SINGULARITIES OF AFFINE EQUIDISTANTS: EXTRINSIC GEOMETRY OF SURFACES IN 4-SPACE.

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Abstract. For a generic embedding of a smooth closed surface M into $\mathbb{R}^4$, the subset of $\mathbb{R}^4$ which is the affine $\lambda-$equidistant of M appears as the discriminant set of a stable mapping $M \times M \to \mathbb{R}^4$, hence their stable singularities are A_k, $k = 2, 3, 4$, and $C^{\pm}_{2,2}$. In this paper, we characterize these stable singularities of $\lambda-$equidistants in terms of the bi-local extrinsic geometry of the surface, leading to a geometrical study of the set of weakly parallel points on M.

1. Introduction

When M is a smooth closed curve on the affine plane $\mathbb{R}^2$, the set of all midpoints of chords connecting pairs of points on M with parallel tangent vectors is called the *Wigner caustic* of M, or the *area evolute* of M, or still, the *affine* $1/2$-*equidistant* of M. The $1/2$-equidistant is generalized to any λ-equidistant, denoted $E_\lambda(M)$, $\lambda \in \mathbb{R}$, by considering all chords connecting pairs of points of M with parallel tangent vectors and the set of all points of these chords which stand in the λ-proportion to their corresponding pair of points on M.

The definition of the affine λ-equidistant of M is generalized to the cases when M is an n-dimensional closed submanifold of $\mathbb{R}^q$, with $q \leq 2n$, by considering the set of all λ-points of chords connecting pairs of points on M whose direct sum of tangent spaces do not coincide with $\mathbb{R}^q$, the so-called *weakly parallel pairs* on M. In the particular case of $M^2 \subset \mathbb{R}^4$, a weakly parallel pair on the surface M can be either 1-parallel (when the tangent spaces span a 3-space) or 2-parallel, which is the case of true parallelism, also called strong parallelism.

Affine equidistants of smooth submanifolds, in particular the Wigner caustic, have a way in mathematical physics and in the definition of affine-invariant global centre symmetry sets of these submanifolds and,

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in every case, precise knowledge of their singularities is an important issue [11, 8, 9, 4, 3, 2]. Thus, stable singularities of affine equidistants of $M^n \subset \mathbb{R}^q$ have been extensively studied [1, 7, 8, 9, 10, 4, 3, 2], culminating in its complete classification for all pairs $(2n, q)$ of nice dimensions [5].

On the other hand, not so much is known with respect to the interpretation for the realization of these stable singularities in terms of the extrinsic geometry of $M^n \subset \mathbb{R}^q$. The case of curves on the plane has long been well understood [1, 7], just as for hypersurfaces [8]. Another instance that has been completely worked out refers to a Lagrangian surface M^2 in symplectic $\mathbb{R}^4$, for its Wigner caustic on shell, that is, the part of its 1/2-equidistant that is close to and contains M [3]. A geometric study of the Wigner caustic on shell for general surfaces in $\mathbb{R}^4$ has also been partly worked out in [10].

In this paper, we extend the extrinsic geometric study of the realization of affine equidistants to the case of general (off-shell) λ-equidistants of any smooth surface $M^2 \subset \mathbb{R}^4$. Our paper is organized as follows:

First, Section 2 reviews basic definitions and characterizations of affine equidistants. The presentation is based on [5]. Then, basic facts on extrinsic geometry of surfaces in 4-space are recalled in Section 3.

Our geometric study is presented in Sections 4 and 5. First, in Section 4 we describe the realization of singularities of affine equidistants in terms of the bi-local extrinsic geometry of the surface. The main result for the case of 1-parallel pairs is presented in Theorem 4.3, while Theorems 4.4 and 4.5 present the results for the 2-parallel case.

Then, this is followed in Section 5 by a complementary study of the set of weakly parallel points on M. We start by using the Grassmannian of 2-planes in 4-space, cf. Propositions 5.1 and 5.2 and Theorem 5.3, leading to the final detailed description of the set of weakly parallel points on M presented in Corollary 5.4 and Theorem 5.7.

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2. Singularities of affine equidistants: overview

In this section, we summarize the material that is presented in [5] in greater detail, in order to describe, characterize and classify the singularities of affine λ-equidistants of smooth submanifolds.

2.1. **Definition of affine equidistants.** Let M be a smooth closed n-dimensional submanifold of the affine space $\mathbb{R}^q$, with $q \leq 2n$. Let α, β be points of M and denote by $\tau_{\alpha-\beta} : \mathbb{R}^q \ni x \mapsto x + (\alpha - \beta) \in \mathbb{R}^q$ the translation by the vector $(\alpha - \beta)$.

Definition 2.1. A pair of points $\alpha, \beta \in M$ $(\alpha \neq \beta)$ is called a **weakly parallel** pair if

$$T_\alpha M + \tau_{\alpha-\beta}(T_\beta M) \neq \mathbb{R}^q.$$

A weakly parallel pair $\alpha, \beta \in M$ is called k**-parallel** if

$$\dim(T_\alpha M \cap \tau_{\alpha-\beta}(T_\beta M)) = k.$$

If $k = n$ the pair $\alpha, \beta \in M$ is called **strongly parallel**, or just **parallel**. We also refer to k as the **degree of parallelism** of the pair (α, β).

Definition 2.2. A **chord** passing through a pair α, β, is the line

$$l(\alpha, \beta) = \{x \in \mathbb{R}^q | x = \lambda\alpha + (1 - \lambda)\beta, \lambda \in \mathbb{R}\},$$

but we sometimes also refer to $l(\alpha, \beta)$ as a chord *joining* α and β.

Definition 2.3. For a given λ, an **affine** λ**-equidistant** of M, $E_\lambda(M)$, is the set of all $x \in \mathbb{R}^q$ such that $x = \lambda\alpha + (1-\lambda)\beta$, for all weakly parallel pairs (α, β) in M. $E_\lambda(M)$ is also called a **momentary equidistant** of M. Whenever M is understood, we write E_λ for $E_\lambda(M)$.

Note that, for any λ, $E_\lambda(M) = E_{1-\lambda}(M)$ and in particular $E_0(M) = E_1(M) = M$. Thus, the case $\lambda = 1/2$ is special:

Definition 2.4. $E_{1/2}(M)$ is called the **Wigner caustic** of M [1, 15].

2.2. **Characterization of affine equidistants by projection.** Consider the product affine space: $\mathbb{R}^q \times \mathbb{R}^q$ with coordinates (x_+, x_-) and the tangent bundle to $\mathbb{R}^q$: $T\mathbb{R}^q = \mathbb{R}^q \times \mathbb{R}^q$ with coordinate system $(x, \dot{x})$ and standard projection $\pi : T\mathbb{R}^q \ni (x, \dot{x}) \to x \in \mathbb{R}^q$.

Definition 2.5. $\forall \lambda \in \mathbb{R} \setminus \{0, 1\}$, a λ**-chord transformation**

$$\Gamma_\lambda : \mathbb{R}^q \times \mathbb{R}^q \to T\mathbb{R}^q \ , \ (x^+, x^-) \mapsto (x, \dot{x})$$

is a linear diffeomorphism defined by:

$$x = \lambda x^+ + (1 - \lambda)x^- \ , \quad \dot{x} = x^+ - x^-. \tag{2.1}$$

Remark 2.6. The choice of linear equation for $\dot{x}$ in (2.1) is not unique, but this is the simplest one. Among other possibilities, the choice $\dot{x} = \lambda x^+ - (1 - \lambda)x^-$ is particularly well suited for the study of affine equidistants of *Lagrangian* submanifolds in symplectic space [4].

Now, let M be a smooth closed n-dimensional submanifold of the affine space $\mathbb{R}^q$ ($2n \geq q$) and consider the product $M \times M \subset \mathbb{R}^q \times \mathbb{R}^q$. Let $\mathcal{M}_\lambda$ denote the image of $M \times M$ by a λ-chord transformation,

$$\mathcal{M}_\lambda = \Gamma_\lambda(M \times M) \ ,$$

which is a $2n$-dimensional smooth submanifold of $T\mathbb{R}^q$.

Then we have the following general characterization:

Theorem 2.7 ([4]). *The set of critical values of the standard projection $\pi : T\mathbb{R}^q \to \mathbb{R}^q$ restricted to $\mathcal{M}_\lambda$ is $E_\lambda(M)$.*

Definition 2.8. $\forall \lambda \in \mathbb{R} \setminus \{0, 1\}$, the **$\lambda$-point map** is the projection

$$\Psi_\lambda : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^q \ , \ (x^+, x^-) \to x = \lambda x^+ + (1-\lambda)x^- \ .$$

Remark 2.9. Because $\Psi_\lambda = \pi \circ \Gamma_\lambda$ we can rephrase Theorem 2.7: *the set of critical values of the projection Ψ_λ restricted to $M \times M$ is $E_\lambda(M)$.*

2.3. **Characterization of affine equidistants by contact.** In the literature, if $M \subset \mathbb{R}^2$ is a smooth curve, the Wigner caustic $E_{1/2}(M)$ has been described in various ways, one of which says that, if $\mathcal{R}_a : \mathbb{R}^2 \to \mathbb{R}^2$ denotes reflection through $a \in \mathbb{R}^2$, then $a \in E_{1/2}(M)$ when M and $\mathcal{R}_a(M)$ are not transversal [1, 15]. We generalize this description for every λ-equidistant of submanifolds of more arbitrary dimensions.

