Regularization of discrete ill-posed problems

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Abstract.

The paper concerns conditioning aspects of finite dimensional problems arisen when the Tikhonov regularization is applied to discrete ill-posed problems. A relation between a regularization parameter and sensitivity of regularized solution is investigated. Moreover, it is shown that choice of regularization parameter optimal with respect to the condition number of finite dimensional operator approximating a compact operator gives regularized discrete solutions convergent to the exact generalized solution when dimension of discrete equation and data error are related in a proper way and the data error tends to 0. As an example the method of truncated singular value decomposition with regularization is considered.

1 Introduction

Let $A \in L(X)$ be a compact operator in a Hilbert space X and let us consider the operator equation

$$Au = f. \tag{1.1}$$

By a generalized solution u^+ we mean the minimal norm solution of the normal equation

$$A^*Au = A^*f \tag{1.2}$$

and the corresponding general inverse operator is denoted by A^+. Assume, that for f the generalized solution $u^+ = A^+f$ exists. If the dimension of the range of A is not finite, the inverse of A as well as the generalized inverse A^+ are not bounded. Thus the problem

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(1.1) is ill-posed in the space X also in the least squares sense. So, for slightly perturbed right hand side f_δ, $\|f - f_\delta\| \leq \delta$, the solution $A^+ f_\delta$ does not necessarily exist, or, if it exists, the distance between these two solutions may be arbitrary large.

In practice we deal with a finite dimensional approximation of the equation (1.1)

$$A_n u_n = f_n, \tag{1.3}$$

where A_n is a linear operator acting between finite dimensional spaces. For simplicity of notation we will mainly consider the case $A_n \in L(X_n)$. Let us assume that the sequence of unitary n-dimensional spaces X_n is a convergent and stable approximation of the space X with uniformly bounded restriction and prolongation operators denoted by $r_n : X \to X_n$, and $p_n : X_n \to X$, respectively. Moreover, let us assume that

$$\|A^*A - p_n A_n^* A_n r_n\| \to 0, \text{ as } n \to \infty, \tag{1.4}$$

and $f_n := r_n f$. The generalized solution u_n for (1.3) exists and $p_n u_n$ converges to u^+ as $n \to \infty$ for the exact data. But this approximation is unstable; a small perturbation of f disturbed convergence.

Let $\{\mu_j^2\}_{j=1}^{\infty}$ and $\{\sigma_j^2(A_n)\}_{j=1}^{n}$ be non increasing sequences of eigenvalues of the operators A^*A and $A_n^*A_n$, respectively, repeated according to their multiplicities. Thus, non-negative μ_j and $\sigma_j(A_n)$ are singular values of A and A_n, respectively. Since, except for the eigenvalues 0, $A_n^*A_n$ and $p_n A_n^* A_n r_n$ have the same eigenvalues, due to the spectral approximation results (cf. [3], th.6) $\sigma_j(A_n)$ converges to μ_j as $n \to 0$. Let $\sigma_{min}(A_n)$ denote square root of the smallest nonzero eigenvalue of $A_n^*A_n$. From ill-posedness of (1.1) it follows $\mu_j \to 0$ as $j \to \infty$ and consequently $\sigma_{min}(A_n) \to 0$ when $n \to \infty$. Thus the condition number $\kappa(A_n)$ of approximating operator A_n (cf. [1], [6]), given by

$$\kappa(A_n) := \frac{\sigma_{max}(A_n)}{\sigma_{min}(A_n)}, \tag{1.5}$$

is large and tends to infinity with n. Thus, we have to deal with very ill conditioned problems (1.3) for large n. The problem (1.3) is well posed in the least squares sense with respect to right hand side perturbations, but its generalized solution is not stable approximation of the exact solution. Discretization of ill-posed problem is sometimes called 'discrete ill-posed problem' (cf. [5]).

Let $f_{n,\delta} = r_n f_\delta$ and u_n^δ be the solution of (1.3) with the right hand side $f_{n,\delta}$. For some kind of discretization it is possible to define a discretization level $n = n(\delta)$ as a function of data error bound δ in such a way that $\|u^+ - p_n u_n^\delta\| \to 0$ as $\delta \to 0$. In this case the discretization is simultaneously a convergent regularization method for the ill-posed problem (1.1) (cf.[4], [11], [12]). Generally, it is recommended to apply some regularization method for solving discrete ill-posed problems (cf.[10]). An additional regularization of (1.3) allows to stabilize the numerical algorithm by decreasing the condition number of corresponding operator, and, as a consequence, allows for use the larger subspace in discretization ($n_{reg} >> n(\delta)$) for given data error δ.

In this paper we concentrate on the Tikhonov regularization method (cf. [14], [1], [4]), which consists in replacing the least squares problem by the following functional minimization problem:

$$\min_{u \in X_n} \{\|A_n u - f_{n,\delta}\|_{X_n}^2 + \alpha \|u\|_{X_n}^2\}. \tag{1.6}$$

The solution of (1.6) denoted by $u_{n,\alpha}^\delta$ is called a regularized solution and it satisfies the equation

$$(A_n^* A_n + \alpha I) u_{n,\alpha}^\delta = A_n^* f_{n,\delta}. \tag{1.7}$$

The parameter α is a regularization parameter and the crucial point of the method is its proper choice. In Section 3 it will be shown how condition number of the problem (1.7) depends on the parameter α, and which choice of α is optimal with respect to minimizing this condition number. The main conclusion is that by the Tikhonow method (1.7) the condition number (1.5) can be only decreased to its square root. In Section 4 a convergence of regularized solution corresponding to the optimal (in the above sense) choice of α to the exact generalized solution u^+ is analyzed. It must be emphasized, however, that parameter choice rules not depending on δ cannot provide a convergent regularization method in the usual sense, i.e. cannot provide convergence of $u_{n,\alpha}^\delta$ to u^+ as $\delta \to 0$ and $n \to \infty$ (cf [4], [8], [9]). In this paper dependence α on δ is replaced by a dependence n on δ. There are established conditions on the approximation of the space and operator and the relation between n and δ which guarantee convergence of regularized discrete solution $u^\delta_{n,\alpha}$ to u^+. As an example the method of truncated singular value decomposition with regularization is considered in Subsection 4.1.

