{aligned}
&\frac{d}{dt}\left(|\nabla\varphi_{xt}|_2^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi_{xt}|_2^2\right) + \frac{\mu}{a}|\nabla^2\varphi_{,xt}|_2^2 + \frac{\nu}{a}|\Delta\varphi_{xt}|_2^2 + \frac{\mu}{a\nu}|\nabla\mathrm{rot}\,\psi_{xt}|_2^2\\
&\le \frac{\varepsilon}{\nu}|v_{,xxx}|_2^2 + \varepsilon_1\nu|\nabla\varphi_{xxx}|_2^2 + \frac{c}{\nu\varepsilon}[|\eta_t|_6^{12}|v|_2^2 + |\eta|_{2,1}^2(|\eta|_{2,1}^4 + 1) + (\|\eta\|_2^4\\
&\quad + |\nabla\varphi_{,xx}|_2^2)|\mathrm{rot}\,\psi_{,xt}|_2^2 + (\|\eta\|_2^4 + |\Delta\varphi|_3^2 + |\nabla v|_3^2)|\nabla\varphi_{,xt}|_2^2 + D_1^2|\nabla\varphi_{,xt}|_3^2\\
&\quad + (\|\eta_t\|_1^2 + D_2^2)|v_{,xx}|_3^2 + (|v|_6^4 + \|\eta\|_2^4 + \|\eta\|_2^2 + D_1^2)|v_{,xt}|_3^2 + |\nabla v|_3^2|\nabla v_{,t}|_2^2\\
&\quad + D_2^{12}|v|_2^2 + D_1^8|v_t|_2^2 + |f_t|_2^2] + \frac{c\nu}{\varepsilon_1}[\|\eta_t\|_1^4|\nabla\varphi_{,xx}|_2^2 + \|\eta\|_2^4|\nabla\varphi_{,xt}|_2^2].
\end{aligned}$$

Let

$$\phi_5 = \phi_5(|\eta|_{2,1}, D_1, D_2, A_1) \quad |v|_2 \le A_1, \quad |v_{,t}|_2 \le D_2.$$

Simplifying, we write (3.44) in the form

$$\begin{aligned}
&a\frac{d}{dt}(|\nabla\varphi_{,xt}|_2^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi_{,xt}|_2^2) + \mu\left(|\nabla\varphi_{,xxt}|_2^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi_{,xxt}|_2^2\right)\\
&\quad + \nu|\Delta\varphi_{,xt}|_2^2 \le \frac{\varepsilon}{\nu}|v_{,xxx}|_2^2 + \frac{\varepsilon_1}{\nu}|\nabla\varphi_{,xxx}|_2^2 + \frac{c}{\nu\varepsilon}\phi_5(|\eta|_{2,1}, D_1, D_2, A_1)\\
&\quad [|\eta|_{2,1}^2 + |\Delta\varphi|_3^2 + |\Delta\varphi|_3^2|\nabla\varphi_{xt}|_2^2 + \|\eta\|_2^4|v_{xt}|_2^2\\
&\quad + (|\mathrm{rot}\,\psi_{,xt}|_2^2 + |\nabla\varphi_{,xt}|_2^2 + |v_{,xt}|_3^2 + |v_{,xx}|_3^2\\
&\quad + |v_{,xt}|_3^2 + |v|_2^2 + |v_{,t}|_2^2 + |\nabla v|_3^2|\nabla\varphi_{,xt}|_2^2 + |\nabla\varphi_{,xx}|_2^2|\mathrm{rot}\,\psi_{,xt}|_2^2)\\
&\quad + |\nabla v|_3^2|\nabla v_{,t}|_2^2 + |f_t|_2^2] + \frac{c\nu}{\varepsilon_1}|\eta|_{2,1}^4(|\nabla\varphi_{,xx}|_2^2 + |\nabla\varphi_{,xt}|_2^2).
\end{aligned} \tag{3.45}$$

Using the interpolations

$$
\begin{aligned}
&\alpha|\operatorname{rot}\psi_{,xt}|_2^2 \le c\alpha|\operatorname{rot}\psi_{,xxt}|_2^{2/3}|\operatorname{rot}\psi_t|_2^{4/3} \le \varepsilon|\operatorname{rot}\psi_{,xxt}|_2^2\\
&\quad + c(1/\varepsilon)\alpha^{3/2}|\operatorname{rot}\psi_{,t}|_2^2,\\
&\alpha|\nabla\varphi_{,xt}|_3^2 \le c\alpha|\nabla\varphi_{,xxt}|_2|\nabla\varphi_{,t}|_2 \le \varepsilon|\nabla\varphi_{,xxt}|_2^2 + c(1/\varepsilon)\alpha^2|\nabla\varphi_{,t}|_2^2,\\
&\alpha|v_{,xx}|_3^2 \le \alpha|v_{,xxx}|_2^{5/3}|v|_2^{1/3} \le \varepsilon|v_{,xxx}|_2^2 + c(1/\varepsilon)\alpha^6|v|_2^2,
\end{aligned}
\tag{3.46}
$$

where α is a parameter, in (3.45) yields

$$
\begin{aligned}
&a\frac{d}{dt}(|\nabla\varphi_{,xt}|_2^2 + \frac{1}{\nu}|\operatorname{rot}\psi_{,xt}|_2^2) + \mu(|\nabla\varphi_{,xxt}|_2^2 + \frac{1}{\nu}|\operatorname{rot}\psi_{,xxt}|_2^2)\\
&\quad + \nu|\Delta\varphi_{,xt}|_2^2 \le \frac{\varepsilon}{\nu}|v_{,xxx}|_2^2 + \nu\varepsilon_1|\nabla\varphi_{,xxx}|_2^2\\
&\quad + \frac{c}{\nu\varepsilon}\phi_6(|\eta|_{2,1}, D_1, D_2, A_1)[|v|_2^2 + |v_t|_2^2 + |\nabla\varphi_{xt}|_2^2\\
&\quad + |\nabla\varphi_{xx}|_2^2|\operatorname{rot}\psi_{xt}|^2 + |\nabla v|_3^2|\nabla\varphi_{,xt}|_2^2\\
&\quad + |\nabla v|_3^2|\nabla v_{,t}|_2^2 + |f_t|_2^2] + \frac{c\nu}{\varepsilon_1}|\eta|_{2,1}^4(|\nabla\varphi_{,xx}|_2^2 + |\nabla\varphi_{,xt}|_2^2),
\end{aligned}
\tag{3.47}
$$

Integrating (3.47) with respect to time yields

$$
\begin{aligned}
&a\bigg(|\nabla\varphi_{xt}(t)|_2^2 + \frac{1}{\nu}|\operatorname{rot}\psi_{xt}(t)|_2^2) + \mu\bigg(|\nabla\varphi_{xxt}|_{2,\Omega^t}^2 + \frac{1}{\nu}|\operatorname{rot}\psi_{xxt}|_{2,\Omega^t}^2\bigg)\\
&\quad + \nu|\Delta\varphi_{xt}|_{2,\Omega^t}^2 \le \frac{\varepsilon}{\nu}|v_{xxx}|_{2,\Omega^t}^2 + \nu\varepsilon_1|\nabla\varphi_{xxx}|_{2,\Omega^t}^2\\
&\quad + \frac{c}{\nu\varepsilon}\phi_6(|\eta|_{2,1,\infty,\Omega^t}, D_1, D_2, A_1)\bigg[A_1^2 + \bar{D}_2^2 + \frac{\Psi^2}{\nu^2}\\
&\quad + \frac{\chi_0^2}{\nu}|\operatorname{rot}\psi_{xt}|_{2,\Omega^t}^2 + \frac{\chi_0^2}{\nu}|v_x|_{3,2,\Omega^t}^2 + |v_x|_{3,\infty,\Omega^t}^2 D_2^2\\
&\quad + |f_t|_{2,\Omega^t}^2\bigg] + \frac{c\nu}{\varepsilon_1}|\eta|_{2,1,\infty,\Omega^t}^4\frac{\Psi^2}{\nu^2}.
\end{aligned}
\tag{3.48}
$$

Using the interpolations and estimates

$$
\begin{aligned}
&\frac{\chi_0^2}{\nu}|\operatorname{rot}\psi_{xt}|_{2,\Omega^t}^2 \le \frac{\chi_0^2}{\nu}|v_{xt}|_{2,\Omega^t}^2 \le \frac{\chi_0^2}{\nu}D_2^2,\\
&D_2^2|v_x|_{3,\infty,\Omega^t}^2 \le \varepsilon_2|v_{xx}|_{2,\infty,\Omega^t}^2 + c(1/\varepsilon_2)D_2^3|v|_{2,\infty,\Omega^t}^2\\
&\le \varepsilon_2|v_{xx}|_{2,\infty,\Omega^t}^2 + c(1/\varepsilon_2)D_2^3A_1^2,\\
&\frac{\chi_0^2}{\nu}|v_x|_{3,2,\Omega^t}^2 \le \varepsilon_3|v_{xx}|_{2,\Omega^t}^2 + c/\varepsilon_3\bigg(\frac{\chi_0^2}{\nu}\bigg)^2|v_x|_{2,\Omega^t}^2\\
&\le \varepsilon_3|v_{xx}|_{2,\Omega^t}^2 + c/\varepsilon_3\bigg(\frac{\chi_0^2}{\nu}\bigg)^2A_1^2
\end{aligned}
\tag{3.49}
$$

in (3.48) implies

$$
\begin{aligned}
&a\left(|\nabla\varphi_{xt}(t)|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_{xt}(t)|_2^2\right)+\mu\left(|\nabla\varphi_{xxt}|_{2,\Omega^t}^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_{xxt}|_{2,\Omega^t}^2\right)\\
&\quad+\nu|\Delta\varphi_{xt}|_{2,\Omega^t}^2\le\frac{\varepsilon}{\nu}|v_{xxx}|_{2,\Omega^t}^2+\nu\varepsilon_1|\nabla\varphi_{xxx}|_{2,\Omega^t}^2\\
&\quad+\varepsilon_2|v_{xx}|_{2,\infty,\Omega^t}^2+\varepsilon_3|v_{xx}|_{2,\Omega^t}^2\\
&\quad+\frac{c}{\nu}\phi_7\left(|\eta|_{2,1,\infty,\Omega^t},D_1,D_2,A_1,\frac{\chi_0}{\sqrt{\nu}},1/\varepsilon,1/\varepsilon_2,1/\varepsilon_3\right)\cdot\\
&\quad\cdot\left[A_1^2+\bar D_2^2+\frac{\Psi^2}{\nu^2}+D_2^2+|f_t|_{2,\Omega^t}^2\right]+\frac{c\nu}{\varepsilon_1}|\eta|_{2,1,\infty,\Omega^t}^4\frac{\Psi^2}{\nu^2}.
\end{aligned}\tag{3.50}
$$

The inequality implies (3.47). This concludes the proof. □

Remark 3.3. Integration of (3.37) with respect to time implies

$$
\begin{aligned}
&a\left(|\nabla\varphi(t)|_{1,1}^2+\frac{1}{\nu}|\mathrm{rot}\,\psi(t)|_{1,1}^2\right)+\mu\left(|\nabla\varphi|_{2,1,2,\Omega^t}^2+\frac{1}{\nu}|\mathrm{rot}\,\psi|_{2,1,2,\Omega^t}^2\right)\\
&\quad+\nu|\nabla\varphi|_{2,1,2,\Omega^t}^2\le\frac{c}{\nu}\phi(|\eta|_{2,1,\infty,\Omega^t},A_1,D_1,D_2)\cdot\\
&\quad\cdot[|\eta|_{2,1,2,\Omega^t}^2+|v_t|_{2,\Omega^t}^2+|\nabla v_t|_{2,\Omega^t}^2+|\Delta\varphi_t|_{3,2,\Omega^t}^2\\
&\quad+|\Delta\varphi|_{3,\infty,\Omega^t}^2(|\nabla\mathrm{rot}\,\psi_t|_{2,\Omega^t}^2+\|\nabla\varphi_t\|_{1,2,\Omega^t}^2)+|\nabla v|_{2,\Omega^t}^2\|\nabla\varphi_t\|_{1,\infty,\Omega^t}^2\\
&\quad+\|\nabla\varphi_t\|_{1,2,\Omega^t}^2D_1^2+|v_t|_{3,2,\Omega^t}^2D_1^2+|f|_{2,\Omega^t}^2+|f_t|_{2,\Omega^t}^2]\\
&\quad+c\nu[|\eta|_{2,1,\infty,\Omega^t}^2(1+|\eta|_{2,1,\infty,\Omega^t}^2)|\Delta\varphi|_{3,2,\Omega^t}^2+\|\eta\|_{2,\infty,\Omega^t}^2\|\nabla\varphi_t\|_{1,2,\Omega^t}^2]\\
&\quad+a\left(|\nabla\varphi(0)|_{1,1}^2+\frac{1}{\nu}|\mathrm{rot}\,\psi(0)|_{1,1}^2\right).
\end{aligned}\tag{3.51}
$$

In view of Lemmas 2.1 and 2.9 we have

$$
\begin{aligned}
&|v_t|_{2,\Omega^t}\le\bar D_2,\quad |\nabla v_t|_{2,\Omega^t}\le D_2,\quad |v_t|_{3,2,\Omega^t}\le c\|v_t\|_{1,2,\Omega^t}\le c\bar D_2,\\
&|\nabla v|_{2,\Omega^t}\le A_1,\quad |\nabla\mathrm{rot}\,\psi_t|_{2,\Omega^t}^2\le|\nabla v_t|_{2,\Omega^t}^2\le D_2^2.
\end{aligned}\tag{3.52}
$$

Using the definition of Ψ and χ_0 in Notation 2.9 we have

$$
\begin{aligned}
&|\Delta\varphi_t|_{3,2,\Omega^t}\le c\|\nabla\varphi_t\|_{2,2,\Omega^t}\le c\frac{\Psi}{\nu},\\
&|\Delta\varphi|_{3,\infty,\Omega^t}\le c\|\nabla\varphi\|_{2,\infty,\Omega^t}\le c\frac{\chi_0}{\sqrt{\nu}},\\
&\|\nabla\varphi_t\|_{1,2,\Omega^t}\le\frac{\Psi}{\nu},\quad \|\nabla\varphi_t\|_{1,\infty,\Omega^t}\le\frac{\chi_0}{\sqrt{\nu}}.
\end{aligned}\tag{3.53}
$$

In virtue of (3.52) and (3.53) inequality (3.51) takes the form

$$
\begin{aligned}
&a\bigg(|\nabla\varphi(t)|_{1,1}^2+\frac{1}{\nu}|\operatorname{rot}\psi(t)|_{1,1}^2\bigg)+\mu\bigg(|\nabla\varphi|_{2,1,2,\Omega^t}^2+\frac{1}{\nu}|\operatorname{rot}\psi|_{2,1,2,\Omega^t}^2\bigg)\\
&\quad+\nu|\nabla\varphi|_{2,1,2,\Omega^t}^2\le\frac{c}{\nu}\phi(|\eta|_{2,1,\infty,\Omega^t},A_1,D_1,D_2)\cdot\\
&\quad\cdot\bigg[|\eta|_{2,1,2,\Omega^t}^2+\bar D_2^2+D_2^2+\frac{\Psi^2}{\nu^2}+\frac{\chi_0^2}{\nu}D_2^2+\frac{\Psi^2}{\nu^2}\frac{\chi_0^2}{\nu}+A_1^2\frac{\chi_0^2}{\nu}\\
&\quad+\frac{\Psi^2}{\nu^2}D_1^2+\bar D_2^2D_1^2+|f|_{2,\Omega^t}^2+|f_t|_{2,\Omega^t}^2\bigg]\\
&\quad+c\nu[|\eta|_{2,1,\infty,\Omega^t}^2(1+|\eta|_{2,1,\infty,\Omega^t}^2)+\|\eta\|_{2,\infty,\Omega^t}^2]\frac{\Psi^2}{\nu^2} \qquad (3.54)\\
&\quad+a\bigg(|\nabla\varphi(0)|_{1,1}^2+\frac{1}{\nu}|\operatorname{rot}\psi(0)|_{1,1}^2\bigg)\\
&\equiv\frac{c}{\nu}\phi_9\bigg(|\eta|_{2,1,\infty,\Omega^t},|\eta|_{2,1,2,\Omega^t},A_1,D_1,D_2,\bar D_2,\frac{\Psi}{\nu},\frac{\chi_0}{\sqrt{\nu}},\\
&\quad|f|_{2,\Omega^t},|f_t|_{2,\Omega^t}\bigg)+c\nu|\eta|_{2,1,\infty,\Omega^t}^2(1+|\eta|_{2,1,\infty,\Omega^t}^2)\frac{\Psi^2}{\nu^2}\\
&\quad+c\bigg(|\nabla\varphi(0)|_{1,1}^2+\frac{1}{\nu}|\operatorname{rot}\psi(0)|_{1,1}^2\bigg).
\end{aligned}
$$

For sufficiently small ε_3 we obtain from (3.39) and (3.54) the inequality

(3.55)

$$
\begin{aligned}
&a\bigg(\|\nabla\varphi(t)\|_1^2+\|\nabla\varphi_t(t)\|_1^2+\frac{1}{\nu}\|\operatorname{rot}\psi(t)\|_1^2+\frac{1}{\nu}\|\operatorname{rot}\psi_t(t)\|_1^2\bigg)\\
&\quad+\mu\bigg(\|\nabla\varphi\|_{2,2,\Omega^t}^2+\frac{1}{\nu}\|\operatorname{rot}\psi\|_{2,2,\Omega^t}^2\bigg)+\mu\bigg(\|\nabla\varphi_t\|_{2,2,\Omega^t}^2+\frac{1}{\nu}\|\operatorname{rot}\psi_t\|_{2,2,\Omega^t}^2\bigg)\\
&\quad+\nu(\|\nabla\varphi\|_{2,2,\Omega^t}^2+\|\nabla\varphi_t\|_{2,2,\Omega^t}^2)\le\frac{\varepsilon}{\nu}|v_{xxx}|_{2,\Omega^t}^2+\nu\varepsilon_1|\nabla\varphi_{xxx}|_{2,\Omega^t}^2\\
&\quad+\varepsilon_2|v_{xx}|_{2,\infty,\Omega^t}^2+\frac{c}{\nu}\phi_{10}+c\nu|\eta|_{2,1,\infty,\Omega^t}^2(1+|\eta|_{2,1,\infty,\Omega^t}^2)\frac{\Psi^2}{\nu^2}\\
&\quad+a\bigg(\|\nabla\varphi(0)\|_1^2+\|\nabla\varphi_t(0)\|_1^2+\frac{1}{\nu}\|\operatorname{rot}\psi(0)\|_1^2+\frac{1}{\nu}\|\operatorname{rot}\psi_t(0)\|_1^2\bigg),
\end{aligned}
$$

where we introduced the new function $\phi_{10}=\phi_8+\phi_9$ and

(3.56)

$$
\phi_{10}=\phi_{10}\bigg(|\eta|_{2,1,\infty,\Omega^t},|\eta|_{2,1,2,\Omega^t},A_1,D_1,D_2,\bar D_2,\frac{\Psi}{\nu},\frac{\chi_0}{\sqrt{\nu}},|f|_{2,\Omega^t},|f_t|_{2,\Omega^t}\bigg)
$$

Now we recall the form of some arguments of ϕ_{10}. From (2.93), (2.95) and

(2.96) we have

$$|\eta(t)|_r \le \exp\bigg(\sqrt{t}\frac{\Psi}{\nu}\bigg)\bigg[\sqrt{t}\frac{\Psi}{\nu} + |\eta(0)|_r\bigg], \quad r\in(1,\infty],$$

$$|\eta(t)|_{2,1} \le \exp(c\sqrt{t}\Phi_0)\bigg[\sqrt{t}\frac{\Psi}{\nu} + |\eta(0)|_{2,1}\bigg],$$

$$\begin{aligned} D_1^2 = c\bigg[&\exp\bigg(c\sqrt{t}\frac{\Psi}{\nu}\bigg)\bigg(t^{2/3}\frac{\Psi}{\nu} + t^{1/6}|\eta(0)|_3^2\bigg) + \frac{\Psi^{4/3}}{\nu^{2/3}} + \frac{\Psi^2}{\nu^{2/3}} \\ &+ \frac{\Psi^{4/3}}{\nu^{2(\varkappa/3-1/6)}}\bigg] + A_2^2, \end{aligned} \tag{3.57}$$

$$D_2^2 = c(a)\exp[cB_1(t)][B_2(t) + |\varrho_0|_\infty |v_t(0)|_2^2],$$

$$\bar{D}_2^2 = c(a)\exp[c\bar{B}_1(t)][\bar{B}_2(t) + |\varphi_0|_\infty |v_t(0)|_2^2],$$

where

$$A_2 = |f|_{3,6,\Omega^t} + |\varrho_0|_\infty^{1/6}|v_0|_6,$$

$$B_1(t) \le |\eta|_{3,2,\Omega^t}^2 + (\|\eta\|_{2,\infty,\Omega^t}^2 + D_1^2 + 1)A_1^2,$$

$$\begin{aligned} B_2(t) \le\ & |\eta_t|_{2,\Omega^t}^2 + \|\eta_t\|_{1,\infty,\Omega^t}^2\bigg(A_1^2\frac{\Psi^2}{\nu^2} + A_1^2 D_1^2 + |f|_{2,\Omega^t}^2\bigg) \\ &+ A_1^2\frac{\Psi^2}{\nu^2} + |f_t|_{2,\Omega^t}^2, \end{aligned}$$

$$\bar{B}_1(t) \le B_1(t) + |\eta|_{2,\Omega^t}^2,$$

$$\bar{B}_2(t) \le B_2(t) + (1 + |\eta|_{\infty,\Omega^t}^2)|f|_{2,\Omega^t}^2,$$

$$A_1^2 = B(T)/(1-e^{-\mu T/2}) + \frac{3}{2}\int\limits_\Omega \bigg(\frac{1}{2}\varrho_0 v_0^2 + \frac{A}{\varkappa-1}\varrho^\varkappa\bigg)dx,$$

$$\begin{aligned} B(T) = c\sup_{k\in\mathbb{N}_0}\exp\bigg(c\int\limits_{kT}^{(k+1)T}\eta^2(t)dt\bigg)&[|\varrho_0|_1^2|f|_{\infty,2,\Omega\times(kT,(k+1)T)}^2 \\ &+ \mu a^\varkappa + |\varrho_0|_1^2|f|_{\infty,1,\Omega^t}^2 + |\varrho_0|_2^2|v_0|_2^2], \end{aligned}$$

$a/2 \le \varrho \le 3a/2$, a, T are given. This ends Remark 3.3.