Definition 2.10. $\forall \lambda \in \mathbb{R} \setminus \{0, 1\}$, a **$\lambda$-reflection** through $a \in \mathbb{R}^q$ is the map

$$\mathcal{R}^\lambda_a : \mathbb{R}^q \to \mathbb{R}^q \ , \ x \mapsto \mathcal{R}^\lambda_a(x) = \frac{1}{\lambda}a - \frac{1-\lambda}{\lambda}x \tag{2.2}$$

Remark 2.11. A λ-reflection through a is not a reflection in the strict sense because $\mathcal{R}^\lambda_a \circ \mathcal{R}^\lambda_a \neq id : \mathbb{R}^q \to \mathbb{R}^q$, instead,

$$\mathcal{R}^{1-\lambda}_a \circ \mathcal{R}^\lambda_a = id : \mathbb{R}^q \to \mathbb{R}^q \ ,$$

so that, if $a = a_\lambda = \lambda a^+ + (1-\lambda)a^-$ is the λ-point of $(a^+, a^-) \in \mathbb{R}^{2q}$,

$$\mathcal{R}^\lambda_{a_\lambda}(a^-) = a^+ \ , \ \mathcal{R}^{1-\lambda}_{a_\lambda}(a^+) = a^- \ .$$

Of course, for $\lambda = 1/2$, $\mathcal{R}^{1/2}_a \equiv \mathcal{R}_a$ is a reflection in the strict sense.

Now, let M be a smooth n-dimensional submanifold of $\mathbb{R}^q$, with $2n \geq q$. Also, let M^+ be a germ of submanifold M around a^+, let M^- be a germ of submanifold M around a^- and let $a = a_\lambda = \lambda a^+ + (1-\lambda)a^-$ be the λ-point of $(a^+, a^-) \in M \times M \subset \mathbb{R}^q \times \mathbb{R}^q$.

Then, the following characterization is immediate:

Proposition 2.12. *The following conditions are equivalent:*

(i) $a \in E_\lambda(M)$

(ii) *M^+ and $\mathcal{R}_a^\lambda(M^-)$ are not transversal at a^+*
(iii) *M^- and $\mathcal{R}_a^{1-\lambda}(M^+)$ are not transversal at a^-.*

Therefore, *the study of the singularities of $E_\lambda(M) \ni 0$ can be proceeded via the study of the contact between M^+ and $\mathcal{R}_0^\lambda(M^-)$ or, equivalently, the contact between $\mathcal{R}_0^{1-\lambda}(M^+)$ and M^-.*

2.4. **Singularities of contact.** Let N_1, N_2 be germs at x of smooth n-dimensional submanifolds of the space $\mathbb{R}^q$, with $2n \geq q$. We describe N_1, N_2 in the following way:

- $N_1 = f^{-1}(0)$, where $f : (\mathbb{R}^q, x) \to (\mathbb{R}^{q-n}, 0)$ is a submersion-germ,
- $N_2 = g(\mathbb{R}^n)$, where $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^q, x)$ is an embedding-germ.

Definition 2.13. A **contact map** between submanifold-germs N_1, N_2 is the following map-germ $\kappa_{N_1,N_2} : (\mathbb{R}^n, 0) \to (\mathbb{R}^{q-n}, 0)$, where $\kappa_{N_1,N_2} = f \circ g$.

Let $\tilde{N}_1, \tilde{N}_2$ be another pair of germs at $\tilde{x}$ of smooth n-dimensional submanifolds of the space $\mathbb{R}^q$, described in the same way as N_1, N_2.

Definition 2.14. The contact of N_1 and N_2 at x is of the same **contact-type** as the contact of $\tilde{N}_1$ and $\tilde{N}_2$ at $\tilde{x}$ if $\exists$ a diffeomorphism-germ $\Phi : (\mathbb{R}^q, x) \to (\mathbb{R}^q, \tilde{x})$ s.t. $\Phi(N_1) = \tilde{N}_1$ and $\Phi(N_2) = \tilde{N}_2$. We denote the contact-type of N_1 and N_2 at x by $\mathcal{K}(N_1, N_2, x)$.

Theorem 2.15 ([14])**.** *$\mathcal{K}(N_1, N_2, x) = \mathcal{K}(\tilde{N}_1, \tilde{N}_2, \tilde{x})$ if and only if the contact maps $f \circ g$ and $\tilde{f} \circ \tilde{g}$ are $\mathcal{K}$-equivalent.*

Definition 2.16. We say that N_1 and N_2 are k-**tangent** at $x = 0$ if

$$\dim(T_0 N_1 \cap T_0 N_2) = k \ .$$

If k is maximal, that is, $k = \dim(T_0 N_1) = \dim(T_0 N_2)$, we say that N_1 and N_2 are **tangent** at 0.

Remark 2.17. In the context of affine equidistants, $E_\lambda(M)$, note that $N_1 = M^+$ and $N_2 = \mathcal{R}_0^\lambda(M^-)$ are k-**tangent** at 0 if and only if $T_{a^+}M^+$ and $T_{a^-}M^-$ are k-**parallel**, where $\lambda a^+ + (1-\lambda)a^- = 0 \in E_\lambda(M)$.

Proposition 2.18 ([5])**.** *If N_1 and N_2 are k-tangent at 0 then the corank of the contact map κ_{N_1,N_2} is k.*

3. Extrinsic geometry of surfaces in 4-space: overview

In this section, we remind basic definitions and results on the extrinsic geometry of smooth surfaces in 4-space. See [12, 13] for details.

Let $f : U \to \mathbb{R}^4$ be a local parametrisation of M, where U is an open subset of $\mathbb{R}^2$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be a positively oriented orthonormal frame in $\mathbb{R}^4$ such that at any $y = (y_1, y_2) \in U$, $\{\mathbf{e}_1(y), \mathbf{e}_2(y)\}$ is a basis for the tangent plane T_pM and $\{\mathbf{e}_3(y), \mathbf{e}_3(y)\}$ is a basis for the normal plane N_pM at $p = f(y)$.

Definition 3.1. The **second fundamental form** of M at p is the vector valued quadratic form $\mathrm{II}_p : T_pM \to N_pM$ associated to the normal component of the second derivative d^2f of f at p, that is,

$$\mathrm{II}_p = \langle d^2f, \mathbf{e}_3\rangle\mathbf{e}_3 + \langle d^2f, \mathbf{e}_4\rangle\mathbf{e}_4.$$

Let $a = \langle \mathbf{e_3}, f_{y_1y_1}\rangle, b = \langle \mathbf{e_3}, f_{y_1y_2}\rangle, c = \langle \mathbf{e_3}, f_{y_2y_2}\rangle, e = \langle \mathbf{e_4}, f_{y_1y_1}\rangle, f = \langle \mathbf{e_4}, f_{y_1y_2}\rangle, g = \langle \mathbf{e_4}, f_{y_2y_2}\rangle.$

Then, with this notation, we can write

$$\mathrm{II}_{\mathrm{p}}(\mathbf{u}) = (au_1^2 + 2bu_1u_2 + cu_2^2)\mathbf{e_3} + (eu_1^2 + 2fu_1u_2 + gu_2^2)\mathbf{e_4},$$

where $\mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2} \in T_pM$.

The matrix $\alpha = \begin{pmatrix} a & b & c \\ e & f & g \end{pmatrix}$ is called the matrix of the second fundamental form with respect to the orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$.

Definition 3.2. The second fundamental form of M at p, along a normal vector field ν is the quadratic form $\mathrm{II}_{\mathrm{p}}^{\nu} : \mathrm{T_pM} \to \mathbb{R}$ defined by

$$\mathrm{II}_p^{\nu}(\mathbf{u}) = \langle \mathrm{II}_{\mathrm{p}}(\mathbf{u}), \mathbf{v}\rangle, \quad \mathbf{u} \in \mathbf{T_pM},\ \mathbf{v} = \nu(p) \in N_pM,$$

where $\mathrm{II}_{\mathrm{p}}(\mathbf{u}) : T_pM \to N_pM$ is the second fundamental form at p.

Let S^1 be the unit circle in T_pM parametrized by $\theta \in [0, 2\pi]$. Denote by γ_θ the curve obtained by intersecting M with the hyperplane at p composed by the direct sum of the normal plane N_pM and the straight line in the tangent direction represented by θ. Such curve is called *normal section of M in the direction θ*.

Definition. The **curvature ellipse** is the image of the mapping

$$\begin{array}{rccc} \eta : & S^1 & \longrightarrow & N_pM \\ & \theta & \longmapsto & \eta(\theta), \end{array},$$

where $\eta(\theta)$ is the curvature vector of γ_θ.