Lemma 3.5. *Let the assumptions of Lemma 3.1. Let Notation 2.10 be applied. Then*

$$\begin{aligned} &\frac{d}{dt}\bigg(\nu|\nabla\varphi_{,xx}|_2^2 + |\mathrm{rot}\,\psi_{,xx}|_2^2\bigg) + \mu(\nu|\nabla^2\varphi_{,xx}|_2^2 + |\nabla\mathrm{rot}\,\psi_{,xx}|_2^2) + \nu^2|\Delta\varphi_{,xx}|_2^2 \\ &\le c\phi(\|\eta\|_2^2)[\|v_{,t}\|_1^2 + \|\eta\|_2^2 + (|v|_6^2 + |\nabla\varphi|_2^2)(|\nabla\varphi_{,xxx}|_2^2 \\ &\quad + |\nabla\varphi_{,xx}|_3^2 + \|\nabla\varphi_{,x}\|_1^2 + |v_{,x}|_6^2) + \|\nabla\varphi_x\|_1^2\|v_x\|_1^2 + (1+\|\eta\|_2^2)\|f\|_1^2], \end{aligned} \tag{3.58}$$

and

$$
\begin{aligned}
&\nu|\nabla\varphi_{xx}(t)|_2^2+|\operatorname{rot}\psi_{,xx}(t)|_2^2+\mu(\nu|\nabla^2\varphi_{,xx}|_{2,2,\Omega^t}^2+|\operatorname{rot}\psi_{,xxx}|_{2,2,\Omega^t}^2)\\
&\quad+\nu^2|\Delta\varphi_{,xx}|_{2,2,\Omega^t}^2\le c\bigg[\exp\Big(\sqrt{t}\Phi_*(t)\Big)\bigg(t\frac{\Psi^2}{\nu^2}+\frac{c_0^2}{\nu^2}\bigg)\|v_{,t}\|_{1,2,\Omega^t}^2\\
&\quad+\phi\bigg(A_1,A_2,\exp\Big(\sqrt{t}\Phi_*(t)\Big)\bigg(\sqrt{t}\frac{\Psi}{\nu}+\frac{c_0}{\nu}\bigg),\frac{\Psi}{\nu},\frac{\chi_0}{\sqrt{\nu}}\bigg)\\
&\quad+\bigg(1+\exp\Big(\sqrt{t}\Phi_*(t)\Big)\bigg(t\frac{\Psi^2}{\nu^2}+\frac{c_0^2}{\nu^2}\bigg)\bigg)\|f\|_{1,2,\Omega^t}^2\bigg]\\
&\quad+c(\nu|\nabla\varphi_{,xx}(0)|_2^2+|\operatorname{rot}\psi_{,xx}(0)|_2^2).
\end{aligned}
\tag{3.59}
$$

Proof. Differentiating (3.2) twice with respect to x, multiplying by $\nabla\varphi_{,xx}$ and integrating over Ω yields

$$
\begin{aligned}
&\frac{a}{2}\frac{d}{dt}|\nabla\varphi_{,xx}|_2^2+\mu|\nabla^2\varphi_{,xx}|_2^2+\nu|\Delta\varphi_{,xx}|_2^2\\
&=-\int_\Omega(\eta v_{,t})_{,xx}\cdot\nabla\varphi_{xx}dx-a_0\int_\Omega\nabla\eta_{xx}\cdot\nabla\varphi_{,xx}dx\\
&\quad-\int_\Omega[(a+\eta)v\cdot\nabla v]_{,xx}\cdot\nabla\varphi_{,xx}dx\\
&\quad+\int_\Omega[(p_\varrho(a)-p_\varrho(a+\eta))\nabla\eta]_{,xx}\cdot\nabla\varphi_{,xx}dx\\
&\quad+\int_\Omega[(a+\eta)f]_{,xx}\cdot\nabla\varphi_{xx}dx.
\end{aligned}
\tag{3.60}
$$

The first term on the r.h.s. is estimated by

$$
\varepsilon|\nabla\varphi_{,xxx}|_2^2+c/\varepsilon\|\eta\|_2^2\|v_t\|_1^2,
$$

the second by

$$
\varepsilon|\nabla\varphi_{,xxx}|_2^2+c/\varepsilon|\nabla\eta_x|_2^2.
$$

Next we examine the third term on the r.h.s. of (3.60). We write it in the form

$$
K\equiv a\int_\Omega(v\cdot\nabla v)_{,xx}\cdot\nabla\varphi_{xx}dx+\int_\Omega(\eta v\cdot\nabla v)_{,xx}\cdot\nabla\varphi_{xx}dx\equiv H+L.
$$

First we consider H. We drop a for simplicity. Then we have

$$H=\int_\Omega [v\cdot\nabla(\nabla\varphi+\mathrm{rot}\,\psi)]_{,xx}\cdot\nabla\varphi_{xx}dx=\int_\Omega (v\cdot\nabla\nabla\varphi)_{,xx}\cdot\nabla\varphi_{xx}dx$$
$$+\int_\Omega (v\cdot\nabla\mathrm{rot}\,\psi)_{,xx}\cdot\nabla\varphi_{xx}dx\equiv I_1+I_2.$$

Consider I_1. We express it in the form

$$I_1=\int_\Omega v_{,xx}\cdot\nabla\nabla\varphi\cdot\nabla\varphi_{,xx}dx+2\int_\Omega v_{,x}\cdot\nabla\nabla\varphi_{,x}\cdot\nabla\varphi_{xx}dx$$
$$+\int_\Omega v\cdot\nabla(\nabla\varphi)_{,xx}\cdot\nabla\varphi_{,xx}dx\equiv I_{11}+I_{12}+I_{13},$$

where

$$I_{13}=\frac{1}{2}\int_\Omega v\cdot\nabla|\nabla\varphi_{,xx}|^2dx=-\frac{1}{2}\int_\Omega \Delta\varphi\cdot|\nabla\varphi_{xx}|^2dx.$$

Hence

$$|I_{13}|\le\varepsilon|\nabla\varphi_{,xx}|_6^2+c/\varepsilon|\Delta\varphi|_3^2|\nabla\varphi_{,xx}|_2^2.$$

Consider I_{12}. Integrating by parts yields

$$I_{12}=-2\int_\Omega v\cdot\nabla\nabla\varphi_{,xx}\cdot\nabla\varphi_{,xx}dx-2\int_\Omega v\cdot\nabla\nabla\varphi_{,x}\cdot\nabla\varphi_{,xxx}dx\equiv I_{12}^1+I_{12}^2,$$

where I_{12}^1 is estimated by the same bound as I_{13} and

$$|I_{12}^2|\le\varepsilon|\nabla\varphi_{,xxx}|_2^2+c/\varepsilon|\nabla\varphi_{,xx}|_3^2|v|_6^2\le\varepsilon|\nabla\varphi_{,xxx}|_2^2+c/\varepsilon|\nabla\varphi_{,xx}|_3^2D_1^2.$$

Finally, we examine

$$I_{11}=-\int_\Omega \Delta\varphi_{,xx}\nabla\varphi\cdot\nabla\varphi_{,xx}dx-\int_\Omega v_{,xx}\cdot\nabla\nabla\varphi_{xx}\cdot\nabla\varphi dx\equiv I_{11}^1+I_{11}^2$$

and

$$|I_{11}^1|\le\varepsilon|\nabla\varphi_{,xxx}|_2^2+c/\varepsilon|\nabla\varphi_{,xx}|_3^2|\nabla\varphi|_6^2,$$
$$|I_{11}^2|\le\varepsilon|\nabla\varphi_{,xxx}|_2^2+c/\varepsilon|v_{,xx}|_3^2|\nabla\varphi|_6^2.$$

Summarizing, we have

$$\begin{aligned}|I_1|\le\varepsilon\|\nabla\varphi_{,xx}\|_1^2+c/\varepsilon[|\Delta\varphi|_3^2|\nabla\varphi_{,xx}|_2^2\\+|\nabla\varphi_{,xx}|_3^2D_1^2+|\nabla\varphi_{,xx}|_3^2|\nabla\varphi|_6^2+|v_{xx}|_3^2|\nabla\varphi|_6^2].\end{aligned}\tag{3.61}$$

Now we estimate I_2,

$$I_2 = \int\limits_\Omega v_{,xx} \cdot \nabla \mathrm{rot}\, \psi \cdot \nabla \varphi_{,xx} dx + 2 \int\limits_\Omega v_{,x} \cdot \nabla \mathrm{rot}\, \psi_{,x} \cdot \nabla \varphi_{,xx} dx$$
$$+ \int\limits_\Omega v \cdot \nabla \mathrm{rot}\, \psi_{,xx} \cdot \nabla \varphi_{,xx} dx \equiv I_{21} + I_{22} + I_{23}.$$

Consider I_{21}. Integrating by parts yields

$$I_{21} = -\int\limits_\Omega \Delta \varphi_{,xx} (\mathrm{rot}\, \psi + G) \cdot \nabla \varphi_{,xx} dx$$
$$- \int\limits_\Omega v_{,xx} (\mathrm{rot}\, \psi + G) \cdot \nabla \nabla \varphi_{,xx} dx \equiv I_{21}^1 + I_{21}^2,$$

where

$$\begin{aligned} |I_{21}^1| &\le \varepsilon |\Delta \varphi_{,xx}|_2^2 + c/\varepsilon |\nabla \varphi_{,xx}|_3^2 |\mathrm{rot}\, \psi + G|_6^2 \\ &\le \varepsilon |\Delta \varphi_{,xx}|_2^2 + c/\varepsilon |\nabla \varphi_{,xx}|_3^2 \cdot (|v|_6^2 + |\nabla \varphi|_6^2) \\ &\le \varepsilon |\Delta \varphi_{,xx}|_2^2 + c/\varepsilon |\nabla \varphi_{,xx}|_3^2 (D_1^2 + |\nabla \varphi|_6^2) \end{aligned}$$

and

$$\begin{aligned} |I_{21}^2| \le \varepsilon |\nabla \varphi_{xxx}|_2^2 + c/\varepsilon |v_{xx}|_3^2 |\mathrm{rot}\, \psi + G|_6^2 &\le \varepsilon |\nabla \varphi_{xxx}|_2^2 + c/\varepsilon |v_{,xx}|_3^2 (|v|_6^2 \\ + |\nabla \varphi|_6^2) \le \varepsilon |\nabla \varphi_{,xxx}|_2^2 + c/\varepsilon |v_{,xx}|_3^2 (D_1^2 + |\nabla \varphi|_6^2). \end{aligned}$$

Integrating by parts in the sum $I_{22} + I_{23}$ we get

$$I_{22} + I_{23} = -\int\limits_\Omega v \cdot \nabla \mathrm{rot}\, \psi_{,xx} \cdot \nabla \varphi_{,xx} dx - 2 \int\limits_\Omega v \cdot \nabla \mathrm{rot}\, \psi_{,x} \cdot \nabla \varphi_{,xxx} dx \equiv J_1 + J_2,$$

where integrating by parts in J_1 yields

$$J_1 = \int\limits_\Omega \Delta \varphi \mathrm{rot}\, \psi_{,xx} \cdot \nabla \varphi_{,xx} dx + \int\limits_\Omega v \mathrm{rot}\, \psi_{,xx} \cdot \nabla^2 \varphi_{,xx} dx \equiv J_{11} + J_{12},$$

where

$$\begin{aligned} |J_{11}| &\le \varepsilon |\nabla \varphi_{,xx}|_6^2 + c/\varepsilon |\Delta \varphi|_3^2 |\mathrm{rot}\, \psi_{,xx}|_2^2, \\ |J_{12}| &\le \varepsilon |\nabla^2 \varphi_{,xx}|_2^2 + c/\varepsilon |\mathrm{rot}\, \psi_{,xx}|_3^2 |v|_6^2 \le \varepsilon |\nabla^2 \varphi_{,xx}|_2^2 + c/\varepsilon |\mathrm{rot}\, \psi_{,xx}|_3^2 D_1^2. \end{aligned}$$

Finally,

$$|J_2| \le \varepsilon |\nabla \varphi_{,xxx}|_2^2 + c/\varepsilon |\mathrm{rot}\, \psi_{,xx}|_3^2 |v|_6^2 \le \varepsilon |\nabla \varphi_{,xxx}|_2^2 + c/\varepsilon |\mathrm{rot}\, \psi_{,xx}|_3^2 D_1^2.$$

Collecting estimates for I_2 yields

$$
\begin{aligned}
(3.62)\qquad |I_2| &\le \varepsilon\|\nabla\varphi_{,xx}\|_1^2 + c/\varepsilon[|\nabla\varphi_{,xx}|_3^2(D_1^2+|\nabla\varphi|_6^2)\\
&\quad + |v_{,xx}|_3^2(D_1^2+|\nabla\varphi|_6^2) + |\Delta\varphi|_3^2|\mathrm{rot}\,\psi_{,xx}|_2^2 + |\mathrm{rot}\,\psi_{,xx}|_3^2D_1^2].
\end{aligned}
$$

Next, we consider the expression

$$
\begin{aligned}
(3.63)\qquad L &= \int_\Omega [\eta v\cdot\nabla v]_{,xx}\cdot\nabla\varphi_{,xx}dx = \int_\Omega [\eta v\cdot\nabla\nabla\varphi]_{,xx}\cdot\nabla\varphi_{xx}dx\\
&\quad + \int_\Omega [\eta v\cdot\nabla\mathrm{rot}\,\psi]_{,xx}\cdot\nabla\varphi_{xx}dx \equiv L_1 + L_2.
\end{aligned}
$$

Performing differentiation in L_1 we have

$$
\begin{aligned}
L_1 &= \int_\Omega [\eta_{xx}v\cdot\nabla\nabla\varphi + \eta v_{xx}\cdot\nabla\nabla\varphi + \eta v\cdot\nabla\nabla\varphi_{xx} + 2\eta_x v_x\cdot\nabla\nabla\varphi\\
&\quad + 2\eta_x v\cdot\nabla\nabla\varphi_x + 2\eta v_x\nabla\nabla\varphi_x]\cdot\nabla\varphi_{,xx}dx \equiv \sum_{i=1}^6 L_{1i}.
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
|L_{11}| &\le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\eta_{,xx}|_2^2|v|_6^2|\nabla\varphi_{,x}|_6^2,\\
|L_{12}| &\le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\eta|_\infty^2|v_{,xx}|_3^2|\nabla\varphi_{,x}|_2^2,\\
L_{13} &= \frac{1}{2}\int_\Omega \eta v\cdot\nabla|\nabla\varphi_{,xx}|^2dx = -\frac{1}{2}\int_\Omega v\cdot\nabla\eta|\nabla\varphi_{,xx}|^2dx\\
&\quad - \frac{1}{2}\int_\Omega \eta\Delta\varphi|\nabla\varphi_{,xx}|^2dx \equiv L_{13}^1 + L_{13}^2.
\end{aligned}
$$

Continuing, we have

$$
\begin{aligned}
|L_{13}^1| &\le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|v|_6^2|\nabla\eta|_6^2|\nabla\varphi_{,xx}|_2^2,\\
|L_{13}^2| &\le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\eta|_\infty^2|\Delta\varphi|_3^2|\nabla\varphi_{,xx}|_2^2,\\
|L_{14}| &\le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\eta_x|_6^2|v_x|_6^2|\nabla\varphi_{,x}|_2^2,\\
|L_{15}| &\le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\eta_x|_6^2|v|_6^2|\nabla\varphi_{,xx}|_2^2,\\
|L_{16}| &\le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\eta|_\infty^2|v_{,x}|_3^2|\nabla\varphi_{,xx}|_2^2.
\end{aligned}
$$

Summarizing the above estimates we get

$$
(3.64)\qquad |L_1| \le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon\|\eta\|_2^2\|\nabla\varphi_x\|_1^2(|v|_6^2 + |v_{xx}|_3^2 + |\Delta\varphi|_3^2 + |v_x|_6^2).
$$

Next we consider L_2 from (3.63). Performing differentiations we obtain

$$\begin{aligned} L_2 = \int_\Omega [&\eta_{,xx} v\cdot\nabla\mathrm{rot}\,\psi + \eta v_{xx}\cdot\nabla\mathrm{rot}\,\psi + \eta v\cdot\nabla\mathrm{rot}\,\psi_{xx} \\ &+ 2\eta_x v_x\cdot\nabla\mathrm{rot}\,\psi + 2\eta_x v\cdot\nabla\mathrm{rot}\,\psi_x + 2\eta v_x\cdot\nabla\mathrm{rot}\,\psi_x]\cdot\nabla\varphi_{,xx}dx \\ \equiv \sum_{i=1}^6 L_{2i}. \end{aligned}$$

Now we estimate the terms from L_2.

$$|L_{21}| \le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\eta_{,xx}|_2^2|v|_6^2|\nabla\mathrm{rot}\,\psi|_6^2.$$

Integrating by parts in L_{22} implies

$$\begin{aligned} L_{22} = &-\int_\Omega \nabla\eta\cdot v_{xx}(\mathrm{rot}\,\psi+G)\cdot\nabla\varphi_{xx}dx - \int_\Omega \eta\Delta\varphi_{xx}(\mathrm{rot}\,\psi+G)\cdot\nabla\varphi_{,xx}dx \\ &-\int_\Omega \eta v_{,xx}(\mathrm{rot}\,\psi+G)\cdot\nabla\nabla\varphi_{,xx}dx \equiv L_{22}^1 + L_{22}^2 + L_{22}^3, \end{aligned}$$

where

$$\begin{aligned} |L_{22}^1| &\le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\nabla\eta|_6^2|v_{,xx}|_2^2|\mathrm{rot}\,\psi+G|_6^2, \\ |L_{22}^2| &\le \varepsilon|\Delta\varphi_{xx}|_2^2 + c/\varepsilon|\eta|_\infty^2|\nabla\varphi_{xx}|_3^2|\mathrm{rot}\,\psi+G|_6^2, \\ |L_{22}^3| &\le \varepsilon|\nabla\varphi_{,xxx}|_2^2 + c/\varepsilon|\eta|_\infty^2|v_{,xx}|_3^2|\mathrm{rot}\,\psi+G|_6^2. \end{aligned}$$

Next we examine L_{23}. Integration by parts yields

$$\begin{aligned} L_{23} = &-\int_\Omega \nabla\eta\cdot v\mathrm{rot}\,\psi_{,xx}\cdot\nabla\varphi_{,xx}dx - \int_\Omega \eta\Delta\varphi\mathrm{rot}\,\psi_{,xx}\cdot\nabla\varphi_{,xx}dx \\ &-\int_\Omega \eta v\cdot\mathrm{rot}\,\psi_{,xx}\nabla^2\varphi_{,xx}dx = L_{23}^1 + L_{23}^2 + L_{23}^3. \end{aligned}$$