Scalar invariants of the extrinsic geometry of surfaces in $\mathbb{R}^4$ can be defined using the coefficients of the second fundamental form. For instance the Gaussian curvature

$$\mathcal{G}_M = ac - b^2 + eg - f^2 \tag{3.1}$$

and the Δ function

$$\Delta_M = \frac{1}{4} det \begin{bmatrix} a & 2b & c & 0 \\ e & 2f & g & 0 \\ 0 & a & 2b & c \\ 0 & e & 2f & g \end{bmatrix}. \tag{3.2}$$

Although neither Δ_M nor $\mathcal{G}_M$ are affine invariants (a chosen metric was used to define them), the following proposition allows for an affine-invariant classification of a point $p \in M \subset \mathbb{R}^4$.

Proposition 3.3 ([3], Proposition 4.18)**.** *The sign of Δ_M is an affine invariant. When $rank\{II_{(p)}\} = 1$, the sign of $\mathcal{G}_M$ is also an affine invariant.*

Definition 3.4. A point $p \in M$ is called
(i) **parabolic** if $\Delta_M(p) = 0$,
(ii) **elliptic** if $\Delta_M(p) > 0$,
(iii) **hyperbolic** if $\Delta_M(p) < 0$.

Definition 3.5. A parabolic point $p \in M$ is called
(i-i) **point of nondegenerate ellipse**, if $rank\{II_{(p)}\} = 2$.
When $rank\{II_{(p)}\} = 1$, p is an inflection point. In this case, it is
(i-ii) **inflection point of real type**, if $\mathcal{G}_M(p) < 0$,
(i-iii) **inflection point of flat type**, if $\mathcal{G}_M(p) = 0$.
(i-iv) **inflection point of imaginary type**, if $\mathcal{G}_M(p) > 0$,

Definition 3.6. A direction $\mathbf{v} \in N_pM$ is a **binormal direction** at p if the second fundamental form $\mathrm{II}_{\mathrm{p}}^{\mathbf{v}}$ along the $\mathbf{v}$ direction is a degenerate quadratic form. In this case, a direction $\mathbf{u} \in T_pM$ in the kernel of $\mathrm{II}_{\mathrm{p}}^{\mathbf{v}}(\mathbf{u})$ is called an **asymptotic direction**.

Definition 3.7. For a surface $M \subset \mathbb{R}^4$, $p \in M$ and $\mathbf{u} \in T_pM$, $\mathbf{v} \in N_pM$, we say that $(\mathbf{u}, \mathbf{v})$ is a **contact pair** of M at p if $\mathbf{v}$ is a binormal direction at p and $\mathbf{u}$ is an asymptotic direction associated to $\mathbf{v}$.

Proposition 3.8 ([13], Lemma 3.2)**.** *Let M be a surface in $\mathbb{R}^4$,*
1) For a hyperbolic point $p \in M$, there are exactly 2 contact pairs at p.
2) For an elliptic point $p \in M$, there are no contact pairs at p.
3) For a parabolic point $p \in M$,
i) if p is a point of nondegenerate ellipse, then there exists only one contact pair at p.
ii) if p is an inflection point, then there exists only one $\mathbf{v} \in N_pM$ such that, for all $\mathbf{u} \in T_pM$, $(\mathbf{u}, \mathbf{v})$ is a contact pair at p.

4. Extrinsic geometry of surfaces in 4-space and singularities of their affine equidistants

We now present the geometric interpretation for the realizations of stable singularities of affine equidistants of surfaces in $\mathbb{R}^4$.

We first recall the following result from [5]:

Theorem 4.1 ([5], Theorem 5.2). *There exists a residual set $\mathcal{S}$ of embeddings $i : M^2 \to \mathbb{R}^4$, such that the map $\Psi_\lambda : M \times M \setminus \Delta \to \mathbb{R}^4$ is locally stable, where $\Psi_\lambda(x, y) = \lambda i(x) + (1 - \lambda)i(y)$ and Δ is the diagonal in $M \times M$.*

Definition 4.2. We say that $i : M^2 \to \mathbb{R}^4$ is a **generic embedding** if $i \in \mathcal{S}$.

Because the codimension of each singularity of Ψ_λ is at most 4, the possible stable singularities of affine equidistants of surfaces in $\mathbb{R}^4$ are:

$$A_1, A_2, A_3, A_4 \text{ for 1-parallelism, } \quad C^{\pm}_{2,2} \text{ for 2-parallelism.}$$

For the reader's convenience, we recall the normal forms of these stable singularities $(\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ in the table below:

Notation	Normal form
A_2	(u_1, u_2, u_3, y^2)
$A_\mu, 2 \le \mu \le 4$	$(u_1, u_2, u_3, y^{\mu+1} + \Sigma_{i=1}^{\mu-1} u_i y^i)$
$C^+_{2,2}$	$(u_1, u_2, x^2 + u_1 y, y^2 + u_2 x)$
$C^-_{2,2}$	$(u_1, u_2, x^2 - y^2, xy + u_1 x + u_2 y)$

We refer to [5], where all possible stable singularities of affine equidistants are classified for submanifolds $M^n \subset \mathbb{R}^q$, with $(2n, q)$ an arbitrary pair of nice dimensions, for all possible degrees of parallelism.

In this paper, we focus on investigating the conditions for realizing these equidistant singularities A_μ, $1 \le \mu \le 4$ and $C^{\pm}_{2,2}$ from the extrinsic geometry of a generic embedding of smooth surface $M \subset \mathbb{R}^4$.

In this specific case we substitute submanifold-germs N_1 and N_2 of Section 2 by $N_1 = M^+$ and $N_2 = \mathcal{R}^\lambda_0(M^-)$, or equivalently by $N_1 = M^-$ and $N_2 = \mathcal{R}^{1-\lambda}_0(M^+)$, where M_+ is the surface-germ of M around $a^+ \in M \subset \mathbb{R}^4$ and M_- is the surface-germ of M around $a^- \in M \subset \mathbb{R}^4$, with $\lambda a^+ + (1 - \lambda)a^- = 0$.

4.1. **Bi-local geometry of weakly parallel pairs and singularities of affine equidistants.** We start by looking at the bi-local geometry of 1-parallel pairs.

Suppose (a^+, a^-) is a pair of 1-parallel points. Then, we can choose coordinates in a neighbourhood of a^+ and a^- as follows:

$$\begin{aligned} \Phi^+ :&(\mathbb{R}^2, 0) \to (\mathbb{R}^4, a^+) \\ &(y, z) \mapsto a^+ + (y, z, \phi(y, z), \psi(y, z)), \end{aligned} \tag{4.1}$$

$j^1\phi(0, 0) = j^1\psi(0, 0) = 0.$

$$\begin{aligned} \Phi^- :&(\mathbb{R}^2, 0) \to (\mathbb{R}^4, a^-) \\ &(u, v) \mapsto a^- + (u, \xi(u, v), \zeta(u, v), v), \end{aligned} \tag{4.2}$$

$j^1\xi(0, 0) = j^1\zeta(0, 0) = 0$. In these coordinates, the local expression of the map $\Psi_\lambda|_{M\times M}$ is given by

$$\begin{aligned} \Psi_\lambda|_{M\times M} \ : \ &(\mathbb{R}^2, 0) \times (\mathbb{R}^2, 0) \to (\mathbb{R}^4, 0) \\ &((y, z), (u, v)) \mapsto (\lambda y + (1-\lambda)u, \lambda z + (1-\lambda)\xi(u, v), \\ &\qquad \lambda\phi(y, z) + (1-\lambda)\zeta(u, v), \lambda\psi(y, z) + (1-\lambda)v) \end{aligned}$$

where, to simplify, we have assumed $\lambda a^+ + (1-\lambda)a^- = 0$, for fixed λ.

In order to construct the contact map, we first reflect (M^-, a^-) through the point 0 to get $\mathcal{R}_0^\lambda(M^-)$, parametrized as

$$\mathcal{R}_0^\lambda(\Phi^-)(u, v) = a^+ - (\frac{(1-\lambda)}{\lambda}u, \frac{(1-\lambda)}{\lambda}\xi(u, v), \frac{(1-\lambda)}{\lambda}\zeta(u, v), \frac{(1-\lambda)}{\lambda}v).$$

The contact map $\mathcal{K}^\lambda : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is then given by

$$\begin{aligned} \mathcal{K}^\lambda(y, z) =&(z + \frac{1-\lambda}{\lambda}\xi(\frac{-\lambda}{1-\lambda}y, \frac{-\lambda}{1-\lambda}\psi(y, z)), \\ &\phi(y, z) + \frac{1-\lambda}{\lambda}\zeta(\frac{-\lambda}{1-\lambda}y, \frac{-\lambda}{1-\lambda}\psi(y, z)). \end{aligned} \tag{4.3}$$

The following theorem distinguishes the A_μ, $1 \le \mu \le 4$ singularities of equidistants, in terms of the bi-local geometry of M.