Continuing,

$$\begin{aligned} |L_{23}^1| &\le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\nabla\eta|_6^2|v|_6^2|\mathrm{rot}\,\psi_{,xx}|_2^2, \\ |L_{23}^2| &\le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\eta|_\infty^2|\Delta\varphi|_3^2|\mathrm{rot}\,\psi_{,xx}|_2^2, \\ |L_{23}^3| &\le \varepsilon|\nabla\varphi_{,xxx}|_2^2 + c/\varepsilon|\eta|_\infty^2|v|_6^2|\mathrm{rot}\,\psi_{,xx}|_3^2. \end{aligned}$$

Next we examine L_{24}. Integration by parts implies

$$\begin{aligned}L_{24} = -2\int_\Omega \nabla\eta_x v_x(\operatorname{rot}\psi + G)\cdot\nabla\varphi_{,xx}dx\\ -2\int_\Omega \eta_x\Delta\varphi_{,x}(\operatorname{rot}\psi+G)\cdot\nabla\varphi_{,xx}dx\\ -2\int_\Omega \eta_x v_x(\operatorname{rot}\psi+G)\cdot\nabla\varphi_{,xxx}dx \equiv L_{24}^1+L_{24}^2+L_{24}^3.\end{aligned}$$

Estimating, we get

$$\begin{aligned}|L_{24}^1| &\le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\eta_{,xx}|_2^2|v_{,x}|_6^2|\operatorname{rot}\psi+G|_6^2,\\ |L_{24}^2| &\le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\eta_{,x}|_6^2|\Delta\varphi_{,x}|_2^2|\operatorname{rot}\psi+G|_6^2,\\ |L_{24}^3| &\le \varepsilon|\nabla\varphi_{,xxx}|_2^2 + c/\varepsilon|\eta_{,x}|_6^2|v_{,x}|_6^2|\operatorname{rot}\psi+G|_6^2.\end{aligned}$$

Next, we have

$$|L_{25}| \le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\eta_{,x}|_6^2|v|_6^2|\operatorname{rot}\psi_{,xx}|_2^2.$$

Finally, we integrate by parts in L_{26}. Then we have

$$\begin{aligned}L_{26} = -2\int_\Omega \eta_{,x}v\cdot\nabla\operatorname{rot}\psi_{,x}\cdot\nabla\varphi_{,xx}dx - 2\int_\Omega \eta v\cdot\nabla\operatorname{rot}\psi_{,xx}\cdot\nabla\varphi_{,xx}dx\\ -2\int_\Omega \eta v\cdot\nabla\operatorname{rot}\psi_{,x}\cdot\nabla\varphi_{,xxx}dx \equiv L_{26}^1+L_{26}^2+L_{26}^3.\end{aligned}$$

Consider L_{26}^2. Integration by parts yields

$$\begin{aligned}L_{26}^2 = 2\int_\Omega \nabla\eta\cdot v\operatorname{rot}\psi_{,xx}\cdot\nabla\varphi_{,xx}dx + 2\int_\Omega \eta\Delta\varphi\operatorname{rot}\psi_{,xx}\cdot\nabla\varphi_{,xx}dx\\ +2\int_\Omega \eta v\cdot\nabla^2\varphi_{xx}\cdot\operatorname{rot}\psi_{xx}dx \equiv F_1+F_2+F_3.\end{aligned}$$

Continuing, we have

$$\begin{aligned}|F_1| &\le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\nabla\eta|_6^2|v|_6^2|\operatorname{rot}\psi_{,xx}|_2^2,\\ |F_2| &\le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\eta|_\infty^2|\Delta\varphi|_3^2|\operatorname{rot}\psi_{,xx}|_2^2,\\ |F_3| &\le \varepsilon|\nabla^2\varphi_{,xx}|_2^2 + c/\varepsilon|\eta|_\infty^2|v|_6^2|\operatorname{rot}\psi_{,xx}|_3^2.\end{aligned}$$

Finally, we have

$$
\begin{aligned}
&|L_{26}^1| \le \varepsilon|\nabla\varphi_{,xx}|_6^2 + c/\varepsilon|\eta_{,x}|_6^2|v|_6^2|\operatorname{rot}\psi_{,xx}|_2^2,\\
&|L_{26}^2| \le \varepsilon\|\nabla\varphi_{,xx}\|_1^2 + c/\varepsilon\|\eta\|_2^2(|v|_6^2+|\Delta\varphi|_3^2)|\operatorname{rot}\psi_{,xx}|_3^2,\\
&|L_{26}^3| \le \varepsilon|\nabla\varphi_{,xxx}|_2^2 + c/\varepsilon|\eta|_\infty^2|v|_6^2|\operatorname{rot}\psi_{,xx}|_3^2.
\end{aligned}
$$

Collecting all estimates for L_2 we have

(3.65)

$$
\begin{aligned}
|L_2| \le \varepsilon\|\nabla\varphi_{,xx}\|_1^2 &+ \frac{c}{\varepsilon}\|\eta\|_2^2[|v|_6^2|\operatorname{rot}\psi_{,x}|_6^2 + (|v_{,xx}|_3^2+|\nabla\varphi_{xx}|_3^2)|\operatorname{rot}\psi+G|_6^2\\
&+(|v|_6^2+|\Delta\varphi|_3^2)|\operatorname{rot}\psi_{,xx}|_3^2 + (|v_x|_6^2+|\Delta\varphi_x|_2^2)|\operatorname{rot}\psi+G|_6^2].
\end{aligned}
$$

Now we collect all estimates for the third term on the r.h.s. of (3.60). Using (3.61), (3.62), (3.63) and (3.64) we have

$$
\begin{aligned}
|H| \le \varepsilon\|\nabla\varphi_{,xx}\|_1^2 &+ c/\varepsilon[|\nabla\varphi_{,xx}|_3^2(|\Delta\varphi|_3^2+|\nabla\varphi|_6^2+D_1^2)\\
&+|v_{,xx}|_3^2(D_1^2+|\nabla\varphi|_6^2)+|\operatorname{rot}\psi_{,xx}|_3^2(D_1^2+|\Delta\varphi|_3^2)]
\end{aligned}
$$

and

$$
\begin{aligned}
|L| \le \varepsilon\|\nabla\varphi_{,xx}\|_1^2 &+ c/\varepsilon\|\eta\|_2^2[\|\nabla\varphi_{,x}\|_1^2(|v|_6^2+|v_{,x}|_6^2+|v_{,xx}|_3^2+|\Delta\varphi|_3^2)\\
&+|\operatorname{rot}\psi_{xx}|_3^2(|v|_6^2+|\Delta\varphi|_3^2)+|\operatorname{rot}\psi|_6^2(|v_{,x}|_6^2+|v_{,xx}|_3^2+|\nabla\varphi_{,xx}|_3^2)].
\end{aligned}
$$

Using that $v=\nabla\varphi+\operatorname{rot}\psi$ we derive

$$
\begin{aligned}
|K| \le |H|+|L| \le \varepsilon\|\nabla\varphi_{,xx}\|_1^2 &+ c/\varepsilon[|\nabla\varphi_{,xx}|_3^2(|\Delta\varphi|_3^2+|\nabla\varphi|_6^2+D_1^2)\\
&+(1+\|\eta\|_2^2)|\operatorname{rot}\psi_{,xx}|_3^2(D_1^2+|\Delta\varphi|_3^2)\\
&+\|\eta\|_2^2\|\nabla\varphi_{,x}\|_1^2(D_1^2+|v_{,x}|_6^2+|v_{,xx}|_3^2+|\Delta\varphi|_3^2)\\
&+\|\eta\|_2^2|\operatorname{rot}\psi|_6^2(|v_{,x}|_6^2+|v_{,xx}|_3^2+|\nabla\varphi_{,xx}|_3^2)].
\end{aligned}
$$

The fourth term on the r.h.s. of (3.60) is estimated by

$$
\varepsilon|\nabla\varphi_{,xxx}|_2^2 + c/\varepsilon\|\eta\|_2^4.
$$

Finally, the last term on the r.h.s. of (3.60) is bounded by

$$
\varepsilon|\nabla\varphi_{,xxx}|_2^2 + c/\varepsilon[|f_{g,x}|_2^2+|\eta_{,x}|_6^2|f|_3^2+|\eta|_\infty^2|f_x|_2^2].
$$

Using the above estimates in (3.60) and assuming that $\varepsilon\le\nu/2$ we obtain the inequality

(3.66)

$$
\begin{aligned}
a\frac{d}{dt}|\nabla\varphi_{,xx}|_2^2 &+ \mu|\nabla^2\varphi_{,xx}|_2^2+\nu|\Delta\varphi_{,xx}|_2^2 \le \frac{c}{\nu}[\|\eta\|_2^2\|v_t\|_1^2\\
&+\|\eta\|_2^2+\|\eta\|_2^4+|\nabla\varphi_{,xx}|_3^2(|\Delta\varphi|_3^2+|\nabla\varphi|_6^2+D_1^2)\\
&+(1+\|\eta\|_2^2)|\operatorname{rot}\psi_{,xx}|_3^2(D_1^2+|\Delta\varphi|_3^2)+\|\eta\|_2^2\|\nabla\varphi_{,x}\|_1^2(D_1^2+|v_{,x}|_6^2\\
&+|v_{,xx}|_3^2+|\Delta\varphi|_3^2)+\|\eta\|_2^2(D_1^2+|\nabla\varphi|_6^2)(|v_{,x}|_6^2+|v_{,xx}|_3^2+|\nabla\varphi_{,xx}|_3^2)\\
&+(1+\|\eta\|_2^2)\|f\|_1^2].
\end{aligned}
$$

Simplifying, we write (3.66) in the form

$$
\begin{aligned}
&a\frac{d}{dt}|\nabla\varphi_{,xx}|_2^2+\mu|\nabla\varphi_{,xxx}|_2^2+\nu|\Delta\varphi_{,xx}|_2^2\le\frac{c}{\nu}[\|\eta\|_2^2\|v_{,t}\|_1^2\\
&\quad+\|\eta\|_2^2(\|\eta\|_2^2+1)+|\nabla\varphi_{,xx}|_3^2(1+\|\eta\|_2^2)(D_1^2+\|\nabla\varphi\|_2^2)\\
&\quad+|\mathrm{rot}\,\psi_{,xx}|_3^2(1+\|\eta\|_2^2)(D_1^2+\|\nabla\varphi\|_2^2)\\
&\quad+\|\eta\|_2^2\|\nabla\varphi_{,x}\|_1^2(D_1^2+|v_{,x}|_6^2+|\Delta\varphi|_3^2)+\|\eta\|_2^2(D_1^2+|\nabla\varphi|_6^2)|v_{,x}|_6^2\\
&\quad+(1+\|\eta\|_2^2)\|f\|_1^2].
\end{aligned}
\tag{3.67}
$$

Differentiating (3.2) twice with respect to x, multiplying by $\mathrm{rot}\,\psi_{,xx}$ and integrating over Ω implies

$$
\begin{aligned}
&\frac{a}{2}\frac{d}{dt}|\mathrm{rot}\,\psi_{,xx}|_2^2+\mu|\nabla\mathrm{rot}\,\psi_{,xx}|_2^2=-\int\limits_\Omega[\eta v_{,t}]_{,xx}\cdot\mathrm{rot}\,\psi_{,xx}dx\\
&\quad-\int\limits_\Omega[(a+\eta)v\cdot\nabla v]_{,xx}\cdot\mathrm{rot}\,\psi_{,xx}dx+\int\limits_\Omega[(a+\eta)f]_{,xx}\cdot\mathrm{rot}\,\psi_{,xx}dx.
\end{aligned}
\tag{3.68}
$$

The first term on the r.h.s. of (3.68) is bounded by

$$
\varepsilon|\mathrm{rot}\,\psi_{,xxx}|_2^2+c/\varepsilon\|\eta\|_2^2\|v_{,t}\|_1^2.
\tag{3.69}
$$

The second term on the r.h.s. of (3.68) is expressed in the form

$$
a\int\limits_\Omega(v\cdot\nabla v)_{,xx}\cdot\mathrm{rot}\,\psi_{,xx}dx+\int\limits_\Omega(\eta v\cdot\nabla v)_{,xx}\cdot\mathrm{rot}\,\psi_{,xx}dx\equiv J+K.
$$

First we consider J. We drop a for simplicity. Then we have

$$
J=\int\limits_\Omega(v\cdot\nabla\nabla\varphi)_{,xx}\cdot\mathrm{rot}\,\psi_{,xx}dx+\int\limits_\Omega(v\cdot\nabla\mathrm{rot}\,\psi)_{,xx}\cdot\mathrm{rot}\,\psi_{xx}dx\equiv J_1+J_2.
$$

Consider J_1,

$$
J_1=\int\limits_\Omega[v_{,xx}\cdot\nabla\nabla\varphi+v\cdot\nabla\nabla\varphi_{xx}+2v_{,x}\nabla\nabla\varphi_{,x}]\cdot\mathrm{rot}\,\psi_{,xx}dx\equiv J_{11}+J_{12}+J_{13}.
$$

Integrating by parts in J_{11} yields

$$
J_{11}=-\int\limits_\Omega\Delta\varphi_{,xx}\nabla\varphi\cdot\mathrm{rot}\,\psi_{,xx}dx-\int\limits_\Omega v_{,xx}\nabla\varphi\cdot\nabla\mathrm{rot}\,\psi_{,xx}dx\equiv J_{11}^1+J_{11}^2,
$$

where

$$|J_{11}^1|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\nabla\varphi_{,xxx}|_2^2|\nabla\varphi|_3^2,$$
$$|J_{11}^2|\le\varepsilon|\mathrm{rot}\,\psi_{,xxx}|_2^2+c/\varepsilon|v_{,xx}|_3^2|\nabla\varphi|_6^2.$$

Integrating by parts in J_{13} yields

$$J_{12}+J_{13}=-\int\limits_\Omega v\cdot\nabla\nabla\varphi_{,xx}\cdot\mathrm{rot}\,\psi_{,xx}dx-2\int\limits_\Omega v\cdot\nabla\nabla\varphi_{,x}\cdot\mathrm{rot}\,\psi_{,xxx}dx\equiv K_1+K_2,$$

where

$$|K_1|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\nabla\varphi_{,xxx}|_2^2|v|_3^2,$$
$$|K_2|\le\varepsilon|\mathrm{rot}\,\psi_{,xxx}|_2^2+c/\varepsilon|\nabla\varphi_{,xx}|_3^2|v|_6^2.$$

Summarizing,

$$\begin{aligned}|J_1|\le\varepsilon\|\mathrm{rot}\,\psi_{,xx}\|_1^2&+c/\varepsilon|\nabla\varphi_{,xxx}|_2^2(|\nabla\varphi|_3^2+|v|_3^2)+c/\varepsilon(|v_{,xx}|_3^2|\nabla\varphi|_6^2\\&+|\nabla\varphi_{,xx}|_3^2|v|_6^2).\end{aligned}$$

Consider J_2. We express it in the form

$$J_2=\int\limits_\Omega[v_{,xx}\cdot\nabla\mathrm{rot}\,\psi+2v_{,x}\cdot\nabla\mathrm{rot}\,\psi_{,x}+v\cdot\nabla\mathrm{rot}\,\psi_{,xx}]\cdot\mathrm{rot}\,\psi_{,xx}dx\equiv J_{21}+J_{22}+J_{23}.$$

Integration by parts in J_{21} gives

$$\begin{aligned}J_{21}=&-\int\limits_\Omega\Delta\varphi_{,xx}(\mathrm{rot}\,\psi+G)\cdot\mathrm{rot}\,\psi_{,xx}dx\\&-\int\limits_\Omega v_{,xx}(\mathrm{rot}\,\psi+G)\cdot\nabla\mathrm{rot}\,\psi_{,xx}dx\equiv J_{21}^1+J_{21}^2,\end{aligned}$$

where

$$|J_{21}^1|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\Delta\varphi_{,xx}|_2^2|\mathrm{rot}\,\psi+G|_3^2,$$
$$|J_{21}^2|\le\varepsilon|\mathrm{rot}\,\psi_{,xxx}|_2^2+c/\varepsilon|v_{,xx}|_3^2|\mathrm{rot}\,\psi+G|_6^2.$$

Integrating by parts in J_{22} implies

$$\begin{aligned}J_{22}+J_{23}=&-\int\limits_\Omega v\cdot\nabla\mathrm{rot}\,\psi_{,xx}\cdot\mathrm{rot}\,\psi_{,xx}dx-2\int\limits_\Omega v\cdot\nabla\mathrm{rot}\,\psi_{,x}\cdot\mathrm{rot}\,\psi_{,xxx}dx\\&\equiv M_1+M_2,\end{aligned}$$

where

$$M_1=-\frac12\int\limits_\Omega v\cdot\nabla|\mathrm{rot}\,\psi_{,xx}|^2dx=\frac12\int\limits_\Omega\Delta\varphi|\mathrm{rot}\,\psi_{,xx}|^2dx$$

and

$$|M_1| \le \varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2 + c/\varepsilon|\Delta\varphi|_2^2|\mathrm{rot}\,\psi_{,xx}|_3^2.$$

Finally,

$$|M_2| \le \varepsilon|\mathrm{rot}\,\psi_{,xxx}|_2^2 + c/\varepsilon|\mathrm{rot}\,\psi_{,xx}|_3^2|v|_6^2.$$

Summarizing,

$$\begin{aligned}|J_2| \le \varepsilon\|\mathrm{rot}\,\psi_{,xx}\|_1^2 &+ c/\varepsilon[|\Delta\varphi_{,xx}|_2^2|\mathrm{rot}\,\psi|_3^2 + |v_{,xx}|_3^2|\mathrm{rot}\,\psi + G|_6^2\\ &+ |\mathrm{rot}\,\psi_{,xx}|_3^2(|\Delta\varphi|_2^2 + |v|_6^2)].\end{aligned}$$

Collecting all estimates for J we get

$$\begin{aligned}|J| \le \varepsilon\|\mathrm{rot}\,\psi_{,xx}\|_1^2 &+ c/\varepsilon[|\nabla\varphi_{,xxx}|_2^2(|\nabla\varphi|_3^2 + |v|_3^2 + |\mathrm{rot}\,\psi + G|_3^2)\\ &+ (|\nabla\varphi_{,xx}|_3^2 + |\mathrm{rot}\,\psi_{,xx}|_3^2)(|v|_6^2 + |\nabla\varphi|_6^2 + |\Delta\varphi|_2^2)]\end{aligned} \tag{3.70}$$

Finally, we consider K. We write it in the form

$$\begin{aligned}K &= \int_\Omega [\eta v\cdot\nabla\nabla\varphi]_{,xx}\cdot\mathrm{rot}\,\psi_{,xx}dx + \int_\Omega [\eta v\cdot\nabla\mathrm{rot}\,\psi]_{,xx}\cdot\mathrm{rot}\,\psi_{,xx}dx\\ &\equiv K_1 + K_2.\end{aligned}$$

Performing differentiations we have

$$\begin{aligned}K_1 = \int_\Omega [&\eta_{xx}v\cdot\nabla\nabla\varphi + \eta v_{,xx}\cdot\nabla\nabla\varphi + \eta v\cdot\nabla\nabla\varphi_{,xx} + 2\eta_{,x}v_{,x}\cdot\nabla\nabla\varphi\\ &+ 2\eta_{,x}v\cdot\nabla\nabla\varphi_{,x} + 2\eta v_{,x}\nabla\nabla\varphi_{,x}]\cdot\mathrm{rot}\,\psi_{,xx}dx \equiv \sum_{i=1}^6 K_{1i}.\end{aligned}$$

Now we estimate the terms K_{1i}, $i = 1,\dots,6$. We have

$$\begin{aligned}|K_{11}| &\le \varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2 + c/\varepsilon|\eta_{,xx}|_2^2|v|_6^2|\nabla\varphi_{,x}|_6^2,\\ |K_{12}| &\le \varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2 + c/\varepsilon|\eta|_\infty^2|v_{,xx}|_2^2|\nabla\varphi_{,x}|_3^2,\\ |K_{13}| &\le \varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2 + c/\varepsilon|\eta|_\infty^2|v|_3^2|\nabla\varphi_{,xxx}|_2^2.\end{aligned}$$

Integrating by parts in K_{14} yields

$$\begin{aligned}K_{14} = -2\int_\Omega \eta_{xx}v\cdot\nabla\nabla\varphi\cdot\mathrm{rot}\,\psi_{,xx}dx &- 2\int_\Omega \eta_{,x}v\cdot\nabla\nabla\varphi_x\cdot\mathrm{rot}\,\psi_{,xx}dx\\ - 2\int_\Omega \eta_{,x}v\cdot\nabla\nabla\varphi\cdot\mathrm{rot}\,\psi_{,xxx}dx &\equiv K_{14}^1 + K_{14}^2 + K_{14}^3,\end{aligned}$$

where

$$|K_{14}^1|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\eta_{,xx}|_2^2|v|_6^2|\nabla\varphi_{,x}|_6^2,$$
$$|K_{14}^2|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\eta_{,x}|_6^2|v|_6^2|\nabla\varphi_{,xx}|_2^2,$$
$$|K_{14}^3|\le\varepsilon|\mathrm{rot}\,\psi_{,xxx}|_2^2+c/\varepsilon|\eta_{,x}|_6^2|v|_6^2|\nabla\varphi_{,x}|_6^2.$$