Theorem 4.3. *Let $a^+ \in M^+$, $a^- \in M^-$, so that $\lambda a^+ + (1-\lambda)a^- = 0$ is a singular point of $\Psi_\lambda|_{M\times M}$. For a pair of vectors $(\mathbf{u}, \mathbf{v})$ in $\mathbb{R}^4$, such that $\mathbf{u}$ is in the direction of 1-parallelism of (a^+, a^-) and $\mathbf{v} \in N_{a^+}M^+ \cap N_{a^-}M^-$ is in the common normal direction, let η_+ and η_- be the normal curvature of M^+ and $\mathcal{R}_0^\lambda(M^-)$ along $\mathbf{v}$ in the common direction $\mathbf{u}$. Then 0 is a singular point of $\Psi_\lambda|_{M\times M}$ of type A_k if and only if*

$$\eta_+^{(j)}(0) \ = \ (-1)^{j+1}\left(\frac{\lambda}{1-\lambda}\right)^{j+1}\eta_-^{(j)}(0)\ ,\ j = 0, \ldots, k-1, \tag{4.4}$$

$$\eta_+^{(k)}(0) \ \neq \ (-1)^{k+1}\left(\frac{\lambda}{1-\lambda}\right)^{k+1}\eta_-^{(k)}(0)\ , \tag{4.5}$$

where $\eta_+^{(j)}$ *and* $\eta_-^{(j)}$ *denote the* j*-order derivatives of* η_+ *and* η_- *respectively.*

Proof. We can solve the first equation $\mathcal{K}_1^\lambda = 0$ in (4.3), as $z = z(y)$, so that the contact map $\mathcal{K}^\lambda$ is $\mathcal{K}$-equivalent to the suspension of

$$\begin{aligned}\theta_\lambda :&\mathbb{R} \to \mathbb{R} \\ &y \mapsto \phi(y, z(y)) + \frac{1-\lambda}{\lambda}\zeta(\frac{-\lambda}{1-\lambda}y, \frac{-\lambda}{1-\lambda}\psi(y, z(y))).\end{aligned} \tag{4.6}$$

The point 0 is a singularity of type A_k of θ_λ if and only if

$$\frac{\partial^j \phi}{\partial y^j}(0) = (-1)^{j-1}\left(\frac{\lambda}{1-\lambda}\right)^{j-1}\frac{\partial^j \zeta}{\partial y^j}(0),\ j = 1, \ldots, k, \tag{4.7}$$

$$\frac{\partial^j \phi}{\partial y^j}(0) \neq (-1)^{j-1}\left(\frac{\lambda}{1-\lambda}\right)^{j-1}\frac{\partial^j \zeta}{\partial y^j}(0),\ j = k+1, \tag{4.8}$$

noting that condition (4.7) for $j = 1$ is the condition of 1-parallelism.

Letting α_+ and α_- be curves in M^+ and $\mathcal{R}_0^\lambda(M^-)$ given by

$$\begin{aligned}\alpha_+(y) =&(y, z(y), \phi(y, z(y), \psi(y, z(y))) \\ \alpha_-(y) =&(y, \frac{\lambda-1}{\lambda}\xi(\frac{\lambda}{\lambda-1}y, \frac{\lambda}{\lambda-1}\psi(y, z(y)), \\ &\frac{\lambda-1}{\lambda}\zeta(\frac{\lambda}{\lambda-1}y, \frac{\lambda}{\lambda-1}\psi(y, z(y))), \psi(y, z(y)))\end{aligned}$$

and letting $\eta_+(y)$ and $\eta_-(y)$ be the projections of the normal curvatures of α_+ and α_- in the common normal direction $\mathbf{v}$, then

$$\eta_+(y) = \frac{\partial^2 \phi}{\partial y^2}(y, z(y)) \text{ and } \eta_-(y) = \frac{\partial^2 \zeta}{\partial y^2}(y, z(y)).$$

So, equations (4.7)-(4.8) reduce to equations (4.4)-(4.5). □

We now look at the bi-local description of 2-parallel pairs.

Suppose (a^+, a^-) is a pair of 2-parallel points. Then, we can choose coordinates in a neighbourhood of a^+ and a^- as follows:

$$\begin{aligned}\Phi^+ :&(\mathbb{R}^2, 0) \to (\mathbb{R}^4, a^+) \\ &(y, z) \mapsto a^+ + (y, z, \phi(y, z), \psi(y, z)),\end{aligned} \tag{4.9}$$

$j^1\phi(0,0) = j^1\psi(0,0) = 0.$

$$\begin{aligned}\Phi^- :&(\mathbb{R}^2, 0) \to (\mathbb{R}^4, a^-) \\ &(u, v) \mapsto a^- + (u, v, \xi(u, v), \zeta(u, v)),\end{aligned} \tag{4.10}$$

$j^1\xi(0,0) = j^1\zeta(0,0) = 0.$

Again, for simplicity we assume that for λ fixed, $\lambda a^+ + (1-\lambda)a^- = 0$. Now the contact map $\mathcal{K}^\lambda : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is

$$\mathcal{K}^\lambda(y,z) = (\phi(y,z) + \frac{1-\lambda}{\lambda}\xi(\frac{-\lambda}{1-\lambda}y, \frac{-\lambda}{1-\lambda}z), \quad \psi(y,z) + \frac{1-\lambda}{\lambda}\zeta(\frac{-\lambda}{1-\lambda}y, \frac{-\lambda}{1-\lambda}z) \tag{4.11}$$

Let the contact surface $\mathcal{C}^\lambda \subset \mathbb{R}^4$ be the graph of the contact map $\mathcal{K}^\lambda$.

If $0 \in \mathcal{C}^\lambda \subset \mathbb{R}^4$ is a singular point of type $C^+_{2,2}$ of the contact map $\mathcal{K}^\lambda$, then $\Delta_{\mathcal{C}^\lambda}(0) < 0$ [13]. It follows that $\mathcal{C}^\lambda$ has two contact pairs at 0. For each of these, we have the following:

Theorem 4.4. *Let $a^+ \in M^+$, $a^- \in M^-$, so that $\lambda a^+ + (1-\lambda)a^- = 0 \in \mathcal{C}^\lambda \subset \mathbb{R}^4$ is a singular point of $\mathcal{K}^\lambda$ of type $C^+_{2,2}$. The pair $(\mathbf{u}, \mathbf{v})$ is a contact pair of $\mathcal{C}^\lambda$ at 0 if and only if one of the following holds.*

(i) The pair $(\mathbf{u}, \mathbf{v})$ is a contact pair of M^+ and of $\mathcal{R}^\lambda_0(M^-)$ at a^+ (equivalently, $(\mathbf{u}, \mathbf{v})$ is a contact pair of M^- and of $\mathcal{R}^{1-\lambda}_0(M^+)$ at a^-).

(ii) The pair $(\mathbf{u}, \mathbf{v})$ is not a contact pair of either M^+ or $\mathcal{R}^\lambda_0(M^-)$ at a^+, but the normal curvatures of M^+ and $\mathcal{R}^\lambda_0(M^-)$ along $\mathbf{u}$ in the direction of $\mathbf{v}$ are in proportion $\frac{\lambda}{1-\lambda}$ at a^+ (equivalently, $(\mathbf{u}, \mathbf{v})$ is not a contact pair of either M^- or $\mathcal{R}^{1-\lambda}_0(M^+)$ at a^-, but the normal curvatures of M^- and $\mathcal{R}^{1-\lambda}_0(M^+)$ along $\mathbf{u}$ in the direction of $\mathbf{v}$ have the proportion $\frac{1-\lambda}{\lambda}$ at a^-).

Proof. Let $(\mathbf{u}, \mathbf{v})$ be a contact pair of the contact surface $\mathcal{C}^\lambda$. Without loss of generality we can take $\mathbf{u} = (1,0,0,0)$ and $\mathbf{v} = (0,0,1,0)$. Then, since $\mathbf{v}$ is a binormal direction, it follows that the hessian of the function germ

$$\mathcal{K}^\lambda_2(y,z) = \psi(y,z) + \frac{1-\lambda}{\lambda}\zeta(\frac{-\lambda}{1-\lambda}y, \frac{-\lambda}{1-\lambda}z)$$

is degenerate and $\mathbf{u}$ is its kernel. Then $\frac{\partial^2 \mathcal{K}^\lambda_2}{\partial y^2}(0) = 0$, hence

$$\frac{\partial^2 \psi}{\partial y^2}(0) = -\frac{\lambda}{1-\lambda}\frac{\partial^2 \zeta}{\partial y^2}(0).$$

As in the proof of Theorem 4.3, either $\frac{\partial^2 \psi}{\partial y^2}(0) = 0$ and $\frac{\partial^2 \zeta}{\partial y^2}(0) = 0$ or they are not zero, but the normal curvatures of M^+ and $\mathcal{R}^\lambda_0(M^-)$ along $\mathbf{v}$ in the direction of $\mathbf{u}$ are proportional. Similar statement holds for M^- and $\mathcal{R}^{1-\lambda}_0(M^+)$. □

If $0 \in \mathcal{C}^\lambda \subset \mathbb{R}^4$ is a singular point of type $C^-_{2,2}$, then $\Delta_{\mathcal{C}^\lambda}(0) > 0$ [13]. It follows that $\mathcal{C}^\lambda$ has no contact pairs at 0. We thus have:

Theorem 4.5. *Let $a^+ \in M^+$, $a^- \in M^-$, so that $\lambda a^+ + (1-\lambda)a^- = 0 \in \mathcal{C}^\lambda \subset \mathbb{R}^4$ is a singular point of type $C^-_{2,2}$. Although $a^+ \in M^+$ and $a^- \in M^-$ are strongly parallel points, both of the following holds true.*

(i) M^+ and $\mathcal{R}_0^\lambda(M^-)$ do not have any common contact pair at a^+ (or equivalently, M^- and $\mathcal{R}_0^{1-\lambda}(M^+)$ do not have any common contact pair at a^-).