Next, we have

$$|K_{15}|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\eta_{,x}|_6^2|v|_6^2|\nabla\varphi_{,xx}|_2^2.$$

Integrating by parts in K_{16} gives

$$\begin{aligned}K_{16}&=-2\int_\Omega\eta_{,x}v\cdot\nabla\nabla\varphi_{,x}\cdot\mathrm{rot}\,\psi_{,xx}dx\\&\quad-2\int_\Omega\eta v\cdot\nabla\nabla\varphi_{,xx}\cdot\mathrm{rot}\,\psi_{,xx}dx\\&\quad-2\int_\Omega\eta v\cdot\nabla\nabla\varphi_{,x}\cdot\mathrm{rot}\,\psi_{,xxx}dx\equiv K_{16}^1+K_{16}^2+K_{16}^3,\end{aligned}$$

where

$$|K_{16}^1|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\eta_{,x}|_6^2|v|_6^2|\nabla\varphi_{,xx}|_2^2,$$
$$|K_{16}^2|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\eta|_\infty^2|v|_3^2|\nabla\varphi_{,xxx}|_2^2,$$
$$|K_{16}^3|\le\varepsilon|\mathrm{rot}\,\psi_{,xxx}|_2^2+c/\varepsilon|\eta|_\infty^2|v|_6^2|\nabla\varphi_{,xx}|_3^2.$$

Summarizing, the above estimates yields

$$\begin{aligned}|K_1|&\le\varepsilon\|\mathrm{rot}\,\psi_{,xx}\|_1^2+c/\varepsilon[\|\eta\|_2^2|v|_6^2\|\nabla\varphi_{,x}\|_1^2+|\eta|_\infty^2|v_{,xx}|_2^2|\nabla\varphi_{,x}|_3^2\\&\quad+|\eta|_\infty^2|v|_6^2\|\nabla\varphi_{,xx}\|_1^2].\end{aligned}\tag{3.71}$$

Finally, we estimate K_2. Performing differentiations we have

$$\begin{aligned}K_2=\int_\Omega&[\eta_{,xx}v\cdot\nabla\mathrm{rot}\,\psi+\eta v_{,xx}\cdot\nabla\mathrm{rot}\,\psi+\eta v\cdot\nabla\mathrm{rot}\,\psi_{,xx}+2\eta_{,x}v_{,x}\cdot\nabla\mathrm{rot}\,\psi\\&+2\eta_{,x}v\cdot\nabla\mathrm{rot}\,\psi_{,x}+2\eta v_{,x}\cdot\nabla\mathrm{rot}\,\psi_{,x}]\cdot\mathrm{rot}\,\psi_{,xx}dx\equiv\sum_{i=1}^6K_{2i}.\end{aligned}$$

Now, we estimate K_{2i}, $i=1,\cdots,6$.

$$|K_{21}|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\eta_{,xx}|_2^2|v|_6^2|\nabla\mathrm{rot}\,\psi|_6^2.$$

To estimate K_{22} we integrate by parts. Then we have

$$K_{22}=-\int\limits_\Omega\nabla\eta\cdot v_{,xx}(\mathrm{rot}\,\psi+G)\cdot\mathrm{rot}\,\psi_{,xx}dx$$
$$-\int\limits_\Omega\eta\Delta\varphi_{,xx}(\mathrm{rot}\,\psi+G)\cdot\mathrm{rot}\,\psi_{,xx}dx$$
$$-\int\limits_\Omega\eta v_{,xx}\cdot(\mathrm{rot}\,\psi+G)\cdot\nabla\mathrm{rot}\,\psi_{,xx}dx\equiv K_{22}^1+K_{22}^2+K_{22}^3,$$

where

$$|K_{22}^1|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\nabla\eta|_6^2|v_{,xx}|_2^2|\mathrm{rot}\,\psi+G|_6^2,$$
$$|K_{22}^2|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\eta|_\infty^2|\nabla\varphi_{,xxx}|_2^2|\mathrm{rot}\,\psi+G|_3^2,$$
$$|K_{22}^3|\le\varepsilon|\mathrm{rot}\,\psi_{,xxx}|_2^2+c/\varepsilon|\eta|_\infty^2|v_{,xx}|_3^2|\mathrm{rot}\,\psi+G|_6^2.$$

Consider K_{23}. Integration by parts yields

$$K_{23}=\frac{1}{2}\int\limits_\Omega\eta v\cdot\nabla|\mathrm{rot}\,\psi_{,xx}|^2dx=-\frac{1}{2}\int\limits_\Omega v\cdot\nabla\eta|\mathrm{rot}\,\psi_{,xx}|^2dx$$
$$-\frac{1}{2}\int\limits_\Omega\eta\Delta\varphi|\mathrm{rot}\,\psi_{,xx}|^2dx\equiv K_{23}^1+K_{23}^2,$$

where

$$|K_{23}^1|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\nabla\eta|_6^2|v|_6^2|\mathrm{rot}\,\psi_{,xx}|_2^2,$$
$$|K_{23}^2|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\eta|_\infty^2|\Delta\varphi|_3^2|\mathrm{rot}\,\psi_{,xx}|_2^2.$$

Integrating by parts in K_{24} gives

$$K_{24}=-2\int\limits_\Omega\eta_{,xx}v\cdot\nabla\mathrm{rot}\,\psi\cdot\mathrm{rot}\,\psi_{,xx}dx-2\int\limits_\Omega\eta_{,x}v\cdot\nabla\mathrm{rot}\,\psi_{,x}\cdot\mathrm{rot}\,\psi_{,xx}dx$$
$$-2\int\limits_\Omega\eta_{,x}v\cdot\nabla\mathrm{rot}\,\psi\cdot\mathrm{rot}\,\psi_{,xxx}dx\equiv K_{24}^1+K_{24}^2+K_{24}^3,$$

where

$$|K_{24}^1|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\eta_{,xx}|_2^2|v|_6^2|\mathrm{rot}\,\psi_{,x}|_6^2,$$
$$|K_{24}^2|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\eta_{,x}|_6^2|v|_6^2|\mathrm{rot}\,\psi_{,xx}|_2^2,$$
$$|K_{24}^3|\le\varepsilon|\mathrm{rot}\,\psi_{,xxx}|_2^2+c/\varepsilon|\eta_{,x}|_6^2|v|_6^2|\mathrm{rot}\,\psi_{,x}|_6^2.$$

Next, we have

$$|K_{25}|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\eta_{,x}|_6^2|v|_6^2|\mathrm{rot}\,\psi_{,xx}|_2^2.$$

Integrating by parts in K_{26} implies

$$K_{26}=-2\int_\Omega \eta_{,x}v\cdot\nabla\mathrm{rot}\,\psi_{,x}\cdot\mathrm{rot}\,\psi_{,xx}dx-2\int_\Omega \eta v\cdot\nabla\mathrm{rot}\,\psi_{,xx}\cdot\mathrm{rot}\,\psi_{xx}dx$$
$$-2\int_\Omega \eta v\cdot\nabla\mathrm{rot}\,\psi_{,x}\cdot\mathrm{rot}\,\psi_{,xxx}dx\equiv K_{26}^1+K_{26}^2+K_{26}^3,$$

where

$$|K_{26}^1|\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\eta_{,x}|_6^2|v|_6^2|\mathrm{rot}\,\psi_{,xx}|_2^2,$$
$$\begin{aligned}K_{26}^2&=-\int_\Omega \eta v\cdot\nabla|\mathrm{rot}\,\psi_{,xx}|^2dx\\&=\int_\Omega\nabla\eta\cdot v|\mathrm{rot}\,\psi_{,xx}|^2dx+\int_\Omega\eta\Delta\varphi|\mathrm{rot}\,\psi_{,xx}|^2dx\\&\equiv D_1+D_2.\end{aligned}$$

Continuing, we have

$$\begin{aligned}|D_1|&\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\nabla\eta|_6^2|v|_6^2|\mathrm{rot}\,\psi_{,xx}|_2^2,\\|D_2|&\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_6^2+c/\varepsilon|\eta|_\infty^2|\Delta\varphi|_3^2|\mathrm{rot}\,\psi_{,xx}|_2^2.\end{aligned}$$

Finally, we estimate

$$|K_{26}^3|\le\varepsilon|\mathrm{rot}\,\psi_{,xxx}|_2^2+c/\varepsilon|\eta|_\infty^2|v|_6^2|\mathrm{rot}\,\psi_{,xx}|_3^2.$$

Summarizing the estimates we get

$$\begin{aligned}|K_2|&\le\varepsilon\|\mathrm{rot}\,\psi_{,xx}\|_1^2+c/\varepsilon\|\eta\|_2^2[(|v|_6^2+|\Delta\varphi|_3^2)\|\mathrm{rot}\,\psi_{,x}\|_1^2\\&\quad+(D_1^2+|\nabla\varphi|_6^2)(|\mathrm{rot}\,\psi_{,xx}|_3^2+|\nabla\varphi_{,xx}|_3^2+|\nabla\varphi_{,xxx}|_2^2)].\end{aligned}\tag{3.72}$$

The last term on the r.h.s. of (3.68) is bounded by

$$\varepsilon|\mathrm{rot}\,\psi_{,xxx}|_2^2+c/\varepsilon(\|f_r\|_1^2+\|\eta\|_2^2\|f\|_1^2).\tag{3.73}$$

Using estimates (3.69)–(3.73) in (3.68) and assuming that ε is sufficiently small we get

$$\begin{aligned}&a\frac{d}{dt}|\mathrm{rot}\,\psi_{,xx}|_2^2+\mu|\nabla\mathrm{rot}\,\psi_{,xx}|_2^2\le c[\|\eta\|_2^2\|v_{,t}\|_1^2\\&\quad+|\nabla\varphi_{,xxx}|_2^2(1+\|\eta\|_2^2)(|\nabla\varphi|_6^2+|v|_6^2)\\&\quad+(|\nabla\varphi_{,xx}|_3^2+|\mathrm{rot}\,\psi_{,xx}|_3^2)(1+\|\eta\|_2^2)(|v|_6^2+\|\nabla\varphi\|_1^2)\\&\quad+\|\nabla\varphi_{,x}\|_1^2\|\eta\|_2^2(|v|_6^2+|v_{,xx}|_2^2)+\|\mathrm{rot}\,\psi_{,x}\|_1^2\|\eta\|_2^2(|v|_6^2+\|\nabla\varphi\|_2^2)\\&\quad+(1+\|\eta\|_2^2)\|f\|_1^2].\end{aligned}\tag{3.74}$$

Multiplying (3.74) by $1/\nu$ and adding to (3.67) yields

(3.75)
$$
\begin{aligned}
&a\frac{d}{dt}\bigg(|\nabla\varphi_{,xx}|_2^2+\frac{1}{\nu}|\operatorname{rot}\psi_{,xx}|_2^2\bigg)+\mu\bigg(|\nabla\varphi_{,xxx}|_2^2+\frac{1}{\nu}|\operatorname{rot}\psi_{,xxx}|_2^2\bigg)\\
&\quad+\nu|\Delta\varphi_{,xx}|_2^2\le\frac{c}{\nu}[\|\eta\|_2^2\|v_{,t}\|_1^2+\|\eta\|_2^2(\|\eta\|_2^2+1)\\
&\quad+|\nabla\varphi_{,xxx}|_2^2(1+\|\eta\|_2^2)(|\nabla\varphi|_6^2+|v|_6^2)\\
&\quad+(1+\|\eta\|_2^2)(|\nabla\varphi_{,xx}|_3^2+|\operatorname{rot}\psi_{,xx}|_3^2)(|v|_6^2+\|\nabla\varphi\|_2^2)\\
&\quad+\|\eta\|_2^2\|\nabla\varphi_{,x}\|_1^2(|v|_6^2+\|\nabla\varphi\|_2^2+\|v_{,x}\|_1^2)+\|\eta\|_2^2\|\operatorname{rot}\psi_{,x}\|_1^2(|v|_6^2+\|\nabla\varphi\|_2^2)\\
&\quad+\|\eta\|_2^2(|v|_6^2+|\nabla\varphi|_6^2)|v_{,x}|_6^2+(1+\|\eta\|_2^2)\|f\|_1^2].
\end{aligned}
$$

First we simplify inequality (3.67)

$$
\begin{aligned}
&a\frac{d}{dt}|\nabla\varphi_{xx}|_2^2+\mu|\nabla\varphi_{xxx}|_2^2+\nu|\Delta\varphi_{xx}|_2^2\\
&\le\frac{c}{\nu}\phi_{11}(\|\eta\|_2)[\|v_t\|_1^2+\|\eta\|_2^2+|\nabla\varphi_{xx}|_3^2(D_1^2+\|\nabla\varphi\|_2^2)\\
&\quad+|\operatorname{rot}\psi_{xx}|_3^2(D_1^2+\|\nabla\varphi\|_2^2)+\|\nabla\varphi_x\|_1^2(D_1^2+|v_x|_6^2+|\Delta\varphi|_3^2)\\
&\quad+(D_1^2+|\nabla\varphi|_6^2)|v_x|_6^2+\|f\|_1^2].
\end{aligned}
\tag{3.76}
$$

Similarly, simplification of (3.74) gives

(3.77)
$$
\begin{aligned}
&a\frac{d}{dt}|\operatorname{rot}\psi_{xx}|_2^2+\mu|\nabla\operatorname{rot}\psi_{xx}|_2^2\le\phi_{12}(\|\eta\|_2)[\|v_t\|_1^2\\
&\quad+|\nabla\varphi_{xxx}|_2^2(|\nabla\varphi|_6^2+|v|_6^2)+(|\nabla\varphi_{xx}|_3^2+|\operatorname{rot}\psi_{xx}|_3^2)(|v|_6^2+\|\nabla\varphi\|_1^2)\\
&\quad+\|\nabla\varphi_x\|_1^2(|v|_6^2+|v_{xx}|_2^2)+\|\operatorname{rot}\psi_x\|_1^2(|v|_6^2+\|\nabla\varphi\|_2^2)\\
&\quad+\|f\|_1^2].
\end{aligned}
$$

From (3.76) and (3.77) we have the clear version of (3.75)

(3.78)
$$
\begin{aligned}
&a\frac{d}{dt}\bigg(|\nabla\varphi_{xx}|_2^2+\frac{1}{\nu}|\operatorname{rot}\psi_{xx}|_2^2\bigg)+\mu\bigg(|\nabla\varphi_{xxx}|_2^2+\frac{1}{\nu}|\operatorname{rot}\psi_{xxx}|_2^2\bigg)\\
&\quad+\nu|\Delta\varphi_{xx}|_2^2\le\frac{c}{\nu}\phi_{13}(\|\eta\|_2)[\|v_t\|_1^2+\|\eta\|_2^2\\
&\quad+|\nabla\varphi_{xxx}|_2^2(|\nabla\varphi|_6^2+|v|_6^2)+(|\nabla\varphi_{xx}|_3^2+|\operatorname{rot}\psi_{xx}|_3^2)(|v|_6^2+\|\nabla\varphi\|_2^2)\\
&\quad+\|\nabla\varphi_x\|_1^2(|v|_6^2+\|\nabla\varphi\|_2^2+\|v_x\|_1^2)+\|\operatorname{rot}\psi_x\|_1^2(|v|_6^2+\|\nabla\varphi\|_2^2)\\
&\quad+(|v|_6^2+|\nabla\varphi|_6^2)|v_x|_6^2+\|f\|_1^2].
\end{aligned}
$$

This inequality implies (3.58).

Integrate (3.78) with respect to time and use Notation 2.9 and the estimates

$|v|_6 \le D_1$, $\|v_t\|_{1,2,\Omega^t} \le \bar D_2$. Then (3.78) implies
(3.79)

$$
\begin{aligned}
&a\bigg(|\nabla\varphi_{xx}(t)|_2^2 + \frac{1}{\nu}|\operatorname{rot}\psi_{xx}(t)|_2^2\bigg) + \mu\bigg(|\nabla\varphi_{xxx}|_{2,\Omega^t}^2 + \frac{1}{\nu}|\operatorname{rot}\psi_{xxx}|_{2,\Omega^t}^2\bigg)\\
&\quad + \nu|\Delta\varphi_{xx}|_{2,\Omega^t}^2 \le \frac{c}{\nu}\phi_{13}(\|\eta\|_{2,\infty,\Omega^t})\bigg[\bar D_2^2 + \|\eta\|_{2,2,\Omega^t}^2\\
&\quad + \frac{\Psi^2}{\nu^2}\bigg(\frac{\chi_0^2}{\nu} + D_1^2\bigg) + (|\nabla\varphi_{xx}|_{3,2,\Omega^t}^2 + |\operatorname{rot}\psi_{xx}|_{3,2,\Omega^t}^2)\bigg(D_1^2 + \frac{\chi_0^2}{\nu}\bigg)\\
&\quad + \frac{\chi_0^2}{\nu}\|v_x\|_{1,2,\Omega^t}^2 + \|\operatorname{rot}\psi_x\|_{1,2,\Omega^t}^2\bigg(D_1^2 + \frac{\chi_0^2}{\nu}\bigg)\\
&\quad + |v_x|_{6,2,\Omega^t}^2\bigg(D_1^2 + \frac{\chi_0^2}{\nu}\bigg) + \|f\|_{1,2,\Omega^t}^2\bigg]\\
&\quad + a\bigg(|\nabla\varphi_{xx}(0)|_2^2 + \frac{1}{\nu}|\operatorname{rot}\psi_{xx}(0)|_2^2\bigg).
\end{aligned}
$$

Examining (3.79) we see that there exists a nonlinear increasing positive function ϕ_{14} depending on $\|\eta\|_{2,\infty,\Omega^t}$, $\frac{\chi_0}{\sqrt{\nu}}$, D_1 such that

$$
\begin{aligned}
&a\bigg(|\nabla\varphi_{xx}(t)|_2^2 + \frac{1}{\nu}|\operatorname{rot}\psi_{xx}(t)|_2^2\bigg)\\
&\quad + \mu\bigg(|\nabla\varphi_{xxx}|_{2,\Omega^t}^2 + \frac{1}{\nu}|\operatorname{rot}\psi_{xxx}|_{2,\Omega^t}^2\bigg) + \nu|\Delta\varphi_{xx}|_{2,\Omega^t}^2\\
&\le \frac{c}{\nu}\phi_{14}\bigg(\|\eta\|_{2,\infty,\Omega^t}, \frac{\chi_0}{\sqrt{\nu}}, D_1\bigg)\bigg[\bar D_2^2 + \|\eta\|_{2,2,\Omega^t}^2 + \frac{\Psi^2}{\nu^2}\\
&\quad + |\nabla\varphi_{xx}|_{3,2,\Omega^t}^2 + |\operatorname{rot}\psi_{xx}|_{3,2,\Omega^t}^2 + \|v_x\|_{1,2,\Omega^t}^2 + \|\operatorname{rot}\psi_x\|_{1,2,\Omega^t}^2\\
&\quad + |v_x|_{6,2,\Omega^t}^2 + \|f\|_{1,2,\Omega^t}^2\bigg] + a\bigg(|\nabla\varphi_{xx}(0)|_2^2 + \frac{1}{\nu}|\operatorname{rot}\psi_{xx}(0)|_2^2\bigg).
\end{aligned}
\tag{3.80}
$$

We have that $|v_{xx}|_2^2 = |\operatorname{rot}\psi_{xx}|_2^2 + |\nabla\varphi_{xx}|_2^2$, because $\int_\Omega \operatorname{rot}\psi_{xx}\cdot\nabla\varphi_{xx}dx = 0$ in view of integrating by parts and the periodic boundary conditions. Then

$$
\begin{aligned}
&\|v_x\|_{1,2,\Omega^t}^2 + \|\operatorname{rot}\psi_x\|_{1,2,\Omega^t}^2 + |v_x|_{6,2,\Omega^t}^2 \le c\|v_x\|_{1,2,\Omega^t}^2\\
&\le c(|v_{xx}|_{2,\Omega^t}^2 + |v_x|_{2,\Omega^t}^2) \le c|v_{xx}|_{2,\Omega^t}^2 + cA_1^2.
\end{aligned}
\tag{3.81}
$$

Next we use the interpolation

$$
\begin{aligned}
|v_{xx}|_{2,\Omega^t}^2 &\le \varepsilon|v_{xxx}|_{2,\Omega^t}^2 + c(1/\varepsilon)|v_x|_{2,\Omega^t}^2\\
&\le \varepsilon|v_{xxx}|_{2,\Omega^t}^2 + c(1/\varepsilon)A_1^2
\end{aligned}
\tag{3.82}
$$