(ii) There is no pair $(\mathbf{u},\mathbf{v}) \in \mathbb{R}^4$ with $\mathbf{u} \in T_{a^+}M^+$ and $\mathbf{v} \in N_{a^+}M^+$, such that the normal curvature along $\mathbf{u}$ in the $\mathbf{v}$ direction of M^+ and of $\mathcal{R}_0^\lambda(M^-)$ are in proportion $\frac{\lambda}{1-\lambda}$ at a^+ (or equivalently, the normal curvature along $\mathbf{u}$ in the $\mathbf{v}$ direction of M^- and of $\mathcal{R}_0^{1-\lambda}(M^+)$ are in proportion $\frac{1-\lambda}{\lambda}$ at a^-).

Remark 4.6. Generically, $\Delta_{\mathcal{C}^\lambda} \neq 0$ because singular points of $\mathcal{C}^\lambda \subset \mathbb{R}^4$ of type $C_{2,3}$ are not unfolded to a stable point of Ψ_λ ([5]).

5. Geometry of the set of weakly parallel points

We now extend our geometric investigations in order to describe the set of weakly parallel points of M, as this set is naturally related to the set of affine equidistants of M and its singularities.

5.1. **Grassmannian investigation of weakly parallel points.** We start by using the Grassmannian $Gr(2,4)$, the space of 2-planes in $\mathbb{R}^4$.

First, we recall the Plücker coordinates for $Gr(2,4)$. Let $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}$ be any basis for $\mathbb{R}^4$ (not necessarily orthonormal or orthogonal, no metric is needed or assumed here). Then, $\mathbf{e_1}\wedge\mathbf{e_2}, \mathbf{e_1}\wedge\mathbf{e_3}, \mathbf{e_1}\wedge\mathbf{e_4}, \mathbf{e_2}\wedge\mathbf{e_3}, \mathbf{e_2}\wedge\mathbf{e_4}, \mathbf{e_3}\wedge\mathbf{e_4}$ is a basis for $\Lambda^2\mathbb{R}^4$ and we denote by $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ the coordinates of an element $\pi \in \Lambda^2\mathbb{R}^4$ in the above basis.

If the bi-vector $\pi \in \Lambda^2\mathbb{R}^4$ with coordinates $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ represents an element in $Gr(2,4)$, then the bi-vector $\pi' \in \Lambda^2\mathbb{R}^4$ with coordinates $(kp_{12}, kp_{13}, kp_{14}, kp_{23}, kp_{24}, kp_{34})$, $0 \neq k \in \mathbb{R}$, represents the same element in $Gr(2,4)$. Thus, defining the equivalence class $[\pi] = \{\pi' \in \Lambda^2\mathbb{R}^4 \mid \pi' = k\pi, k \in \mathbb{R}^*\}$, it follows that $[\pi] \in \mathbb{P}(\Lambda^2\mathbb{R}^4)$ has homogeneous coordinates $[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}]$.

However, not every element $[\pi] \in \mathbb{P}(\Lambda^2\mathbb{R}^4)$ lies in $Gr(2,4)$. π is in $Gr(2,4)$ iff π is an elementary bi-vector, i.e. $\pi = \mathbf{u} \wedge \mathbf{v}$, for some $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$. Thus $[\pi] \in Gr(2,4)$ iff

$$\pi \wedge \pi = 0.$$

In terms of the above coordinates, this translates into the equation

$$p_{12}p_{34} + p_{23}p_{14} - p_{13}p_{24} = 0. \tag{5.1}$$

The homogeneous coordinates $[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}]$ subject to constraint (5.1) are the Plücker coordinates of $[\pi] \in Gr(2,4)$ with respect to the basis $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}$ of $\mathbb{R}^4$. It follows that $dim_{\mathbb{R}}(Gr(2,4)) = 4$.

Now, consider the Gauss map

$$G : M \to Gr(2,4) \ , \ \mathbb{R}^4 \supset M \ni a \mapsto [T_a M] \in Gr(2,4).$$

The Gauss map fails to be injective precisely for (non-diagonal) strongly parallel pairs, i.e, $a_1 \neq a_2 \in M$, such that $G(a_1) = G(a_2)$. Thus, for a residual set of embeddings $M \subset \mathbb{R}^4$, $G : M \to Gr(2,4)$ is an immersion with transversal double points and such a $[\pi] \in G(M)$ whose neighborhood in $G(M)$ is not homeomorphic to $\mathbb{R}^2$ is the common tangent plane for a (non-diagonal) 2-parallel pair $(a_1, a_2) \in M \times M$.

Consider also the double Gauss map:

$$G \times G : M \times M \to Gr(2,4) \times Gr(2,4) \ , \ (a_1, a_2) \mapsto ([\pi_1], [\pi_2])$$

Then, $[\pi_1]$ and $[\pi_2]$ are weakly parallel, iff

$$\pi_1 \wedge \pi_2 = 0. \tag{5.2}$$

And we denote

$$W = \{([\pi_1], [\pi_2]) \in Gr(2,4) \times Gr(2,4) | \ \pi_1 \wedge \pi_2 = 0\}.$$

In terms of the Plücker coordinates for $Gr(2,4)$,

$$[\pi_1] = [p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}] \ , \ p_{12}p_{34} + p_{23}p_{14} - p_{13}p_{24} = 0 \ , \tag{5.3}$$

$$[\pi_2] = [q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, q_{34}] \ , \ q_{12}q_{34} + q_{23}q_{14} - q_{13}q_{24} = 0 \ , \tag{5.4}$$

condition (5.2) translates into

$$p_{12}q_{34} + p_{34}q_{12} + p_{14}q_{23} + p_{23}q_{14} - p_{13}q_{24} - p_{24}q_{13} = 0. \tag{5.5}$$

Thus, equations (5.3), (5.4) and (5.5) define coordinates for an element $([\pi_1], [\pi_2])$ of the 7-dimensional subvariety $W \subset (Gr(2,4) \times Gr(2,4))$.

We denote by W_{reg} the set of smooth points of W, and by $Sing(W)$ the set of singular points of W.

Proposition 5.1. *Away from the diagonal, W is a smooth hypersurface of $Gr(2,4) \times Gr(2,4)$.*

Proof. First, note that each of the equations (5.3) and (5.4) define smooth submanifolds $Gr(2,4) \subset \mathbb{P}(\Lambda^2\mathbb{R}^4)$ and, similarly, equation (5.5) defines a smooth submanifold of $\mathbb{P}(\Lambda^2\mathbb{R}^4) \times \mathbb{P}(\Lambda^2\mathbb{R}^4)$. Thus, W is singular only where these three submanifolds of $\mathbb{P}(\Lambda^2\mathbb{R}^4) \times \mathbb{P}(\Lambda^2\mathbb{R}^4)$ do not intersect transversaly. By straightforward computation, we see that the rank of the matrix of the derivatives of equations (5.3), (5.4) and (5.5) is not maximal iff $\forall \ 1 \leq i < j \leq 4, \ p_{ij}/q_{ij} = k \in \mathbb{R}^*$. It follows that $Sing(W) = \{([\pi_1], [\pi_2]) \in Gr(2,4) \times Gr(2,4) \mid [\pi_1] = [\pi_2]\}$. □

Now, as $Gr(2,4) \times Gr(2,4)$ fibers (trivially) over $Gr(2,4)$, say, via the first projection Pr_1, this induces a sub-bundle $W \to Gr(2,4)$, $([\pi_1],[\pi_2]) \mapsto [\pi_1]$, which may not be trivial. Its typical fiber $W_{[\pi_1]}$ is a 3-variety, which can locally be described as follows.

Chose a basis $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}$ for $\mathbb{R}^4$ such that $[\pi_1] = [\mathbf{e_1} \wedge \mathbf{e_2}]$. Then, $[\pi_1] = [1,0,0,0,0,0]$, and $[\pi_2] = [q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, q_{34}] \in W_{[\pi_1]}$ iff $q_{12}q_{34} + q_{23}q_{14} - q_{13}q_{24} = 0$ and $q_{34} = 0$, that is,

$$[\pi_2] \in W_{[\pi_1]} \iff [\pi_2] = [q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, 0] \ , \ q_{23}q_{14} - q_{13}q_{24} = 0 \ ,$$

or equivalently,

$$[\pi_2] \in W_{[\pi_1]} \iff [\pi_2] = [1, \alpha, \beta, \gamma, \delta, 0] \ , \ \beta\gamma - \alpha\delta = 0 \ , \tag{5.6}$$

in other words, close to $\alpha = \beta = \gamma = \delta = 0$,

$$W_{[\pi_1]} = \{(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4 \mid \alpha\delta - \beta\gamma = 0\} \ . \tag{5.7}$$

Thus, we have a refinement of Proposition 5.1, that is,

Proposition 5.2. *In a neighborhood of $[\pi_2] = [\pi_1]$, the 3-variety $W_{[\pi_1]}$ is a cone.*

The following theorem, which follows from standard transversality arguments, describes how affine equidistants $E_\lambda(M)$ are related to the intersection of W and $G(M) \times G(M)$.