Next we use

$$
|\nabla\varphi_{xx}|_{3,2,\Omega^t} \le c\|\nabla\varphi_{xx}\|_{1,2,\Omega^t} \le c\frac{\Psi}{\nu}.
\tag{3.83}
$$

Finally, we use the interpolation
(3.84)

$$\begin{aligned}|\operatorname{rot}\psi_{xx}|^2_{3,2,\Omega^t}&\le\varepsilon|\operatorname{rot}\psi_{xxx}|^2_{2,\Omega^t}+c(1/\varepsilon)|\operatorname{rot}\psi_x|^2_{2,\Omega^t}\\&\le\varepsilon|\operatorname{rot}\psi_{xxx}|^2_{2,\Omega^t}+c(1/\varepsilon)|v_x|^2_{2,\Omega^t}\le\varepsilon|\operatorname{rot}\psi_{xxx}|^2_{2,\Omega^t}+c(1/\varepsilon)A_1^2.\end{aligned}$$

Exploiting (3.81)–(3.84) in (3.80) and assuming that ε is sufficiently small we see that there exists function ϕ_{15} such that
(3.85)

$$\begin{aligned}&a\bigg(|\nabla\varphi_{xx}(t)|_2^2+\frac{1}{\nu}|\operatorname{rot}\psi_{xx}(t)|_2^2\bigg)+\mu\bigg(|\nabla\varphi_{xxx}|^2_{2,\Omega^t}+\frac{1}{\nu}|\operatorname{rot}\psi_{xxx}|^2_{2,\Omega^t}\bigg)\\&\quad+\nu|\Delta\varphi_{xx}|^2_{2,\Omega^t}\le\frac{c}{\nu}\phi_{15}\bigg(\|\eta\|_{2,\infty,\Omega^t},\|\eta\|_{2,2,\Omega^t},\frac{\chi_0}{\sqrt{\nu}},\frac{\Psi}{\nu},\\&\quad A_1,D_1,\bar D_2,\|f\|_{1,2,\Omega^t}\bigg)+a\bigg(|\nabla\varphi_{xx}(0)|_2^2+\frac{1}{\nu}|\operatorname{rot}\psi_{xx}(0)|_2^2\bigg).\end{aligned}$$

The above inequality implies (3.59). This concludes the proof. □

Remark 3.4. From (3.55) and (3.85) we have
(3.86)

$$\begin{aligned}&a(\nu|\nabla\varphi(t)|^2_{2,1}+|\operatorname{rot}\psi(t)|^2_{2,1})+\mu(\nu|\nabla\varphi|^2_{3,1,2,\Omega^t}+|\operatorname{rot}\psi|^2_{3,1,2,\Omega^t})\\&\quad+\nu^2|\nabla\varphi|^2_{3,1,2,\Omega^t}\le\phi_{16}(D_1,D_2,\bar D_2,A_1,\frac{\Psi}{\nu},\frac{\chi_0}{\sqrt{\nu}},|\eta|_{2,1,2,\Omega^t},|\eta|_{2,1,\infty,\Omega^t})\\&\quad+c(1+|\eta|^2_{2,1,\infty,\Omega^t})|f|^2_{1,1,2,\Omega^t}+c(\nu|\nabla\varphi(0)|^2_{2,1}+|\operatorname{rot}\psi(0)|^2_{2,1,}).\end{aligned}$$

4 Estimate and Existence

Lemma 4.1. *Let Notation 2.10 hold. Let $\nu>0$ be given. Assume that $\eta(0)\in L_\infty(\Omega)$, $\eta(0),\nabla\varphi(0),\operatorname{rot}\psi(0)\in\Gamma_1^2(\Omega)$, $f\in L_2(0,T;H^1(\Omega))$, $f_t\in L_2(\Omega^T)$, $f\in L_6(0,T;L_3(\Omega))$, $f\in L_1(0,T;L_\infty(\Omega))$, $\nu|\nabla\varphi(0)|^2_{2,1}\le\text{const}$. Then for sufficiently large ν there exists a constant A depending on all above assumptions such that*

$$\chi_1(t)+\chi_2(t)+\Psi(t)\le A,\quad t\le T,\tag{4.1}$$

where T is proportional to ν.

Proof. Using (2.92)–(2.96), (3.79) and Notation 2.10 in (3.86) we have

$$\begin{aligned}&\chi_1^2+\chi_2^2+\Psi^2\le\phi_{20}(\exp(t^{1/2}\Phi_*)\bigg(t^{1/2}\frac{\Psi}{\nu}+c_0/\nu\bigg),\\&\phi\bigg(\frac{\Psi}{\nu^\alpha}\bigg),\frac{\Psi}{\nu},\frac{\chi_0}{\sqrt{\nu}},A_1,A_2|f|_{1,1,2,\Omega^t}\bigg)\\&\qquad+c(\nu|\nabla\varphi(0)|^2_{2,1}+|\operatorname{rot}\psi(0)|^2_{2,1,}),\end{aligned}\tag{4.2}$$

where $\phi(0)=0$.
To show (4.1) we take constant A so large that

$$\phi_{20}(\exp(t^{1/2}\Phi_*)\frac{c_0}{\nu},0,0,0,A_1,A_2,|f|_{1,1,2,\Omega^t})+c\Phi_0^2(0)<A. \tag{4.3}$$

Then for ν sufficiently large there exists a constant A such that (see [T, Ch. 1, Sect. 1.3])

$$\begin{aligned}&\phi_{20}(\exp(t^{1/2}\Phi_*)\bigg(t^{1/2}\frac{A}{\nu}+\frac{c_0}{\nu}\bigg),\phi\bigg(\frac{A}{\nu^\alpha}\bigg),\frac{A}{\nu},\frac{A}{\sqrt{\nu}},\\&A_1,A_2,|f|_{1,1,2,\Omega^t})+c\Phi_0^2(0)\le A.\end{aligned} \tag{4.4}$$

Hence (4.1) holds. This concludes the proof. □

Corollary 4.2. *Let the assumptions of Lemma 4.1 hold. Then (4.4) implies*
(4.5)

$$\begin{aligned}&|\nabla\varphi(t)|_{2,1}^2+\frac{1}{\nu}|\operatorname{rot}\psi(t)|_{2,1}^2+\mu|\nabla\varphi|_{3,1,2,\Omega^t}^2+\nu|\nabla\varphi|_{3,1,2,\Omega^t}^2+\frac{\mu}{\nu}|\operatorname{rot}\psi|_{3,1,2,\Omega^t}^2\\&\le\frac{1}{\nu}\phi(A)+|\nabla\varphi(0)|_{2,1}^2+\frac{1}{\nu}|\operatorname{rot}\psi(0)|_{2,1}^2.\end{aligned}$$

From (4.5) and estimates for η (see Lemmas 2.9, 2.10) we have

$$|\eta|_{2,1,\infty,\Omega^t}\le\frac{1}{\nu}\phi(A) \tag{4.6}$$

where

$$|\eta(0)|_{2,1}\le c_0/\nu$$

and

$$|\nabla\varphi(t)|_{2,1}^2+\nu|\nabla\varphi|_{3,1,2,\Omega^t}^2\le\frac{1}{\nu}\phi(A)+|\nabla\varphi(0)|_{2,1}^2+\frac{1}{\nu}|\operatorname{rot}\psi(0)|_{2,1}^2, \tag{4.7}$$

so quantities η and φ can be made in these norms as small as we want for sufficiently large ν.

Theorem 4.3. *Let the assumptions of Lemma 4.1 hold. Then there exists a solution to problem (1.7)–(1.9) such that $\psi,\varphi\in L_\infty(0,t;\Gamma_1^2(\Omega))\cap L_2(0,t;\Gamma_1^3(\Omega))$, $t\le T$ as long as estimate (4.1) holds.*

Proof. For T sufficiently small there exists a solution to problem (1.7)–(1.9) in the above spaces. Having estimate (4.1) for $t\le T$ the local solution can be extended in time up to time $t=T$. This ends the proof. □

5 Global estimate and existence

In Section 4 we proved long time estimate for solutions to problem (1.7)–(1.9), where the estimate time is proportional to ν. Since ν is finite we have only finite time estimate. Hence there exist finite time solutions. Therefore, to prove global existence we need additional differential inequality. This is derived in this Section.

Lemma 5.1. *Assume that φ, ψ, η are solutions to problem (3.1)–(3.3). Let the assumptions of Corollary 4.2 hold. Then*

$$
\begin{aligned}
&\frac{d}{dt}\bigg(|\nabla\varphi|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi|_2^2\bigg)+\frac{\mu}{a}|\nabla^2\varphi|_2^2+\frac{\nu}{a}|\Delta\varphi|_2^2+\frac{\mu}{a\nu}|\nabla\mathrm{rot}\,\psi|_2^2\\
&\le\frac{c}{\nu}[|\eta|_2^2+|\eta|_3^2|v_{,t}|_2^2+|\Delta\varphi|_3^2|v|_2^2+|v|_3^2|\nabla v|_2^2+|f|_2^2].
\end{aligned}\tag{5.1}
$$

Proof. Multiplying (3.2) by $\nabla\varphi$ and integrating over Ω yields

(5.2)

$$
\begin{aligned}
\frac{a}{2}\frac{d}{dt}|\nabla\varphi|_2^2+\mu|\nabla^2\varphi|_2^2+\nu|\Delta\varphi|_2^2&=-a_0\int_\Omega\nabla\eta\cdot\nabla\varphi dx-\int_\Omega\eta v_{,t}\cdot\nabla\varphi dx\\
&-\int_\Omega(a+\eta)v\cdot\nabla v\cdot\nabla\varphi dx+\int_\Omega[p_\varrho(a)-p_\varrho(a+\eta)]\nabla\eta\cdot\nabla\varphi dx\\
&+\int_\Omega(a+\eta)f\cdot\nabla\varphi dx.
\end{aligned}
$$

Integrating by parts the first term on the r.h.s. is bounded by

$$\varepsilon|\Delta\varphi|_2^2+c/\varepsilon|\eta|_2^2,$$

the second by

$$\varepsilon|\nabla\varphi|_6^2+c/\varepsilon|\eta|_3^2|v_{,t}|_2^2$$

the third term on the r.h.s. of (5.2) is expressed in the form

$$-a\int_\Omega v\cdot\nabla v\cdot\nabla\varphi dx-\int_\Omega\eta v\cdot\nabla v\cdot\nabla\varphi dx\equiv I_1+I_2.$$

integrating by parts in I_1 yields

$$I_1=a\int_\Omega\Delta\varphi v\cdot\nabla\varphi dx+\int_\Omega v\cdot v\cdot\nabla\nabla\varphi dx\equiv I_{11}+I_{12},$$

where

$$|I_{11}| \le \varepsilon|\nabla\varphi|_6^2 + c/\varepsilon|\Delta\varphi|_3^2|v|_2^2,$$
$$|I_{12}| \le \varepsilon|\nabla^2\varphi|_2^2 + c/\varepsilon|v|_6^2|v|_3^2.$$

Next we examine I_2. Then we have

$$|I_2| \le \varepsilon|\nabla\varphi|_6^2 + c/\varepsilon|\eta|_\infty^2|v|_3^2|\nabla v|_2^2.$$

The fourth term on the r.h.s. of (5.2) is estimated by

$$\varepsilon|\nabla\varphi|_6^2 + c/\varepsilon|\eta|_3^2|\nabla\eta|_2^2.$$

Finally, the last term on the r.h.s. of (5.2) equals

$$I_3 = a\int\limits_\Omega f_g\cdot\nabla\varphi dx + \int\limits_\Omega \eta f\cdot\nabla\varphi dx.$$

Hence

$$|I_3| \le \varepsilon|\nabla\varphi|_2^2 + c/\varepsilon|f_g|_2^2 + \varepsilon|\nabla\varphi|_6^2 + c/\varepsilon|\eta|_3^2|f|_2^2.$$

Employing the above estimates in (5.2) and assuming that ε is sufficiently small yields

$$\begin{aligned} a\frac{d}{dt}|\nabla\varphi|_2^2 + \mu|\nabla^2\varphi|_2^2 + \nu|\Delta\varphi|_2^2 &\le \frac{c}{\nu}[|\eta|_2^2 + |\eta|_3^2|v_{,t}|_2^2 + |\Delta\varphi|_3^2|v|_2^2 \\ &+ |v|_6^2|v|_3^2 + |\eta|_\infty^2|v|_3^2|\nabla v|_2^2 + |\eta|_3^2|\nabla\eta|_2^2 + |f_g|_2^2 + |\eta|_3^2|f|_2^2]. \end{aligned} \tag{5.3}$$

Multiplying (3.2) by $\operatorname{rot}\psi$ and integrating the result over Ω implies

$$\begin{aligned} \frac{a}{2}\frac{d}{dt}|\operatorname{rot}\psi|_2^2 + \mu|\nabla\operatorname{rot}\psi|_2^2 &= -\int\limits_\Omega \eta v_{,t}\cdot\operatorname{rot}\psi dx - \int\limits_\Omega (a+\eta)v\cdot\nabla v\cdot\operatorname{rot}\psi dx \\ &+ \int\limits_\Omega (a+\eta)f\cdot\operatorname{rot}\psi dx. \end{aligned} \tag{5.4}$$

We estimate the first term on the r.h.s. of (5.4) by

$$\varepsilon|\operatorname{rot}\psi|_6^2 + c/\varepsilon|\eta|_3^2|v_{,t}|_2^2.$$

The second term on the r.h.s. of (5.4) is expressed in the form

$$-a\int\limits_\Omega v\cdot\nabla v\cdot\operatorname{rot}\psi dx - \int\limits_\Omega \eta v\cdot\nabla v\cdot\operatorname{rot}\psi dx \equiv J_1 + J_2,$$

where

$$|J_1| \le \varepsilon |\mathrm{rot}\,\psi|_6^2 + c/\varepsilon |v|_3^2 |\nabla v|_2^2,$$
$$|J_2| \le \varepsilon |\mathrm{rot}\,\psi|_6^2 + c/\varepsilon |\eta|_\infty^2 |v|_3^2 |\nabla v|_2^2.$$

Finally, the last term on the r.h.s. of (5.4) takes the form

$$a \int_\Omega f_r \cdot \mathrm{rot}\,\psi dx + \int_\Omega \eta f \cdot \mathrm{rot}\,\psi dx \equiv J_3.$$

Hence,

$$|J_3| \le \varepsilon |\mathrm{rot}\,\psi|_2^2 + c/\varepsilon |f_r|_2^2 + \varepsilon |\mathrm{rot}\,\psi|_6^2 + c/\varepsilon |\eta|_3^2 |f|_2^2.$$

Employing the above estimates in (5.4) and using that ε is sufficiently small we derive the inequality

$$\begin{aligned} &\frac{a}{2}\frac{d}{dt}|\mathrm{rot}\,\psi|_2^2 + \mu |\nabla \mathrm{rot}\,\psi|_2^2 \le c[|\eta|_3^2 |v_{,t}|_2^2 + |v|_3^2 |\nabla v|_2^2 + |\eta|_\infty^2 |v|_3^2 |\nabla v|_2^2 \\ &\quad + |f_r|_2^2 + |\eta|_3^2 |f|_2^2]. \end{aligned} \tag{5.5}$$

Multiplying (5.5) by $1/\nu$, adding to (5.3) and using that $|\eta| \le a/2$ we obtain (5.1). This concludes the proof. □

Lemma 5.2. *Assume that φ, ψ, η are solutions to problem (3.1)–(3.3). Use (3.2) in the form (3.11). Let the assumptions of Corollary 4.2 hold. Then*

$$\begin{aligned} &\frac{d}{dt}\left(|\nabla \varphi_{,t}|_2^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi_{,t}|_2^2\right) + \frac{\mu}{a}|\nabla^2 \varphi_{,t}|_2^2 + \frac{\mu}{a}|\Delta \varphi_{,t}|_2^2 + \frac{\mu}{a\nu}|\nabla \mathrm{rot}\,\psi_{,t}|_2^2 \\ &\le \frac{c}{\nu}[|\eta_{,t}|_2^2 + |\eta|_{2,1}^2(|\nabla \mathrm{rot}\,\psi|_3^2 + |\Delta\varphi|_3^2) \\ &\quad + |\Delta\varphi|_3^2 |v_{,t}|_2^2 + |v_{,t}|_3^2 |v|_6^2 + |f_t|_2^2] + c\nu |\eta|_{2,1}^2 |\Delta\varphi|_3^2. \end{aligned} \tag{5.6}$$

Proof. Differentiate (3.13) with respect to t, multiply by $\nabla \varphi_{,t}$ and integrate over Ω. Then we have

(5.7)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\nabla\varphi_{,t}|_2^2 + \frac{\mu}{a}|\nabla^2\varphi_{,t}|_2^2 + \frac{\nu}{a}|\Delta\varphi_{,t}|_2^2 = -\int_\Omega \left(\frac{a_0}{a+\eta}\nabla\eta\right)_{,t} \cdot \nabla\varphi_{,t} dx \\ &\quad - \frac{\mu}{a}\int_\Omega \left(\frac{\eta}{a+\eta}\Delta v\right)_{,t} \cdot \nabla\varphi_{,t} dx - \frac{\nu}{a}\int_\Omega \left(\frac{\eta}{a+\eta}\nabla\Delta\varphi\right)_{,t} \cdot \nabla\varphi_{,t} dx \\ &\quad - \int_\Omega (v \cdot \nabla v)_{,t} \cdot \nabla\varphi_{,t} dx + \int_\Omega \left[\frac{1}{a+\eta}(p_\varrho(a) - p_\varrho(a+\eta))\nabla\eta\right]_{,t} \cdot \nabla\varphi_{,t} dx \\ &\quad + \int_\Omega f_{g,t} \cdot \nabla\varphi_t dx. \end{aligned}$$

The first term on the r.h.s. is bounded by

$$\begin{aligned}&c\int_\Omega(|\nabla\eta_{,t}|\,|\nabla\varphi_{,t}|+|\eta_{,t}|\,|\nabla\eta|\nabla\varphi_{,t}|)dx\\&\le\varepsilon|\nabla^2\varphi_{,t}|_2^2+c/\varepsilon|\eta_{,t}|_2^2+\varepsilon|\nabla\varphi_{,t}|_6^2+c/\varepsilon|\eta_{,t}|_3^2|\nabla\eta|_2^2.\end{aligned}$$

The second term on the r.h.s. of (5.7) equals

$$-\frac{\mu}{a}\int_\Omega\left(\frac{\eta}{a+\eta}\Delta\mathrm{rot}\,\psi\right)_{,t}\cdot\nabla\varphi_{,t}dx-\frac{\mu}{a}\int_\Omega\left(\frac{\eta}{a+\eta}\Delta\nabla\varphi\right)_{,t}\cdot\nabla\varphi_{,t}dx\equiv I_1+I_2.$$

Using estimates of I_1 and I_2 in the proof of Lemma 3.2 we have

$$|I_1|\le\varepsilon\|\nabla\varphi_{,t}\|_1^2+c/\varepsilon(|\eta|_{2,1}^4|\nabla\mathrm{rot}\,\psi|_2^2+|\eta|_{2,1}^2|\nabla\mathrm{rot}\,\psi|_3^2+\|\eta\|_2^2|\nabla\mathrm{rot}\,\psi_{,t}|_2^2)$$

and

$$\begin{aligned}|I_2|&\le\varepsilon\|\nabla\varphi_{,t}\|_1^2+c/\varepsilon(|\eta|_{2,1}^4|\nabla^2\varphi|_2^2+|\eta|_{2,1}^2|\nabla^2\varphi|_3^2+\|\eta\|_2^2|\nabla\varphi_{,t}|_2^2)\\&\quad+\frac{\mu}{a}\int_\Omega\frac{\eta}{a+\eta}|\nabla^2\varphi_{,t}|^2dx,\end{aligned}$$

where the last term is absorbed by the second term on the l.h.s. of (5.7). Consider the third term on the r.h.s. of (5.7). Repeating the proof of estimate of I_3 in the proof of Lemma 3.2 we obtain

$$\begin{aligned}|I_3|&\le\nu\varepsilon\|\nabla\varphi_{,t}\|_1^2+\frac{\nu c}{\varepsilon}(|\eta|_{2,1}^4|\Delta\varphi|_2^2+|\eta|_{2,1}^2|\Delta\varphi|_3^2+\|\eta\|_2^2|\nabla\varphi_t|_3^2)\\&\quad+\frac{\nu}{a}\int_\Omega\frac{\eta}{a+\eta}|\Delta\varphi_{,t}|^2dx,\end{aligned}$$

where the last term is absorbed by the last term on the l.h.s. of (5.7). We write the fourth term on the r.h.s. of (5.7) in the form

$$\begin{aligned}I_4&=\int_\Omega v_{,t}\cdot\nabla v\cdot\nabla\varphi_{,t}dx+\int_\Omega v\cdot\nabla v_{,t}\cdot\nabla\varphi_{,t}dx=-\int_\Omega\Delta\varphi_{,t}v\cdot\nabla\varphi_{,t}dx\\&\quad-\int_\Omega v_{,t}v\cdot\nabla^2\varphi_{,t}dx-\int_\Omega\Delta\varphi v_{,t}\cdot\nabla\varphi_{,t}dx-\int_\Omega vv_{,t}\cdot\nabla^2\varphi_{,t}dx.\end{aligned}$$