Theorem 5.3. *Let $M \subset \mathbb{R}^4$ be a generic embedding and (a,b) be a weakly parallel pair on M.*

(i) Let (a,b) be a 1-parallel pair, so that $(G(a), G(b)) \in W_{reg}$. If $\Psi_\lambda|_{M\times M} : (\mathbb{R}^2 \times \mathbb{R}^2, (a,b)) \to (\mathbb{R}^4, \lambda a + (1-\lambda)b)$ has a stable singularity (of type A_k, $k = 1,2,3,4$), then $G(M) \times G(M)$ is transverse to W_{reg} at $(G(a), G(b))$.

(ii) Let (a,b) be a 2-parallel pair, so that $(G(a), G(b)) \in Sing(W)$. If $\Psi_\lambda|_{M\times M} : (\mathbb{R}^2 \times \mathbb{R}^2, (a,b)) \to (\mathbb{R}^4, \lambda a + (1-\lambda)b)$ has a stable singularity (of type $C^{\pm}_{2,2}$) then (a,b) is a transversal double point of the Gauss map.

5.2. **Geometric description of the set of weakly parallel points.** We emphasize that, from Theorem 5.3, for generic embeddings of smooth closed surfaces in $\mathbb{R}^4$ there are only double points of Gauss map. There are no triple, quadruple... points of the Gauss map, generically.

Therefore we obtain the following corollary of Theorem 5.3:

Corollary 5.4. *For generic embeddings of smooth closed surfaces in $\mathbb{R}^4$, strongly parallel (nonidentical) points come only in pairs and there are only finite numbers of such pairs.*

An interesting question, whose answer is unknown to us, is whether there exists any embedded compact surface $M \subset \mathbb{R}^4$ without non-identical 2-parallel points, in other words, such that the Gauss map $G : M \to Gr(2,4)$ is injective.

Notation 5.5. For $p \in M$, let $\mathcal{W}_p \subset M$ denote the set of weakly parallel points to p and let $\mathcal{W}_p^q$ denote the germ of $\mathcal{W}_p$ at $q \in M$.

Remark 5.6. It is easy to see that $G(\mathcal{W}_p) \subset W$ where the latter is described in Propositions 5.1 and 5.2.

Then, the following theorem describes $\mathcal{W}_p^q$ in all possible situations.

Theorem 5.7. *For a generic embedding of M into $\mathbb{R}^4$, cf. Definition 4.2 and Theorem 4.1, the following hold.*

(1) If q is 1-parallel to p, then $\mathcal{W}_p^q$ is a germ of smooth curve.

(2) If q is 2-parallel to p, then:

(i) If q is an elliptic point of M, then $\mathcal{W}_p^q = \{q\}$.

(ii) If q is a parabolic point of M, then $\mathcal{W}_p^q$ is a singular curve with a cusp singularity at q which is tangent to the asymptotic direction at q (this is generic for $q = p$, as a generic embedding has a parabolic point, or in a 1-parameter family of embeddings for $q \neq p$, cf. Remark 4.6).

(iii) If q is a hyperbolic point of M, then $\mathcal{W}_p^q$ is a singular curve with a transversal double point at q so that each branch of $\mathcal{W}_p^q$ is a smooth curve tangent to an asymptotic direction at q.

Proof. If the points $p, q \in M$ are 1-parallel then the germs of M at $p = (p_1, p_2, p_3, p_4)$ and at $q = (q_1, q_2, q_3, q_4)$ can be parametrized in the following way $F(x, y) = (p_1 + x, p_2 + y, p_3 + f_3(x, y), p_4 + f_4(x, y))$ and $G(u, v) = (q_1 + u, q_2 + g_2(u, v), q_3 + g_3(u, v), q_4 + v)$ respectively, where f_3, f_4, g_2, g_4 are smooth function-germs vanishing at $(0, 0)$ such that $df_3|_{(0,0)} = df_4|_{(0,0)} = dg_2|_{(0,0)} = dg_3|_{(0,0)} = 0$. The point $G(u, v)$ is weakly parallel to p if the Jacobian of the map

$$(x, y, u, v) \mapsto \lambda F(x, y) + (1 - \lambda)G(u, v) \tag{5.8}$$

vanishes at the point $(0, 0, u, v)$. The Jacobian of the map (5.8) at $(0, 0, u, v)$ has the form $\frac{\partial g_3}{\partial u}(u, v)$. Generically $d(\frac{\partial g_3}{\partial u})|_{(0,0)} \neq 0$, therefore $\mathcal{W}_p^q$ is a germ at q of a smooth curve.

If the points $p, q \in M$ are 2-parallel then the germs of M at $p = (p_1, p_2, p_3, p_4)$ and at $q = (q_1, q_2, q_3, q_4)$ can be parametrized in the following way $F(x, y) = (p_1 + x, p_2 + y, p_3 + f_3(x, y), p_4 + f_4(x, y))$ and $G(u, v) = (q_1 + u, q_2 + v, q_3 + g_3(u, v), q_4 + g_4(u, v))$ respectively, where f_3, f_4, g_3, g_4 are smooth function-germs vanishing at $(0, 0)$ such that $df_3|_{(0,0)} = df_4|_{(0,0)} = dg_3|_{(0,0)} = dg_4|_{(0,0)} = 0$.

The point $G(u, v)$ is weakly parallel to p if the Jacobian of the map (5.8) vanishes at $(0, 0, u, v)$. It is easy to see that the Jacobian of the map (5.8) at $(0, 0, u, v)$ is $Jac(g_3, g_4)(u, v)$, i.e. the Jacobian of the map (g_3, g_4) at (u, v). It is also easy to see $d(Jac(g_3, g_4))|_{(0,0)}$ vanishes.

The Hessian of the function $(u, v) \mapsto Jac(g_3, g_4)(u, v)$ at $(0, 0)$ is equal to $4\Delta_M(q)$. Therefore if q is an elliptic point, then $\mathcal{W}_p^q = \{q\}$, if q is a parabolic point, then $\mathcal{W}_p^q$ is a singular curve with a cusp singularity at q which is tangent to the asymptotic direction at q, and finally if q is a hyperbolic point, then $\mathcal{W}_p^q$ consists of the crossing of two smooth curves at q, each one tangent to an asymptotic direction at q.

We can also interpret the above calculations in terms of singularities of projections into planes. In fact, let $\rho_p : M \to N_pM$ be the projection of M into the 2-plane $N_pM = \mathbb{R}^2$, which is fixed.

Then the singular set of the projection,

$$\Sigma\rho_p = \{q \in M | \text{ there exists some } \mathbf{v} \in T_qM, \mathbf{v} \in \ker\rho_p\}$$

coincides with the set $\mathcal{W}_p$. Given $q \in \mathcal{W}_p$, we use the above local parametrizations to study $\mathcal{W}_p^q$.

If points $p, q \in M$ are 1-parallel then the germs of M at $p = (p_1, p_2, p_3, p_4)$ and at $q = (q_1, q_2, q_3, q_4)$ can be parametrized respectively by $F(x, y) = (p_1 + x, p_2 + y, p_3 + f_3(x, y), p_4 + f_4(x, y))$ and $G(u, v) = (q_1 + u, q_2 + g_2(u, v), q_3 + g_3(u, v), q_4 + v)$. The normal plane of M at p is the plane $[(0, 0, 1, 0), (0, 0, 0, 1)]$. Hence, the germ at q of the projection $\rho_p : M \to N_pM$ is given by

$$\rho_p \circ G(u, v) = (q_3 + g_3(u, v), q_4 + v).$$

Thus, as above, $\mathcal{W}_p^q$ is smooth in a neighbourhood of q if and only if $(g_{3uu}(0), g_{3uv}(0)) \neq (0, 0)$ and this condition is satisfied for generic embeddings of M. On the other hand, if points $p, q \in M$ are 2-parallel, the germ at q of the projection is given by

$$\rho_p \circ G(u, v) = (q_3 + g_3(u, v), q_4 + g_4(u, v)),$$

and we proceed as above. □

Because the case $(2 - ii)$ above for $q \neq p$ is only generic in a 1-parameter family of embeddings, according to Definition 4.2 and Theorem 4.1, we now analyze its bifurcation set.

Proposition 5.8. *Let $I \ni t$ be an open interval containing 0, with M_t a generic smooth 1-parameter family of smooth surface embeddings in $\mathbb{R}^4$ such that the points p_t and q_t in M_t are strongly parallel $\forall t \in I$ and q_0 is a parabolic point of $M_0 \subset \mathbb{R}^4$. Let $\mathcal{W}_{p_t}^{q_t} \subset M_t$ denote the germ of weakly parallel points to p_t at q_t. Then, $\mathcal{W}_{p_t}^{q_t}$ is described by the Whitney umbrella*

$$2u^2 - 3v^3 - 2tv^2 = 0 \ , \tag{5.9}$$

such that a curve $\mathcal{C}_{t=t_0}$ on this surface in $\mathbb{R}^3$ has a smooth branch and an isolated point if $t_0 < 0$, or is a cusp if $t_0 = 0$, or is a looped curve with a transversal self-crossing if $t_0 > 0$. These three cases correspond to the point $q_{t_0} = (u, v) = (0, 0)$ being an isolated point for $t_0 < 0$ (elliptic case), a cuspidal point for $t_0 = 0$ (parabolic case), or a transversal double point for $t_0 > 0$ (hyperbolic case).

Proof. Following the same notation of the proof of Theorem 5.7, with t denoting the parameter of the family of embeddings and assuming q_t is parabolic for $t = 0$, the germ of M_t at q_t can be put after translation to the form[1] $g_t(u, v) = (u, v, g_t^3(u, v), g_t^4(u, v))$, where $g_t^3(u, v) = u^2 + v^3 + tv^2 + V_t(u, v)$ and $g_t^4(u, v) = uv + W_t(u, v)$, with V_t and W_t of third or higher order in (u, v) for all t.

The point p_t being 2-parallel to q_t, the germ of M_t at p_t is, after translation, of the general form $f_t(x, y) = (x, y, f_t^3(x, y), f_t^4(x, y))$, with f_t^3 and f_t^4 of second order in (u, v) for all t.

Thus, as before, $g_t(u, v)$ is weakly parallel to p_t if the Jacobian of the map (5.8) vanishes at $(0, 0, u, v)$ and this Jacobian is the same as the Jacobian of the map (g_t^3, g_t^4) at (u, v), which is of the form $J(u, v, t) = 2u^2 - 3v^3 - 2tv^2 + R_t(u, v)$, where R_t is of third or higher order in (u, v) for all t. We now apply the following lemma:

Lemma 5.9. *The Jacobian $J(u, v, t) = 2u^2 - 3v^3 - 2tv^2 + R_t(u, v)$, with R_t of third or higher order in (u, v), $\forall t$, can be put for small t, by a smooth near-identity change of coordinates of the form $(u, v, t) \mapsto (U(u, v, t), V(u, v, t), t)$, to the normal form $H(U, V, t) = 2U^2 - 3V^3 - 2tV^2(1 + \phi(U, V, t))$, with ϕ a smooth function satisfying $\phi(0, 0, t) = 0$, for small t.*

Proof. Start by writing $R_t(u, v) = t\big(\psi_3(v, t) + u\psi_2(v, t) + 2u^2\psi_1(u, v, t)\big)$, where ψ_3 is of order at least 3 in v, $\forall t$, ψ_2 is of order at least 2 in v, $\forall t$, and $\psi_1(0, 0, t) = 0$. Then, $J(u, v; t) = 2u^2(1 + t\psi_1(u, v, t)) - 3v^3(1 - t\widetilde{\psi}_3(v, t)) - 2tv^2(1 - u\widetilde{\psi}_2(v, t))$, where $\widetilde{\psi}_3(v, t) = \psi_3(v, t)/3v^3$, $\widetilde{\psi}_2(v, t) =$

[1]In general, the tangent plane to M_t at q_t will change with t, but we can adopt an orthonormal moving frame such that $T_{q_t}M_t = span < (1, 0, 0, 0), (0, 1, 0, 0) >, \forall t$.

$\psi_2(v,t)/2v^2$. Thus, setting $V(u,v,t) = V(v,t) = v\sqrt[3]{1 - t\widetilde{\psi}_3(v,t)}$ and $U(u,v,t) = u\sqrt{1 + t\psi_1(u,v,t)}$, we note that $(u,v,t) \mapsto (U,V,t)$ is a near-identity transformation for small t, therefore invertible, so that we can write $J(u,v,t) = H(U,V,t) = 2U^2 - 3V^3 - 2tV^2(1 + \phi(U,V,t))$, where ϕ is a smooth function satisfying $\phi(0,0,t) = 0$, for small t. □

It follows that, for small t and in a neighborhood of $(U,V) = (0,0)$, the curve $\mathcal{C}'_{t=t_0}$, which is obtained as the section $\{H(U,V,t = t_0) = 0\}$, is a small deformation of the curve $\mathcal{C}_{t=t_0}$, which is obtained as the section $\{h(u,v,t = t_0) = 0\}$, where $h(u,v,t) = 2u^2 - 3v^3 - 2tv^2$. In particular, for $t_0 = 0$ the curve $\mathcal{C}'_{t=t_0}$ is a cusp, just as $\mathcal{C}_{t=t_0}$, for $t_0 < 0$ the curve $\mathcal{C}'_{t=t_0}$ has a smooth branch and an isolated point at $(0,0)$, just as $\mathcal{C}_{t=t_0}$, and for $t_0 > 0$ the curve $\mathcal{C}'_{t=t_0}$ is a looped curve with a transversal self-crossing at $(0,0)$, just like $\mathcal{C}_{t=t_0}$. □

Remark 5.10. In the same vein, when the embedding is fixed and $q = p$, if a smooth curve $I \ni s \mapsto p(s) \in M$ is transversal to the smooth curve of parabolic points on M at a parabolic point $p(0)$, then by slightly adapting the above reasoning we can easily see that the family of germs $\mathcal{W}_{p(s)}^{p(s)}$ is described by the Whitney umbrella (5.9).

Remark 5.11. As a last remark, we note that two distinct points $q, q' \in \mathcal{W}_p$ need not be weakly parallel to each other. For instance, if $[\pi_1] = G(p) = [\mathbf{e_1} \wedge \mathbf{e_2}]$, we may have that $G(q) = [\mathbf{e_1} \wedge \mathbf{e_3}]$ and $G(q') = [\mathbf{e_2} \wedge \mathbf{e_4}]$. We also note that, if (p,q) is a strongly parallel pair $(p \neq q)$, the local geometry of p and q can be distinct (one elliptic, the other hyperbolic, etc), thus in general $\mathcal{W}_p^q$ and $\mathcal{W}_q^p$ can be of distinct types.

5.3. **Illustrations.** We now provide examples of Theorem 5.7 and Proposition 5.8, this latter in the form of Remark 5.10.

Example 5.12. Let us consider the following embedding of a torus into the affine space $\mathbb{R}^4$([6]), $F(x,y) = (f_1(x,y), f_2(x,y), f_3(x,y), f_4(x,y))$,

$$f_1(x,y) = \cos(x)\left(1 - \frac{\cos(y)}{10}\right) + \frac{1}{10}\sin(x)\sin(y),$$

$$f_2(x,y) = \left(1 - \frac{\cos(y)}{10}\right)\sin(x) - \frac{1}{10}\cos(x)\sin(y),$$

$$f_3(x,y) = \cos(2x)\left(1 - \frac{2\cos(y)}{5}\right) + \frac{4}{5}\sin(2x)\sin(y),$$

$$f_4(x,y) = \left(1 - \frac{2\cos(y)}{5}\right)\sin(2x) - \frac{4}{5}\cos(2x)\sin(y).$$

The curves of parabolic points on this torus are given by

$$y = \pm 2\arctan\left(\sqrt{\frac{1}{5}\left(-4+\sqrt{41}\right)}\right).$$

Fig. 1 presents the curve of weakly parallel points on the x, y-plane to a hyperbolic point (π, π) (or elliptic point $(0, 0)$). All points marked by black dots on Fig. 1 are strongly parallel. Elliptic points $(0, 0)$ and $(\pi, 0)$ are isolated points of the the curve. There are transversal self-intersections of the curve in hyperbolic points $(0, \pi)$ and (π, π).

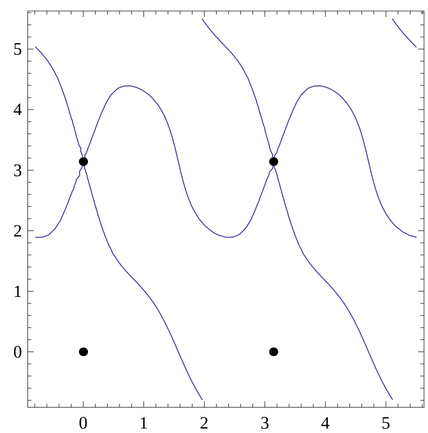

Figure 1. Set of weakly parallel points to an elliptic or hyperbolic point.

Example 5.13. Let us again consider the torus from Example 5.12. In Figures 2 to 4 we preset the bifurcation of $\mathcal{W}_p^p$ - the germ at a point p of the curve of weakly parallel points to p - when we change p from a hyperbolic point to a parabolic point and then to an elliptic point. For p we chose a point with the following coordinates on the (x, y)-plane:

$$\left(s, 2\arctan\left(\sqrt{\frac{1}{5}\left(-4+\sqrt{41}\right)}\right)+s\right)$$

For $s = 0$ the point p is parabolic and at this parabolic point (marked by a black dot) the curve has a cusp singularity, cf. Fig. 3, which also shows the curve of weakly parallel points to this parabolic point.

For sufficiently small positive s the point p is hyperbolic (cf. Fig. 2) and for sufficiently small negative s the point p is elliptic (cf. Fig 4). The dotted lines on Figs. 2-4 are lines of parabolic points. From the figures we see that the bifurcation of the set $\mathcal{W}_p^p$ when we change s is diffeomorphic to the Whithney umbrella, which is presented on Fig. 5.