Hence, we have

$$\begin{aligned}|I_4|&\le\varepsilon|\nabla\varphi_{,t}|_6^2+c/\varepsilon(|\Delta\varphi_{,t}|_2^2|v|_3^2+|\Delta\varphi|_3^2|v_{,t}|_2^2)+\varepsilon|\nabla^2\varphi_{,t}|_2^2\\&\quad+c/\varepsilon|v_{,t}|_3^2|v|_6^2.\end{aligned}$$

We estimate the fifth term on the r.h.s. of (5.7) by

$$|I_5| \le c\int\limits_\Omega (|\eta_{,t}|\,|\nabla\eta| + |\eta|\,|\nabla\eta_t|)|\nabla\varphi_{,t}|dx \le \varepsilon|\nabla\varphi_{,t}|_6^2 + c/\varepsilon|\eta|_{2,1}^4.$$

Finally, we estimate the last term on the r.h.s. of (5.7) by

$$\varepsilon|\nabla\varphi_{,t}|_6^2 + c/\varepsilon|f_{g,t}|_2^2.$$

Employing the above estimates in (5.7) and assuming that ε is sufficiently small we have

$$\begin{aligned}
&\frac{d}{dt}|\nabla\varphi_{,t}|_2^2 + \frac{\mu}{a}\|\nabla\varphi_{,t}\|_1^2 + \frac{\nu}{a}|\Delta\varphi_{,t}|_2^2 \le \frac{c}{\nu}[|\eta_{,t}|_2^2 + |\eta|_{2,1}^4\\
&\quad + |\eta|_{2,1}^4(|\nabla\mathrm{rot}\,\psi|_2^2 + |\Delta\varphi|_2^2) + |\eta|_{2,1}^2(|\nabla\mathrm{rot}\,\psi|_3^2 + |\Delta\varphi|_3^2)\\
&\quad + \|\eta\|_2^2(|\nabla\mathrm{rot}\,\psi_{,t}|_2^2 + |\nabla^2\varphi_{,t}|_2^2) + |\Delta\varphi_{,t}|_2^2|v|_3^2\\
&\quad + |\Delta\varphi|_3^2|v_{,t}|_2^2 + |v_{,t}|_3^2|v|_6^2 + |f_t|_2^2] + c\nu(|\eta|_{2,1}^4|\Delta\varphi|_2^2 + |\eta|_{2,1}^2|\Delta\varphi|_3^2\\
&\quad + \|\eta\|_2^2|\nabla\varphi_{,t}|_3^2).
\end{aligned} \tag{5.8}$$

Differentiate (3.13) with respect to t, multiply by $\mathrm{rot}\,\psi_{,t}$ and integrate over Ω. Then we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\mathrm{rot}\,\psi_{,t}|_2^2 + \frac{\mu}{a}|\nabla\mathrm{rot}\,\psi_{,t}|_2^2 = -\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\Delta v\right)_{,t}\cdot\mathrm{rot}\,\psi_{,t}dx\\
&\quad - \frac{\nu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\Delta\nabla\varphi\right)_{,t}\cdot\mathrm{rot}\,\psi_{,t}dx - \int\limits_\Omega(v\cdot\nabla v)_{,t}\cdot\mathrm{rot}\,\psi_{,t}dx\\
&\quad + \int\limits_\Omega f_{r,t}\cdot\mathrm{rot}\,\psi_{,t}dx.
\end{aligned} \tag{5.9}$$

Now we examine the particular terms from the r.h.s. of (5.9). Looking for the estimate of the first term on the r.h.s. of (3.18) we see that the first term on the r.h.s. of (5.9) is estimated by

$$\begin{aligned}
&\varepsilon\|\mathrm{rot}\,\psi_{,t}\|_1^2 + c/\varepsilon[|\eta|_{2,1}^4(|\nabla\mathrm{rot}\,\psi|_2^2 + |\Delta\varphi|_2^2) + |\eta|_{2,1}^2(|\nabla\mathrm{rot}\,\psi|_3^2 + |\Delta\varphi|_3^2)\\
&\quad + \|\eta\|_2^2(|\mathrm{rot}\,\psi_{,t}|_3^2 + |\Delta\varphi_{,t}|_3^2)] + \frac{\mu}{a}\int\limits_\Omega\frac{\eta}{a+\eta}|\nabla\mathrm{rot}\,\psi_{,t}|^2dx,
\end{aligned}$$

where we used Corollary 4.2 and the last term is absorbed by the second term on the l.h.s. of (5.9).

Looking for the estimate of the second term on the r.h.s. of (3.18) (term I_3) we see that the second term on the r.h.s. of (5.9) is bounded by

$$\varepsilon|\mathrm{rot}\,\psi_{,t}|_6^2 + \frac{c\nu^2}{\varepsilon}(|\eta|_{2,1}^4|\Delta\varphi|_2^2 + |\nabla\eta_{,t}|_2^2|\Delta\varphi|_3^2 + \|\eta\|_2^2|\Delta\varphi_t|_2^2).$$

The third term on the r.h.s. of (5.9) equals

$$J = -\int_\Omega (v\cdot\nabla v_t + v_t\cdot\nabla v)\cdot\mathrm{rot}\,\psi_{,t}dx \equiv J_1 + J_2.$$

First we examine J_1. We write it in the form

$$J_1 = -\int_\Omega v\cdot\nabla\mathrm{rot}\,\psi_{,t}\cdot\mathrm{rot}\,\psi_{,t}dx - \int_\Omega v\cdot\nabla\nabla\varphi_{,t}\cdot\mathrm{rot}\,\psi_{,t}dx \equiv J_{11} + J_{12},$$

where

$$J_{11} = -\frac{1}{2}\int_\Omega v\cdot\nabla|\mathrm{rot}\,\psi_{,t}|^2dx = \frac{1}{2}\int_\Omega \Delta\varphi|\mathrm{rot}\,\psi_{,t}|^2dx$$

so

$$|J_{11}| \le \varepsilon|\mathrm{rot}\,\psi_{,t}|_6^2 + c/\varepsilon|\Delta\varphi|_3^2|\mathrm{rot}\,\psi_{,t}|_2^2.$$

Integrating by parts in J_{12} yields

$$J_{12} = \int_\Omega \nabla v\cdot\nabla\varphi_{,t}\cdot\mathrm{rot}\,\psi_{,t}dx$$

so

$$|J_{12}| \le \varepsilon|\mathrm{rot}\,\psi_{,t}|_6^2 + c/\varepsilon|\nabla v|_3^2|\nabla\varphi_{,t}|_2^2.$$

Next, we examine J_2. Integration by parts implies

$$J_2 = \int_\Omega \Delta\varphi_{,t}v\cdot\mathrm{rot}\,\psi_{,t}dx + \int_\Omega v_{,t}v\cdot\nabla\mathrm{rot}\,\psi_{,t}dx.$$

Hence, we have

$$|J_2| \le \varepsilon\|\mathrm{rot}\,\psi_{,t}\|_1^2 + c/\varepsilon(|\Delta\varphi_{,t}|_2^2|v|_3^2 + |v_{,t}|_3^2|v|_6^2).$$

Finally, the last term on the r.h.s. of (5.9) is bounded by

$$\varepsilon|\mathrm{rot}\,\psi_{,t}|_2^2 + c/\varepsilon|f_{r,t}|_2^2.$$

Employing the above estimates in (5.9) and using that ε is sufficiently small we derive the inequality

(5.10)
$$\begin{aligned}&\frac{d}{dt}|\mathrm{rot}\,\psi_{,t}|_2^2+\frac{\mu}{a}|\nabla\mathrm{rot}\,\psi_{,t}|_2^2\le c[|\eta|_{2,1}^2(|\eta|_{2,1}^2+1)|\nabla\mathrm{rot}\,\psi|_3^2\\&\quad+(\|\eta\|_2^2+|\Delta\varphi|_3^2)|\mathrm{rot}\,\psi_{,t}|_3^2+|\eta|_{2,1}^4|\Delta\varphi|_3^2+|\nabla\eta|_3^2|\Delta\varphi_{,t}|_2^2+|\nabla v|_2^2|\nabla\varphi_{,t}|_3^2\\&\quad+|v_{,t}|_3^2|v|_6^2+|f_t|_2^2]+c\nu^2[|\eta|_{2,1}^2(|\eta|_{2,1}^2+1)|\Delta\varphi|_3^2+\|\eta\|_2^2|\Delta\varphi_{,t}|_2^2].\end{aligned}$$

Multiplying (5.10) by $1/\nu$, adding to (5.8) and using that $c(\|\eta\|_2^2+|\Delta\varphi|_3^2)<\frac{\mu}{a}$ in view of Corollary 4.2 we obtain the inequality

$$\begin{aligned}&\frac{d}{dt}\left(|\nabla\varphi_{,t}|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_{,t}|_2^2\right)+\frac{\mu}{a}|\nabla^2\varphi_{,t}|_2^2+\frac{\nu}{a}|\Delta\varphi_{,t}|_2^2+\frac{\mu}{a\nu}|\nabla\mathrm{rot}\,\psi_{,t}|_2^2\\&\le\frac{c}{\nu}[|\eta_{,t}|_2^2+|\eta|_{2,1}^2(|\Delta\varphi|_2^2+|\nabla\mathrm{rot}\,\psi|_3^2+|\Delta\varphi|_3^2)\\&\quad+|\Delta\varphi|_3^2|v_t|_2^2+|v_t|_3^2|v|_6^2+|f_t|_2^2]+c\nu|\eta|_{2,1}^2|\Delta\varphi|_3^2,\end{aligned}\tag{5.11}$$

where we used that $c\|\eta\|_2^2<1$, $c\|v(t)\|_1^2<c^2$. Then (5.11) implies inequality (5.6). This concludes the proof. □

Remark 5.1. From (5.1) and (5.6) under the restrictions

$$c\|v(t)\|_1^2<c^2,\quad c(\|\eta\|_2^2+|\Delta\varphi|_3^2)<\frac{\mu}{a},\quad c\|\eta\|_2^2<1$$

which holds in view of Corollary 4.2 we obtain the inequality

$$\begin{aligned}&\frac{d}{dt}\left(|\nabla\varphi|_2^2+|\nabla\varphi_{,t}|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_{,t}|_2^2\right)\\&\quad+\frac{\mu}{a}(|\nabla^2\varphi|_2^2+|\nabla^2\varphi_{,t}|_2^2)+\frac{\nu}{a}(|\Delta\varphi|_2^2+|\Delta\varphi_{,t}|_2^2)\\&\quad+\frac{\mu}{a\nu}(|\nabla\mathrm{rot}\,\psi|_2^2+|\nabla\mathrm{rot}\,\psi_{,t}|_2^2)\\&\le\frac{c}{\nu}\Big[|\eta|_3^2|v_{,t}|_2^2+|\eta|_2^2+|\eta_t|_2^2+|\eta|_{2,1}^2(|\nabla\mathrm{rot}\,\psi|_3^2+|\Delta\varphi|_3^2)\\&\quad+|\Delta\varphi|_3^2|v_{,t}|_2^2+|v|_3^2|\nabla v|_2^2+|v_{,t}|_3^2|v|_6^2+|f|_2^2+|f_t|_2^2\Big]\\&\quad+c\nu|\eta|_{2,1}^2|\Delta\varphi|_3^2.\end{aligned}\tag{5.12}$$

Lemma 5.3. *Assume that φ, ψ, η are solutions to problem (3.1)–(3.3). Let the assumptions of Corollary 4.2 hold. Then*

(5.13)
$$\begin{aligned}&\frac{d}{dt}\left(|\nabla\varphi_{,x}|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_{,x}|_2^2\right)+\frac{\mu}{a}|\nabla^2\varphi_{,x}|_2^2+\frac{\nu}{a}|\Delta\varphi_{,x}|_2^2+\frac{\mu}{\nu a}|\nabla\mathrm{rot}\,\psi_{,x}|_2^2\\&\le\frac{c}{\nu}[|\eta|_\infty^2|v_{,t}|_2^2+|\nabla\eta|_2^2+|v|_6^2|\nabla v|_3^2+|f|_2^2].\end{aligned}$$

Proof. Differentiate (3.2) with respect to x, multiply by $\nabla\varphi_x$ and integrate over Ω. Then we get

(5.14)

$$\begin{aligned}\frac{a}{2}\frac{d}{dt}|\nabla\varphi_{,x}|_2^2+\mu|\nabla^2\varphi_{,x}|_2^2+\nu|\Delta\varphi_{,x}|_2^2&=-\int\limits_\Omega(\eta v_{,t})_{,x}\cdot\nabla\varphi_{,x}dx\\&-a_0\int\limits_\Omega\nabla\eta_{,x}\cdot\nabla\varphi_{,x}dx+\int\limits_\Omega[(a+\eta)v\cdot\nabla v]_{,x}\cdot\nabla\varphi_{,x}dx\\&+\int\limits_\Omega[(p_\varrho(a)-p_\varrho(a+\eta))\nabla\eta]_{,x}\cdot\nabla\varphi_{,x}dx+\int\limits_\Omega[(a+\eta)f]_{,x}\cdot\nabla\varphi_{,x}dx.\end{aligned}$$

After integration by parts in the first term on the r.h.s. of (5.14) we bound it by

$$\varepsilon|\nabla\varphi_{,xx}|_2^2+c/\varepsilon|\eta|_\infty^2|v_{,t}|_2^2,$$

the second term is bounded by

$$\varepsilon|\nabla\varphi_{,xx}|_2^2+c/\varepsilon|\nabla\eta|_2^2.$$

After integration by parts the third term on the r.h.s. of (5.14) is estimated by

$$\varepsilon|\nabla\varphi_{,xx}|_2^2+c/\varepsilon(1+|\eta|_\infty^2)|v|_6^2|\nabla v|_3^2$$

and the fourth term by

$$\varepsilon|\nabla\varphi_{,xx}|_2^2+c/\varepsilon|\eta|_\infty^2|\nabla\eta|_2^2.$$

Finally, the last term on the r.h.s. of (5.14) equals

$$I_1=-a\int\limits_\Omega f_g\cdot\nabla\varphi_{,xx}dx-\int\limits_\Omega\eta f\nabla\varphi_{,xx}dx$$

which is estimated by

$$|I_1|\le\varepsilon|\nabla\varphi_{,xx}|_2^2+c/\varepsilon(|f_g|_2^2+|\eta|_\infty^2|f|_2^2).$$

Employing the above estimates in (5.14) and assuming that ε is sufficiently small we obtain the inequality

$$\begin{aligned}\frac{d}{dt}|\nabla\varphi_{,x}|_2^2+\frac{\mu}{a}|\nabla^2\varphi_{,x}|_2^2+\frac{\nu}{a}|\Delta\varphi_{,x}|_2^2&\le\frac{c}{\nu}[|\eta|_\infty^2|v_{,t}|_2^2+|\nabla\eta|_2^2\\&+|v|_6^2|\nabla v|_3^2+|\eta|_\infty^2|\nabla\eta|_2^2+(1+|\eta|_\infty^2)|f|_2^2].\end{aligned}\tag{5.15}$$

Differentiate (3.2) with respect to x, multiply by $\mathrm{rot}\,\psi_{,x}$ and integrate over Ω. Then we derive

$$\begin{aligned}\frac{a}{2}\frac{d}{dt}|\mathrm{rot}\,\psi_{,x}|_2^2+\mu|\nabla\mathrm{rot}\,\psi_{,x}|_2^2&=-\int\limits_\Omega(\eta v_{,t})_{,x}\cdot\mathrm{rot}\,\psi_{,x}dx\\&+\int\limits_\Omega[(a+\eta)v\cdot\nabla v]_{,x}\cdot\mathrm{rot}\,\psi_{,x}dx+\int\limits_\Omega[(a+\eta)f]_{,x}\cdot\mathrm{rot}\,\psi_{,x}dx.\end{aligned}\tag{5.16}$$

Integrating by parts in the terms from the r.h.s. of the above inequality we obtain

$$\begin{aligned}\frac{d}{dt}|\mathrm{rot}\,\psi_{,x}|_2^2+\frac{\mu}{a}|\mathrm{rot}\,\psi_{,x}|_2^2&\le c[|\eta|_\infty^2|v_{,t}|_2^2+(1+|\eta|_\infty^2)|v|_6^2|\nabla v|_3^2\\&+(1+|\eta|_\infty^2)|f|_2^2].\end{aligned}\tag{5.17}$$

Multiplying (5.17) by $1/\nu$ and adding to (5.16) yields

$$\begin{aligned}&\frac{d}{dt}\bigg(|\nabla\varphi_{,x}|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_{,x}|_2^2\bigg)+\frac{\mu}{a}|\nabla^2\varphi_{,x}|_2^2+\frac{\nu}{a}|\Delta\varphi_{,x}|_2^2+\frac{\mu}{\nu a}|\nabla\mathrm{rot}\,\psi_{,x}|_2^2\\&\le\frac{c}{\nu}[|\eta|_\infty^2|v_t|_2^2+(1+|\eta|_\infty^2)|v|_6^2|\nabla v|_3^2+(1+|\eta|_\infty^2)|\nabla\eta|_2^2\\&\quad+(1+|\eta|_\infty^2)|f|_2^2].\end{aligned}\tag{5.18}$$

Using that $|\eta|_\infty\le a/2$ the above inequality implies (5.13). This concludes the proof. □

Remark 5.2. We use the restriction introduced in Remark 5.1. Then (5.12) and (5.13) imply the inequality

$$\begin{aligned}&\frac{d}{dt}\bigg(\|\nabla\varphi\|_1^2+|\nabla\varphi_t|_2^2+\frac{1}{\nu}\|\mathrm{rot}\,\psi\|_1^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_{,t}|_2^2\bigg)\\&\quad+\frac{\mu}{a}(\|\nabla\varphi\|_2^2+\|\nabla\varphi_{,t}\|_1^2)+\frac{\nu}{a}(\|\Delta\varphi\|_1^2+|\Delta\varphi_{,t}|_2^2)\\&\quad+\frac{\mu}{a\nu}(\|\mathrm{rot}\,\psi\|_2^2+|\mathrm{rot}\,\psi_{,t}|_2^2)\\&\le\frac{c}{\nu}[|\eta|_\infty^2|v_{,t}|_2^2+|\eta|_{1,1}^2+|v|_3^2|\nabla v|_2^2+|v_{,t}|_3^2|v|_6^2+|v|_6^2|\nabla v|_3^2\\&\quad+|f|_2^2+|f_t|_2^2].\end{aligned}\tag{5.19}$$

Lemma 5.4. *Assume that φ, ψ, η are solutions to problem (3.1)–(3.3) and*

Corollary 4.2 holds. Then

(5.20)

$$\frac{d}{dt}\bigg(|\nabla\varphi_{,xt}|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_{,xt}|_2^2\bigg)+\frac{\mu}{a}|\nabla^2\varphi_{,xt}|_2^2+\frac{\nu}{a}|\Delta\varphi_{,xt}|_2^2+\frac{\mu}{\nu a}|\nabla\mathrm{rot}\,\psi_{,xt}|_2^2$$
$$\le\nu\varepsilon|\nabla\varphi_{,xxx}|_2^2+\frac{c\nu}{\varepsilon}(\|\eta_{,t}\|_1^2|\nabla\varphi_{,xx}|_2^2+\|\eta\|_2^2|\nabla\varphi_{,xt}|_2^2)+\frac{c}{\nu}[|\eta_{,xt}|_2^2+|\eta|_{2,1}^4$$
$$+\|\eta_t\|_1^2|\Delta v|_3^2+\|v\|_2^2\|v_{,t}\|_1^2+|f_t|_2^2]+c\nu\|\eta_{,t}\|_1^2|\nabla\varphi_{,xx}|_2^2.$$

Proof. Differentiate (3.11) with respect to x and t, multiply by $\nabla\varphi_{,xt}$ and integrate over Ω. Then we have

(5.21)

$$\frac{1}{2}\frac{d}{dt}|\nabla\varphi_{,xt}|_2^2+\frac{\mu}{a}|\nabla^2\varphi_{,xt}|_2^2+\frac{\nu}{a}|\Delta\varphi_{,xt}|_2^2=-\int\limits_\Omega\bigg(\frac{a_0}{a+\eta}\nabla\eta\bigg)_{,xt}\cdot\nabla\varphi_{,xt}dx$$
$$-\frac{\mu}{a}\int\limits_\Omega\bigg(\frac{\eta}{a+\eta}\Delta v\bigg)_{,xt}\cdot\nabla\varphi_{,xt}dx-\frac{\nu}{a}\int\limits_\Omega\bigg(\frac{\eta}{a+\eta}\nabla\Delta\varphi\bigg)_{,xt}\cdot\nabla\varphi_{,xt}dx$$
$$+\int\limits_\Omega(v\cdot\nabla v)_{,xt}\cdot\nabla\varphi_{,xt}dx+\int\limits_\Omega\bigg[\frac{1}{a+\eta}(p_\varrho(a)-p_\varrho(a+\eta))\nabla\eta\bigg]_{,xt}\cdot\nabla\varphi_{,xt}dx$$
$$+\int\limits_\Omega f_{g,xt}\cdot\nabla\varphi_{,xt}dx.$$

Now, we examine the particular terms from the r.h.s. of (5.21). We repeat the proof of Lemma 3.4. Let I_1 be the first term on the r.h.s. of (5.21). Integrating by parts we estimate it by

$$|I_1|\le\varepsilon|\nabla\varphi_{,xxt}|_2^2+c/\varepsilon(|\eta_{,t}|_4^2|\nabla\eta|_4^2+|\nabla\eta_t|_2^2).$$

Let I_2 be the second term on the r.h.s. of (5.21). In view of Corollary 4.2 we can assume that $\|\eta\|_2\le1$. Then from the proof of Lemma 3.4 we have

$$|I_2|\le\varepsilon(|\nabla\mathrm{rot}\,\psi_{,xt}|_2^2+\|\nabla\varphi_{,xt}\|_1^2)+c/\varepsilon[\|\eta_t\|_1^2|\Delta v|_3^2+\|\eta\|_2^2(|v_{,xt}|_3^2$$
$$+|\varphi_{,xt}|_2^2+|\nabla\varphi_{,xt}|_2^2)]+\frac{\mu}{a}\int\limits_\Omega\frac{\eta}{a+\eta}|\nabla^2\varphi_{,xt}|^2dx,$$

where the last term is absorbed by the second term on the l.h.s. of (5.21). Let I_3 be the third term on the r.h.s. of (5.21). Then, from the proof of Lemma 3.4, we have

$$|I_3|\le\nu\varepsilon(|\nabla\varphi_{,xxt}|_2^2+|\nabla\varphi_{,xxx}|_2^2)+\frac{c\nu}{\varepsilon}(\|\eta_t\|_1^2|\nabla\varphi_{,xx}|_2^2+\|\eta\|_2^2|\nabla\varphi_{,xt}|_2^2)$$
$$+\frac{\nu}{a}\int\limits_\Omega\frac{\eta}{a+\eta}|\Delta\varphi_{,xt}|^2dx,$$

where the last term is absorbed by the third term on the l.h.s. of (5.21) and Corollary 4.2 is used.