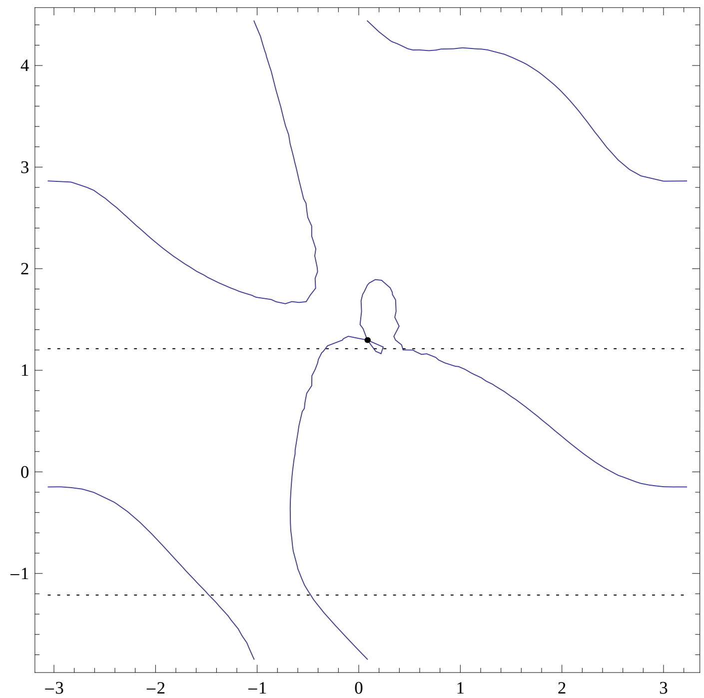

Figure 2. Set of weakly parallel points to a hyperbolic point (s=0.085).

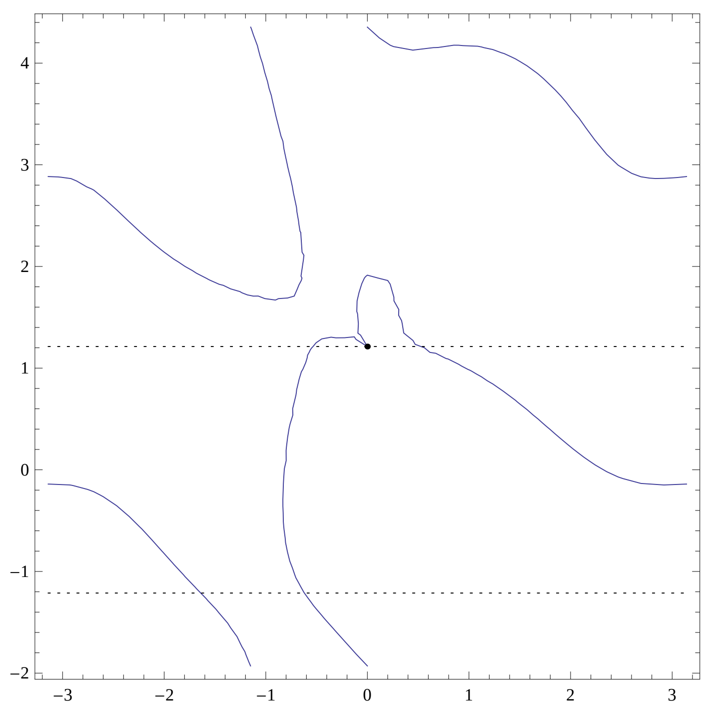

Figure 3. Set of weakly parallel points to a parabolic point (s=0).

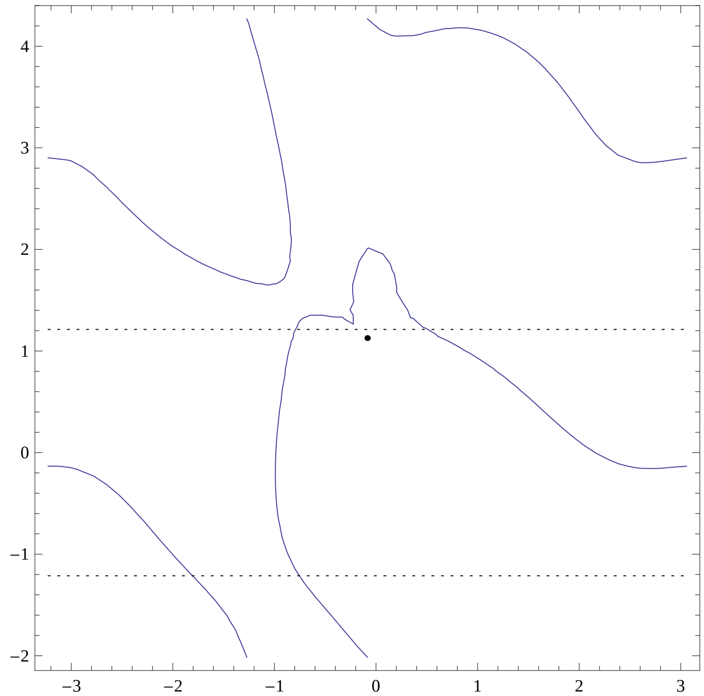

Figure 4. Set of weakly parallel points to an elliptic point ($s = -0.085$).

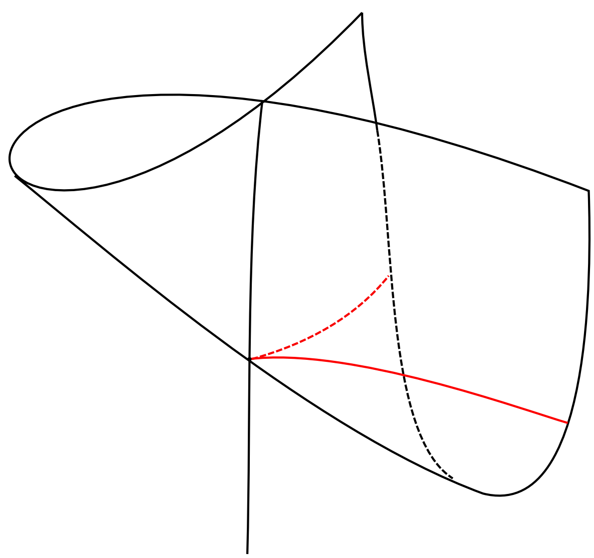

Figure 5. The bifurcation of the germ, at a parabolic point p, of the set of weakly parallel points to p.

References

[1] M. V. Berry, *Semi-classical mechanics in phase space: A study of Wigner's function*, Philos. Trans. R. Soc. Lond. A 287 (1977) 237-271.

[2] M. Craizer, W. Domitrz, P. de M. Rios *Even-dimensional improper affine spheres*, J. Math. Anal. Appl. 421 (2015) 1803–1826.

[3] W. Domitrz, M. Manoel, P. de M. Rios, *The Wigner caustic on shell and singularities of odd functions*, J. Geometry and Physics 71 (2013) 58-72.

[4] W. Domitrz, P. de M. Rios, *Singularities of equidistants and global centre symmetry sets of Lagrangian submanifolds*, Geom. Dedicata 169 (2014) 361-382.

[5] W. Domitrz, P. de M. Rios, M. A. S. Ruas, *Singularities of affine equidistants: projections and contacts*, J. Singul. 10 (2014), 67-81.

[6] R. A. Garcia, D. Mochida, M. C. Romero Fuster, M. A. S. Ruas, *Inflection points and topology of surfaces in 4-space*, Trans. Amer. Math. Soc. 352 (2000), no. 7, 3029-3043.

[7] P. J. Giblin, P. A. Holtom, *The centre symmetry set*, Geometry and Topology of Caustics, Banach Center Publications, Vol 50, Warsaw, 1999, 91-105.

[8] P. J. Giblin, V. M. Zakalyukin, *Singularities of systems of chords*, Funct. Anal. Appl. 36 (2002) 220-224.

[9] P. J. Giblin, V. M. Zakalyukin, *Singularities of centre symmetry sets*, Proc. London Math. Soc. 90 (2005) 132-166.

[10] P. Giblin, S. Janeczko, *Geometry of curves and surfaces through the contact map*, Topology Appl. 159 (2012) 466-475.

[11] S. Janeczko, *Bifurcations of the center of symmetry*, Geom. Dedicata 60 (1996) 9-16.

[12] J. A. Little, *On singularities of submanifolds of higher dimensional Euclidean spaces*, Ann. Mat. Pura Appl. (4) 83 (1969) 261-335.

[13] D. K. H. Mochida, M. C. Romero Fuster, M. A. S. Ruas, *Geometry of surfaces in* 4*-space from a contact viewpoint*, Geom. Dedicata 54 (1995) 323-332.

[14] J. A. Montaldi, *On contact between submanifolds*, Michigan Math. J. 33 (1986) 195-199.

[15] A. M. Ozorio de Almeida, J. Hannay, *Geometry of Two Dimensional Tori in Phase Space: Projections, Sections and the Wigner Function*, Annals of Physics 138 (1982) 115-154.

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