Consider the fourth term on the r.h.s. of (5.21). We can express it in the form

$$I_4 = -\int\limits_\Omega v_{,t}\cdot\nabla v\cdot\nabla\varphi_{,xt}dx - \int\limits_\Omega v\cdot\nabla v_{,t}\cdot\nabla\varphi_{,xxt}dx \equiv I_4^1 + I_4^2,$$

where

$$\begin{aligned}|I_4^1| &\le \varepsilon|\nabla\varphi_{,xt}|_2^2 + c/\varepsilon|v_{,t}|_6^2|v_{,x}|_3^2,\\ |I_4^2| &\le \varepsilon|\nabla\varphi_{,xxt}|_2^2 + c/\varepsilon|v|_\infty^2|v_{,xt}|_2^2.\end{aligned}$$

The fifth term on the l.h.s. of (5.21) is estimated by

$$|I_5| \le \varepsilon|\nabla\varphi_{,xxt}|_2^2 + c/\varepsilon|\eta|_{2,1}^4(1+|\eta|_{2,1}^2).$$

Finally, the last term on the r.h.s. of (5.21) is bounded by

$$|I_6| \le \varepsilon|\nabla\varphi_{,xxt}|_2^2 + c/\varepsilon|f_{,t}|_2^2.$$

Employing the above estimates in (5.21), assuming that ε is sufficiently small and that $|\eta|_{2,1}\le 1$ in view of Corollary 4.2, we obtain

$$\begin{aligned}&\frac{d}{dt}|\nabla\varphi_{,xt}|_2^2 + \frac{\mu}{a}|\nabla^2\varphi_{,xt}|_2^2 + \frac{\nu}{a}|\Delta\varphi_{,xt}|_2^2 \le \frac{c}{\nu}[|\eta_{,xt}|_2^2 + |\eta|_{2,1}^4\\ &\quad + \|\eta_t\|_1^2|\Delta v|_3^2 + \|\eta\|_2^2(|v_{,xt}|_3^2 + \|\varphi_{,xt}\|_1^2)\\ &\quad + \|v\|_2^2\|v_{,t}\|_1^2 + |f_t|_2^2] + \nu\varepsilon|\nabla\varphi_{,xxx}|_2^2 + \frac{c\nu}{\varepsilon}(\|\eta_{,t}\|_1^2|\nabla\varphi_{,xx}|_2^2\\ &\quad + \|\eta\|_2^2|\nabla\varphi_{,xt}|_2^2).\end{aligned}\tag{5.22}$$

Differentiate (3.13) with respect to x and t, multiply the result by $\operatorname{rot}\psi_{,xt}$ and integrate over Ω. Then we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\operatorname{rot}\psi_{,xt}|_2^2 + \frac{\mu}{a}|\nabla\operatorname{rot}\psi_{,xt}|_2^2 = -\frac{\mu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\Delta v\right)_{,xt}\cdot\operatorname{rot}\psi_{,xt}dx\\ &\quad -\frac{\nu}{a}\int\limits_\Omega\left(\frac{\eta}{a+\eta}\Delta\nabla\varphi\right)_{,xt}\cdot\operatorname{rot}\psi_{,xt}dx + \int\limits_\Omega(v\cdot\nabla v)_{,xt}\cdot\operatorname{rot}\psi_{,xt}dx\\ &\quad + \int\limits_\Omega f_{,xt}\cdot\operatorname{rot}\psi_{,xt}dx.\end{aligned}\tag{5.23}$$

Let I_1 be the first term on the r.h.s. of (5.23). Repeating the proof of estimate of I_1 from the proof of Lemma 3.4 we have

$$\begin{aligned}|I_1| &\le \varepsilon(|\operatorname{rot}\psi_{,xxt}|_2^2 + |\nabla\varphi_{,xxt}|_2^2) + c/\varepsilon(|\eta_t|_6^2|\Delta v|_3^2 + \|\eta\|_2^4|\operatorname{rot}\psi_{,xt}|_2^2)\\ &\quad + \frac{\mu}{a}\int\limits_\Omega\frac{\eta}{a+\eta}|\nabla\operatorname{rot}\psi_{,xt}|^2dx,\end{aligned}$$

where the last term is absorbed by the last term on the l.h.s. of (5.23).
Let I_2 be the second term on the r.h.s. of (5.23). Then from the proof of Lemma 3.4 we have

$$\begin{aligned}|I_2| &\le \varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + \frac{c\nu^2}{\varepsilon}|\eta_t|_6^2|\Delta\nabla\varphi|_3^2 + \varepsilon_1\nu^2|\nabla\varphi_{,xxt}|_2^2\\ &\quad + c/\varepsilon_1\|\eta\|_2^4|\mathrm{rot}\,\psi_{,xt}|_2^2.\end{aligned}$$

Next, we examine the third term on the r.h.s. ov (5.23). We express it in the form

$$I_3 = -\int\limits_\Omega v_{,t}\cdot\nabla v\cdot\mathrm{rot}\,\psi_{,xxt}dx - \int\limits_\Omega v\cdot\nabla v_{,t}\cdot\mathrm{rot}\,\psi_{,xxt}dx \equiv I_3^1 + I_3^2,$$

where

$$\begin{aligned}|I_3^1| &\le \varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + c/\varepsilon|v_{,t}|_6^2|\nabla v|_3^2,\\ |I_3^2| &\le \varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + c/\varepsilon|v|_\infty^2|v_{,xt}|_2^2.\end{aligned}$$

Finally, the last term on the r.h.s. of (5.23) is bounded by

$$\varepsilon|\mathrm{rot}\,\psi_{,xxt}|_2^2 + c/\varepsilon|f_t|_2^2.$$

Employing the above estimates in (5.23) and assuming that ε is sufficiently small we derive the inequality
(5.24)

$$\begin{aligned}&\frac{d}{dt}|\mathrm{rot}\,\psi_{,xt}|_2^2 + \frac{\mu}{a}|\nabla\mathrm{rot}\,\psi_{,xt}|_2^2 \le \varepsilon|\nabla\varphi_{,xxt}|_2^2 + c/\varepsilon(|\eta_{,t}|_6^2|\Delta v|_3^2\\ &\quad + \|\eta\|_2^4|\mathrm{rot}\,\psi_{,xt}|_2^2) + \varepsilon_1\nu^2|\nabla\varphi_{,xxt}|_2^2 + c/\varepsilon_1\|\eta\|_2^4|\mathrm{rot}\,\psi_{,xt}|_2^2 + c\nu^2|\eta_t|_6^2|\Delta\nabla\varphi|_2^2\\ &\quad + c\|v_t\|_1^2\|v\|_2^2 + c|f_t|_2^2.\end{aligned}$$

Multiplying (5.24) by $1/\nu$, adding to (5.22) and choosing ε and ε_1 small we obtain

$$\begin{aligned}&\frac{d}{dt}\left(|\nabla\varphi_{,xt}|_2^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi_{,xt}|_2^2\right) + \frac{\mu}{a}|\nabla^2\varphi_{,xt}|_2^2 + \frac{\nu}{a}|\Delta\varphi_{,xt}|_2^2\\ &\quad + \frac{\mu}{a\nu}|\nabla\mathrm{rot}\,\psi_{,xt}|_2^2 \le \nu\varepsilon|\nabla\varphi_{,xxx}|_2^2\\ &\quad + \frac{c\nu}{\varepsilon}(\|\eta_{,t}\|_1^2|\nabla\varphi_{,xx}|_2^2 + \|\eta\|_2^2|\nabla\varphi_{,xt}|_2^2)\\ &\quad + \frac{c}{\nu}\Big[|\eta_{,xt}|_2^2 + |\eta|_{2,1}^4 + \|\eta_{,t}\|_1^2|\Delta v|_3^2 + \|\eta\|_2^2|v_{,xt}|_3^2 + |f_t|_2^2\Big]\\ &\quad + c\nu\|\eta_{,t}\|_1^2|\nabla\varphi_{,xx}|_2^2,\end{aligned}\tag{5.25}$$

where terms $\|\eta\|_2^2\|\nabla\varphi_{,xt}\|_1^2$, $\|\eta\|_2^4|\mathrm{rot}\,\psi_{,xt}|_2^2$ are absorbed by the l.h.s. terms in view of Corollary 4.2. Hence we derive (5.20) from (5.25). This concludes the proof. □

Lemma 5.5. *Assume that φ, ψ, η are solutions to problem (3.1)–(3.3). Let the assumptions of Corollary 4.2 hold. Then*

(5.26)
$$\frac{d}{dt}\bigg(|\nabla\varphi_{,xx}|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_{,xx}|_2^2\bigg)+\frac{\mu}{a}|\nabla^2\varphi_{,xx}|_2^2+\frac{\nu}{a}|\Delta\varphi_{,xx}|_2^2+\frac{\mu}{\nu a}|\nabla\mathrm{rot}\,\psi_{,xx}|_2^2$$
$$\le\frac{c}{\nu}[\|\eta\|_2^2\|v_{,t}\|_1^2+(1+\|\eta\|_2^2)\|v\|_2^4+\|f\|_1^2].$$

Proof. Differentiating (3.2) twice with respect to x, multiplying by $\nabla\varphi_{,xx}$ and integrating over Ω yields

(5.27)
$$\frac{a}{2}\frac{d}{dt}|\nabla\varphi_{,xx}|_2^2+\mu|\nabla^2\varphi_{,xx}|_2^2+\nu|\Delta\varphi_{,xx}|_2^2=-\int\limits_\Omega[\eta v_{,t}]_{,xx}\cdot\nabla\varphi_{,xx}dx$$
$$-a_0\int\limits_\Omega\nabla\eta_{,xx}\cdot\nabla\varphi_{xx}dx-\int\limits_\Omega[(a+\eta)v\cdot\nabla v]_{,xx}\cdot\nabla\varphi_{,xx}dx$$
$$+\int\limits_\Omega[(p_\varrho(a)-p_\varrho(a+\eta))\nabla\eta]_{,xx}\cdot\nabla\varphi_{,xx}dx+\int\limits_\Omega[(a+\eta)f]_{,xx}\cdot\nabla\varphi_{xx}dx.$$

The first term on the r.h.s. of (5.27) is bounded by
$$\varepsilon|\nabla\varphi_{,xxx}|_2^2+c/\varepsilon\|\eta\|_2^2\|v_{,t}\|_1^2,$$
the second by
$$\varepsilon|\nabla\varphi_{,xxx}|_2^2+c/\varepsilon|\nabla\eta_{,x}|_2^2.$$
We write the third term in the form
$$a\int\limits_\Omega(v\cdot\nabla v)_{,x}\cdot\nabla\varphi_{,xxx}dx+\int\limits_\Omega(\eta v\cdot\nabla v)_{,x}\cdot\nabla\varphi_{,xxx}dx\equiv I_1+I_2,$$
where
$$|I_1|\le\varepsilon|\nabla\varphi_{,xxx}|_2^2+c/\varepsilon(|v_{,x}|_3^2|v_{,x}|_6^2+|v|_\infty^2|v_{,xx}|_2^2)$$
and
$$|I_2|\le\varepsilon|\nabla\varphi_{,xxx}|_2^2+c/\varepsilon(|\eta_{,x}|_6^2|v|_\infty^2|v_{,x}|_3^2+|\eta|_\infty^2|v_{,x}|_3^2|v_{,x}|_6^2$$
$$+|\eta|_\infty^2|v|_\infty^2|v_{,xx}|_2^2).$$
The fourth term is estimated by
$$\varepsilon|\nabla\varphi_{,xxx}|_2^2+c/\varepsilon\|\eta\|_2^4.$$
Finally, the last term is bounded by
$$\varepsilon|\nabla\varphi_{,xxx}|_2^2+c/\varepsilon[|f_{g,x}|_2^2+|\eta_{,x}|_6^2|f|_3^2+|\eta|_\infty^2|f_{,x}|_2^2].$$

Utilizing the estimates in (5.27) yields

(5.28)
$$\begin{aligned}&\frac{d}{dt}|\nabla\varphi_{,xx}|_2^2+\frac{\mu}{a}|\nabla^2\varphi_{,xx}|_2^2+\frac{\nu}{a}|\Delta\varphi_{,xx}|_2^2\le\frac{c}{\nu}[\|\eta\|_2^2\|v_{,t}\|_1^2\\&\quad+\|\eta\|_2^2+\|\eta\|_2^4+|\eta_{,x}|_6^2|v|_\infty^2|v_{,x}|_3^2+(1+|\eta|_\infty^2)(|v_{,x}|_3^2|v_{,x}|_6^2+|v|_\infty^2|v_{,xx}|_2^2)\\&\quad+\|f\|_1^2].\end{aligned}$$

Differentiating (3.2) twice with respect to x, multiplying by $\mathrm{rot}\,\psi_{,xx}$ and integrating over Ω implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\mathrm{rot}\,\psi_{,xx}|_2^2+\frac{\mu}{a}|\nabla\mathrm{rot}\,\psi_{,xx}|_2^2=-\int_\Omega[\eta v_{,t}]_{,xx}\cdot\mathrm{rot}\,\psi_{,xx}dx\\&-\int_\Omega[(a+\eta)v\cdot\nabla v]_{,xx}\cdot\mathrm{rot}\,\psi_{,xx}dx+\int_\Omega[(a+\eta)f]_{,xx}\cdot\mathrm{rot}\,\psi_{,xx}dx\end{aligned}\tag{5.29}$$

The first term on the r.h.s. is bounded by

$$\varepsilon|\mathrm{rot}\,\psi_{,xxx}|_2^2+c/\varepsilon\|\eta\|_2^2\|v_t\|_1^2$$

We express the second term on the r.h.s. of (5.29) in the form

$$a\int_\Omega(v\cdot\nabla v)_{,x}\cdot\mathrm{rot}\,\psi_{,xxx}dx+\int_\Omega(\eta v\cdot\nabla v)_{,x}\cdot\mathrm{rot}\,\psi_{,xxx}dx\equiv J_1+J_2,$$

where

$$\begin{aligned}|J_1|&\le\varepsilon|\mathrm{rot}\,\psi_{,xx}|_2^2+c/\varepsilon(|v_{,x}|_3^2|v_{,x}|_6^2+|v|_\infty^2|v_{,xx}|_2^2),\\|J_2|&\le\varepsilon|\mathrm{rot}\,\psi_{,xxx}|_2^2+c/\varepsilon(|\eta_{,x}|_6^2|v|_6^2|v_{,x}|_6^2+|\eta|_\infty^2|v_{,x}|_3^2|v_{,x}|_6^2\\&\qquad+|\eta|_\infty^2|v|_\infty^2|v_{,xx}|_2^2).\end{aligned}$$

Finally, the last term on the r.h.s. of (5.29) is estimated by

$$\varepsilon|\mathrm{rot}\,\psi_{,xxx}|_2^2+c/\varepsilon(|f_{r,x}|_2^2+|\eta_{,x}|_6^2|f|_3^2+|\eta|_\infty^2|f_{,x}|_2^2).$$

Utilizing the estimates in (5.29) and choosing that ε is sufficiently small we have

$$\frac{d}{dt}|\mathrm{rot}\,\psi_{,xx}|_2^2+\frac{\mu}{a}|\nabla\mathrm{rot}\,\psi_{,xx}|_2^2\le c[\|\eta\|_2^2\|v_{,t}\|_1^2+\|v\|_2^4(1+\|\eta\|_2^2)+\|f\|_1^2].\tag{5.30}$$

Multiply (5.30) by $1/\nu$ and add to (5.28). Then we obtain (5.26). This concludes the proof. □

Theorem 5.6. *Let φ, ψ, η be solutions to problem (3.1)–(3.3). Let the assumptions of Corollary 4.2 hold. Assume that there exists such time T that*

$$-\frac{\mu}{a}T + c\int\limits_0^T \|v(t)\|_2^2 dt < 0. \tag{5.31}$$

Assume that there exists such relation between T, $\int_0^T \|v(t)\|_2^2 dt$, $\eta(t)$, $f(t)$, $|\nabla\varphi(0)|_{2,1} + \frac{1}{\sqrt{\nu}}|\mathrm{rot}\,\psi(0)|_{2,1}$ that

$$\begin{aligned} &\exp\bigg(c\int\limits_0^T \|v(t)\|_2^2 dt\bigg)\frac{c}{\nu}\int\limits_0^T (|\eta|_{2,1}^2 + |f|_{2,1}^2)dt \\ &\quad + \exp\bigg(-\frac{\mu}{a}T + c\int\limits_0^T \|v(t)\|_2^2\bigg)X^2(0) \le X^2(0), \end{aligned} \tag{5.32}$$

where $X^2(t) = \nu|\nabla\varphi(t)|_{2,1}^2 + |\mathrm{rot}\,\psi(t)|_{2,1}^2 + \|\eta(t)\|_2^2$. Then for any $k \in \mathbb{N}_0$ we have

$$X^2(kT) \le X^2(0). \tag{5.33}$$

Moreover, assuming that $|\eta(0)|_{2,1} \le c_0/\nu$, $\|f_g(t)\|_1^2 \le f_0^2 e^{-\alpha t}$, c_0, f_0, α constants there exists a global solution (φ, ψ, η) to problem (3.1)–(3.3) such that in any time interval $[kT, (k+1)T]$ it is a solution described by Corollary 4.2.

Proof. Adding (5.20) and (5.26) we have

$$\begin{aligned} &\frac{d}{dt}\bigg(|\nabla\varphi_{,x}|_2^2 + |\nabla\varphi_{,xx}|_2^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi_{,xt}|_2^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi_{,xx}|_2^2\bigg) \\ &\quad + \frac{\mu}{a}(|\nabla^2\varphi_{,xt}|_2^2 + |\nabla^2\varphi_{,xx}|_2^2) + \frac{\nu}{a}(|\Delta\varphi_{,xt}|_2^2 + |\Delta\varphi_{,xx}|_2^2) \\ &\quad + \frac{\mu}{\nu a}(|\nabla\mathrm{rot}\,\psi_{,xt}|_2^2 + |\nabla\mathrm{rot}\,\psi_{,xx}|_2^2) \\ &\le c\nu(\|\eta_t\|_1^2|\nabla\varphi_{,xx}|_2^2 + \|\eta\|_2^2|\nabla\varphi_{,xt}|_2^2) \\ &\quad + c\nu\|\eta_t\|_1^2|\Delta\nabla\varphi|_2^2 + \frac{c}{\nu}[|\eta|_{2,1}^2 + \|\eta_t\|_1^2|\Delta v|_3^2 + \|\eta\|_2^2\|v_t\|_1^2 \\ &\quad + (1 + \|\eta\|_2^2)(\|v\|_2^2\|v_{,t}\|_1^2 + \|v\|_2^4) + \|f\|_1^2 + |f_t|_2^2]. \end{aligned} \tag{5.34}$$

In view of Corollary 4.2 we obtain from (5.34) the inequality

$$
\begin{aligned}
&\frac{d}{dt}\left(|\nabla\varphi_{,xt}|_2^2+|\nabla\varphi_{,xx}|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_{,xt}|_2^2+\frac{1}{\nu}|\mathrm{rot}\,\psi_{,xx}|_2^2\right)\\
&\quad+\frac{\mu}{a}(|\nabla^2\varphi_{,xt}|_2^2+|\nabla^2\varphi_{,xx}|_2^2)+\frac{\nu}{a}(|\Delta\varphi_{,xt}|_2^2+|\Delta\varphi_{,xx}|_2^2)\\
&\quad+\frac{\mu}{\nu a}(|\nabla\mathrm{rot}\,\psi_{,xt}|_2^2+|\nabla\mathrm{rot}\,\psi_{,xx}|_2^2)\\
&\le\frac{c}{\nu}[|\eta|_{2,1}^2+\|\eta_t\|_1^2|\Delta v|_3^2+\|\eta\|_2^2\|v_{,t}\|_1^2+(1+\|\eta\|_2^2)(\|v\|_2^2\|v_t\|_1^2\\
&\quad+\|v\|_2^4)+\|f\|_1^2+|f_t|_2^2].
\end{aligned}
\tag{5.35}
$$

From (5.19), (5.35) and Corollary 4.2 we have

$$
\begin{aligned}
&\frac{d}{dt}\left(|\nabla\varphi|_{2,1}^2+\frac{1}{\nu}|\mathrm{rot}\,\psi|_{2,1}^2\right)+\frac{\mu}{a}|\nabla\varphi|_{3,1}^2+\frac{\nu}{a}|\nabla\varphi|_{3,1}^2+\frac{1}{a\nu}|\mathrm{rot}\,\psi|_{3,1}^2\\
&\le\frac{c}{\nu}[|\eta|_{2,1}^2+\|v\|_2^4+\|v\|_2^2\|v_t\|_1^2+|f|_{1,1}^2].
\end{aligned}
\tag{5.36}
$$

Introduce the quantity

$$
X_1^2=\nu|\nabla\varphi|_{2,1}^2+|\mathrm{rot}\,\psi|_{2,1}^2
\tag{5.37}
$$

We also need a similar differential inequality for η. From (3.1) and (3.2) we have

$$
\begin{aligned}
&\nabla\eta_t+a_0\nabla\eta=-\nabla(v\cdot\nabla\eta)-a\nabla\Delta\varphi-\nabla(\eta\Delta\varphi)-(a+\eta)v_t\\
&\quad+\mu\Delta v+\nu\nabla\Delta\varphi-(a+\eta)v\cdot\nabla v+(p_\varrho(a)-p_\varrho(a+\eta))\nabla\eta\\
&\quad+(a+\eta)f.
\end{aligned}
\tag{5.38}
$$

Multiplying (5.38) by $\nabla\eta$ and integrating over Ω yields

(5.39)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla\eta|_2^2+a_0|\nabla\eta|_2^2=-\int_\Omega\nabla(v\cdot\nabla\eta)\cdot\nabla\eta dx\\
&\quad+\varepsilon|\nabla\eta|_2^2+c/\varepsilon[|\nabla\Delta\varphi|_2^2+|\nabla\varphi_t|_2^2+|v_t|_2^2|\eta|_\infty^2+|\Delta\nabla\varphi|_2^2+\nu^2|\nabla\Delta\varphi|_2^2\\
&\quad+|v|_6^2X_1^2+|v|_6^2|\nabla v|_3^2|\eta|_\infty^2+|f_g|_2^2+|f|_2^2|\eta|_\infty^2]+c|\eta|_\infty^2|\nabla\eta|_2^2.
\end{aligned}
$$

In view of Corollary 4.2 $|\eta|_\infty$ is bounded and small. Then (5.39) takes the form

$$
\begin{aligned}
&\frac{d}{dt}|\nabla\eta|_2^2+a_0|\nabla\eta|_2^2\le-\int_\Omega\nabla(v\cdot\nabla\eta)\cdot\nabla\eta dx+c[|\nabla\Delta\varphi|_2^2+|\nabla\varphi_t|_2^2\\
&\quad+\nu^2|\nabla\Delta\varphi|_2^2+|f_g|_2^2]+c[|v_t|_2^2+|v|_6^2|\nabla v|_3^2+|f|_2^2]\|\eta\|_2^2\\
&\quad+c|v|_6^2X_1^2.
\end{aligned}
\tag{5.40}
$$

Consider the first term on the r.h.s. of (5.40). Performing differentiation it takes the form

$$I = -\int\limits_\Omega v\cdot\nabla\nabla\eta\nabla\eta dx - \int\limits_\Omega \nabla v\cdot\nabla\eta\nabla\eta dx \equiv I_1 + I_2,$$

where

$$I_1 = -\frac{1}{2}\int\limits_\Omega v\cdot\nabla|\nabla\eta|^2 dx = \frac{1}{2}\int\limits_\Omega \Delta\varphi|\nabla\eta|^2 dx.$$

Hence

$$|I_1| \le \varepsilon|\nabla\eta|_2^2 + c/\varepsilon|\Delta\varphi|_\infty^2|\nabla\eta|_2^2.$$

Similarly, we have

$$|I_2| \le \varepsilon|\nabla\eta|_2^2 + c/\varepsilon|\nabla v|_\infty^2|\nabla\eta|_2^2.$$

Employing the estimates in (5.40) and using that $|v|_2 \le c$ in view of the energy estimate (see Lemma 2.1)we have

(5.41)
$$\begin{aligned}\frac{d}{dt}|\nabla\eta|_2^2 + a_0|\nabla\eta|_2^2 &\le c[|\nabla v|_\infty^2 + |\Delta\varphi|_\infty^2 + |v_t|_2^2 + |v|_6^2|\nabla v|_3^2\\ &+ |f|_2^2]\|\eta\|_2^2 + c[|\nabla\Delta\varphi|_2^2 + |\nabla\varphi_t|_2^2 + \nu^2|\nabla\Delta\varphi|_2^2 + |f_g|_2^2] + c|v|_6^2X_1^2.\end{aligned}$$

Differentiate (5.38) with respect to x, multiply by $\nabla\eta_x$ and integrate over Ω. Then we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla\eta_x|_2^2 + a_0|\nabla\eta_x|_2^2 &= -\int\limits_\Omega(\nabla(v\cdot\nabla\eta))_{,x}\cdot\nabla\eta_x dx\\ &+ c[|\nabla\Delta\varphi_{,x}|_2^2 + \|\nabla\varphi_t\|_1^2 + \|v_t\|_1^2\|\eta\|_2^2 + \nu^2|\nabla\Delta\varphi_{,x}|_2^2\\ &+ |[(a+\eta)v\cdot\nabla v]_{,x}|^2] + \int\limits_\Omega[(p_\varrho(a) - p_\varrho(a+\eta))\nabla\eta]_{,x}\cdot\nabla\eta_x dx\\ &+ \int\limits_\Omega[(a+\eta)f]_{,x}\cdot\nabla\eta_{,x}dx.\end{aligned} \tag{5.42}$$

Carrying out differentiations in the first term on the r.h.s. of (5.42) yields

$$I_1 = -\int\limits_\Omega(v\cdot\nabla\nabla\eta_x + \nabla v\cdot\nabla\eta_x + v_x\cdot\nabla\nabla\eta + \nabla v_x\cdot\nabla\eta)\cdot\nabla\eta_x dx.$$

Hence

$$\begin{aligned}|I_1| &\le \varepsilon|\nabla\eta_x|_2^2 + c/\varepsilon[|\nabla v|_\infty^2|\nabla\eta_x|_2^2 + |v_{,x}|_\infty^2|\nabla^2\eta|_2^2 + |\nabla v_x|_3^2|\nabla\eta|_6^2\\ &+ |\Delta\varphi|_\infty^2|\nabla\eta_x|_2^2] \le \varepsilon|\nabla\eta_x|_2^2 + c/\varepsilon(|\nabla v|_\infty^2 + |\nabla v_x|_3^2 + |\Delta\varphi|_\infty^2)\|\nabla\eta\|_1^2.\end{aligned}$$

Consider the fifth term under the square bracket on the r.h.s. of (5.42). Then we obtain

$$
\begin{aligned}
I_3 &= \int_\Omega [(a+\eta)v\cdot\nabla v]_{,x}^2 dx \le a\int_\Omega (v\cdot\nabla v)_{,x}^2 dx + \int_\Omega (\eta v\cdot\nabla v)_{,x}^2 dx \\
&\le a\int_\Omega |\nabla v|^4 dx + a\int_\Omega |v|^2|\nabla v_x|^2 dx + \int_\Omega |\eta_x|^2|v|^2|\nabla v|^2 dx \\
&\quad + \int_\Omega |\eta|^2|\nabla v|^4 dx + \int_\Omega |\eta|^2|v|^2|\nabla v_x|^2 dx \equiv \sum_{i=1}^5 I_{3i}.
\end{aligned}
$$

Continuing, we have

$$
\begin{aligned}
&I_{31} \le c|\nabla v|_6^2|\nabla v|_3^2 \le c\|v\|_2^4 \le c\|v\|_2^2X_1^2\\
&I_{32} \le c|v|_\infty^2|\nabla v_x|_2^2 \le c|v|_\infty^2\|v\|_2^2 \le c|v|_\infty^2X_1^2\\
&I_{33} \le c|\eta_x|_6^2|v|_\infty^2|\nabla v|_3^2 \le \|\nabla\eta\|_1^2|v|_\infty^2\|v\|_2^2,\\
&I_{34}+I_{35} \le |\eta|_\infty^2(\|v\|_2^4+|v|_\infty^2\|v\|_2^2) \le c\|v\|_2^4\|\eta\|_2^2
\end{aligned}
$$

Hence

$$
I_3 \le c\|v\|_2^2X_1^2 + c\|v\|_2^4\|\eta\|_2^2.
$$

The last but one term on the r.h.s. of (5.42) is bounded by

$$
\begin{aligned}
I_4 &\le c\int_\Omega |\eta|\,|\nabla\eta_x|^2 dx + c\int_\Omega |\nabla\eta|^2|\nabla\eta_x| dx\\
&\le \varepsilon|\nabla\eta_x|_2^2 + c/\varepsilon|\eta|_\infty^2|\nabla\eta_x|_2^2 + c|\nabla\eta|_4^4.
\end{aligned}
$$

Finally, the last term on the r.h.s. of (5.42) is estimated by

$$
I_5 \le \varepsilon|\nabla\eta_x|_2^2 + c/\varepsilon(|f_{g,x}|_2^2 + |\eta_x|_4^2|f|_4^2 + |\eta|_\infty^2|f_{,x}|_2^2).
$$

Employing the above estimates in (5.42) and assuming that ε is sufficiently small implies

$$
\begin{aligned}
\frac{d}{dt}|\nabla\eta_x|_2^2 + a_0|\nabla\eta_x|_2^2 &\le c[|\nabla v|_\infty^2 + |\nabla v_x|_3^2 + |\Delta\varphi|_\infty^2 + \|v_t\|_1^2 + \|v\|_2^4\\
&\quad + \|f\|_1^2]\|\eta\|_2^2 + c(\|\nabla\varphi_t\|_1^2 + |\Delta\nabla\varphi_x|_2^2 + \nu^2\|\Delta\nabla\varphi\|_1^2 + |f_{g,x}|^2)\\
&\quad + c\|v\|_2^2X_1^2,
\end{aligned}
\tag{5.43}
$$

where the terms $|\eta|_\infty^2|\nabla\eta_x|_2^2 + |\nabla\eta|_4^4$ are absorbed by the second term on the l.h.s. in view of Corollary 4.2.

From (5.41) and (5.43) we obtain (where we used that $\int_\Omega \eta dx = 0$)

$$\begin{aligned}(5.44)\qquad &\frac{d}{dt}\|\eta\|_2^2 + a_0\|\eta\|_2^2 \le c[|\nabla v|_\infty^2 + |v_{,xx}|_3^2 + |\Delta\varphi|_\infty^2 + \|v_t\|_1^2 + \|v\|_2^4\\ &\quad + \|f\|_1^2]\|\eta\|_2^2 + c(\|\nabla\varphi_t\|_1^2 + \|\nabla\varphi_{,xx}\|_1^2 + \nu^2\|\nabla\varphi_{,xx}\|_1^2 + \|f_g\|_1^2)\\ &\quad + c\|v\|_2^2 X_1^2\end{aligned}$$

Using definition of X_1 (see (5.37)) we write (5.36) in the form

$$(5.45)\qquad \frac{d}{dt}X_1^2 + \frac{\mu}{a}X_1^2 + \nu^2|\nabla\varphi|_{3,1}^2 \le c\|v\|_2^2 X_1^2 + c(|\eta|_{2,1}^2 + |f|_{1,1}^2).$$

Introduce the notation

$$X_2^2 = X_1^2 + \|\eta\|_2^2, \quad a_* = \min\left\{a_0, \frac{\mu}{a}\right\}.$$

Then (5.44) and (5.45) imply the inequality after appropriate summing

$$\begin{aligned}(5.46)\qquad &\frac{d}{dt}X_2^2 + a_*X_2^2 + \nu^2|\nabla\varphi|_{3,1}^2 \le c(|v|_{3,1}^2 + |\Delta\varphi|_\infty^2 + \|v\|_2^4 + \|f\|_1^2)X_2^2\\ &\quad + c(|\eta|_{2,1}^2 + \|f_g\|_1^2).\end{aligned}$$

Let

$$\begin{aligned}G^2(t) &= |v(t)|_{3,1}^2 + |\Delta\varphi(t)|_\infty^2 + \|v(t)\|_2^4 + \|f(t)\|_1^2,\\ K^2(t) &= |\eta|_{2,1}^2 + \|f_g(t)\|_1^2.\end{aligned}$$

Then (5.46) takes the following form because the integral $|\nabla\varphi|_{3,1}$ is absorbed by the last term on the l.h.s. of (5.46)

$$(5.47)\qquad \frac{d}{dt}X_2^2 + a_*X_2^2 \le cG^2X_2^2 + K^2.$$

Integrating (5.47) from 0 to t we have

$$\begin{aligned}(5.48)\qquad X_2^2(t) &\le \exp\left[c\int_0^t G^2(t')dt'\right]\int_0^t K^2dt'\\ &\quad + \exp\left[-a_*t + c\int_0^t G^2(t')dt'\right]X_2^2(0).\end{aligned}$$

Setting $t = T$ and assuming that

$$(5.49)\qquad -\frac{a_*}{2}T + c\int_0^T G^2(t)dt \le 0$$

we obtain from (5.48) the inequality

$$X_2^2(T)\le\exp\bigg[c\int\limits_0^TG^2(t)dt\bigg]\int\limits_0^TK^2dt+\exp\bigg(-\frac{a_*T}{2}\bigg)X_2^2(0). \tag{5.50}$$

Consider (5.50). In view of Corollary 4.2 we have that
(5.51)

$$X_2^2(T)\le\exp(\chi_1^2+\chi_2^2+\Psi^2)\bigg(\frac{\Psi^2}{\nu^2}+T|\eta(0)|_{2,1}^2+e^{-\frac{a_*}{2}T}\int\limits_0^T\|f_g(t)\|_1^2e^{\frac{a_*}{2}t}dt\bigg)$$
$$+\exp\bigg(-\frac{a_*}{2}T\bigg)X_2^2(0).$$

Assuming that

$$|\eta(0)|_{2,1}\le\frac{c_0}{\nu},\quad T\le\nu,\quad \|f_g(t)\|_1^2\le f_0^2e^{-\alpha t},\quad t\le T \tag{5.52}$$

we obtain

$$e^{-\frac{a_*}{2}T}\int\limits_0^T\|f_g(t)\|_1^2e^{\frac{a_*}{2}t}dt\le f_0^2\int\limits_0^Te^{-\alpha t+\frac{a_*}{2}t}dt\le cf_0^2e^{-\alpha T}.$$

Therefore, for ν sufficiently large we obtain from (5.51) that

$$X_2^2(T)\le c/\nu \tag{5.53}$$

Since $X_2^2(T)\ge|v(T)|_{2,1}^2+|\eta(T)|_{2,1}^2$ we can consider problem (3.1)–(3.3) with small initial data at $t=T$. Let $X=X_2$.
Hence, in view of [BSZ, Z1, Z2, VZ] we have existence of global regular solutions to (3.1)–(3.3) with small initial data at $t=T$ described by (5.53).
□

The existence of solutions to problem (3.1)–(3.3) in the time interval $[T,\infty)$ can be made by the step by step in time approach presented in [Z3].
Therefore, we have

Theorem 5.7. *Let the assumptions of Lemma 4.1 hold. Let T be so large and estimate in Lemma 4.1 so appropriate that (5.49) hold. Assume restrictions (5.52). Assume that ν is sufficiently large. Then there exists a global solution to problem (3.1)–(3.3) such that*

$$\eta\in L_\infty(T_1,T_2;\Gamma_1^2(\Omega)),\quad \nabla\varphi,\operatorname{rot}\psi\in L_\infty(T_1,T_2;\Gamma_1^2(\Omega))\cap L_2(T_1,T_2;\Gamma_1^3(\Omega)),$$

where for $(T_1,T_2)\subset(0,T)$, the solution is described by Lemma 4.1 and for $T_1>T$, we have that there exists $T_0>0$ that $(T_1,T_2)=(T+kT_0,T+(k+1)T_0)$ for any $k\in\mathbb{N}_0$.

Remark 5.3. In view of Lemma 4.1 the constant A (see (4.1)) may depend on time with a growth less than T. Then (5.49) may always hold.

References

[BIN] Besov, O.V.; Il'in, V.P.; Nikolskij, S.M.: Integral representation of functions and imbedding theorems, Moscow 1975 (in Russian).

[BSZ] Burczak, J.; Shibata, Y.; Zajączkowski, W.M.: Local and global solutions near equilibria via the energy method.

[LSU] Ladyzhenskaya, O.A.; Solonnikov, V.A.; Uraltseva, N.N.: Linear and quasilinear equations of parabolic type, Nauka, Moscow 1967 (in Russian).

[MN1] Matsumura, A.; Nishida, T.: The initial value problem for equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ. 20 (1980), 67–104.

[MN2] Matsumura, A.; Nishida, T.: The initial boundary value problem for the equations of motion of compressible viscous and heat-conductive fluids, Proc. Japan Acad. Ser. A55 (1979), 337–342.

[MN3] Matsumura, A.; Nishida, T.: Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids, Commun. Math. Phys. 89 (1983), 445–464.

[T] Tao, T.: Nonlinear dispersive equations: local and global analysis.

[V] Valli, A.: Periodic and stationary solutions for compressible Navier-Stokes equations via stability method, Ann. Sc. Norm. Super. Pisa (IV) 10 (1983), 607–647.

[VZ] Valli, A.; Zajączkowski, W.M.: Navier-Stokes equations for compressible fluids: global existence and qualitative properties of the solutions in the general case, Commun. Math. Phys. 103 (1986), 259–296.

[Za] Zadrzyńska, E.: Free boundary problems for nonstationary Navier-Stokes equations, Dissertationes Math. 424 (2004), pp. 135.

[Z1] Zajączkowski, W.M.: On nonstationary motion of a compressible barotropic viscous fluid bounded by a free surface, Dissertationes Math. 324 (1993).

[Z2] Zajączkowski, W.M.: On nonstationary motion of a compressible barotropic viscous capillary fluid bounded by a free surface, SIAM J. Math. Anal. 25 (1994), 1–84.

[Z3] Zajączkowski, W.M.: Some stability problem to the Navier-Stokes equations in the periodic case, JMAA.