2 The Tikhonov regularization method in a finite dimensional space

Let n be a fixed discretization level. Let $(\sigma_j, \varphi_j, \psi_j)$ be a singular system for the finite dimensional operator A_n where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$ are nonzero singular values of A_n, $\{\varphi_j\}_{j=1}^n$ and $\{\psi_j\}_{j=1}^n$ are complete orthonormal systems of eigenvectors of $A_n^* A_n$ and $A_n A_n^*$, respectively, corresponding to the eigenvalues σ_j^2, $j = 1, \cdots, n$. In Sections 2 and 3, to shorten notation we omit the parameter n in the singular system of A_n. Moreover, we will write the scalar product in X_n $(\cdot,\cdot)$ instead $(\cdot,\cdot)_n$ when no confusion can arise. The singular value decomposition of A_n has the form

$$A_n x = \sum_{j=1}^k \sigma_j (x, \varphi_j)\psi_j, \quad x \in X_n. \tag{2.1}$$

Similarly,

$$A_n^* A_n x = \sum_{j=1}^k \sigma_j^2 (x, \varphi_j)\varphi_j. \tag{2.2}$$

According to the above notation the generalized solution of (1.3) has the form

$$u_n = \sum_{j=1}^k \frac{(f_n, \psi_j)}{\sigma_j}\varphi_j, \tag{2.3}$$

where $\{(f_n, \psi_j)\}$ are the expansion coefficients of $f_n \in X_n$ in the basis $\{\psi_j\}$:

$$f_n = \sum_{j=1}^n (f_n, \psi_j)\psi_j. \tag{2.4}$$

Let u_n^δ be the generalized solution of (1.3) for a perturbed right hand side $f_{n,\delta} := r_n f_\delta$. Since for A_n the condition number $\kappa(A_n) = \frac{\sigma_1}{\sigma_k}$, we have

$$\frac{\|u_n - u_n^\delta\|}{\|u_n\|} \leq \frac{\sigma_1}{\sigma_k} \frac{\|f_n - f_{n,\delta}\|}{\|f_n\|}. \tag{2.5}$$

It is reasonable to require that the regularization parameter α in (1.7) is such that the regularized solution $u_{n,\alpha}$ is far less sensitive to perturbation on f_n than u_n.

3 Sensitivity of regularized solutions

Let n be still fixed. According to our notation σ_k is the smallest nonzero singular value of A_n.

LEMMA 3.1. *If $u_{n,\alpha}$ and $u^\delta_{n,\alpha}$ are solutions of (1.7) for the right hand side f_n and $f_{n,\delta}$, respectively, then*

$$\frac{\|u_{n,\alpha}-u^\delta_{n,\alpha}\|}{\|u_{n,\alpha}\|} \le \kappa_{n,\alpha}\frac{\|f_n-f_{n,\delta}\|}{\|f_n\|}, \tag{3.1}$$

where

$$\kappa_{n,\alpha} = \begin{cases} \frac{\sigma_1+\alpha/\sigma_1}{\sigma_k+\alpha/\sigma_k} & \textit{if } \alpha\in(0,\sigma_k^2), \\ \frac{\sigma_1^2+\alpha}{2\sigma_1\sqrt{\alpha}} & \textit{if } \alpha\in[\sigma_k^2,\sigma_1\sigma_k], \\ \frac{\sigma_k^2+\alpha}{2\sigma_k\sqrt{\alpha}} & \textit{if } \alpha\in[\sigma_1\sigma_k,\sigma_1^2], \\ \frac{\sigma_k+\alpha/\sigma_k}{\sigma_1+\alpha/\sigma_1} & \textit{if } \alpha\in(\sigma_1^2,\infty). \end{cases} \tag{3.2}$$

PROOF. Under the above notations the solutions of (1.7) for the right hand side f_n and $f_{n,\delta}$ are given by the formulas

$$u_{n,\alpha} = \sum_{j=1}^n \frac{\sigma_j}{\sigma_j^2+\alpha}(f_n,\psi_j)\varphi_j, \tag{3.3}$$

$$u^\delta_{n,\alpha} = \sum_{j=1}^n \frac{\sigma_j}{\sigma_j^2+\alpha}(f_{n,\delta},\psi_j)\varphi_j. \tag{3.4}$$

Thus $u_{n,\alpha}$ and $u^\delta_{n,\alpha}$ are solutions of the equation $B_{n,\alpha}v_n = g_n$, where $B_{n,\alpha}\in L(X_n)$,

$$B_{n,\alpha}x = \sum_{j=1}^n \frac{\sigma_j^2+\alpha}{\sigma_j}(x,\varphi_j)\psi_j \quad x\in X_n, \tag{3.5}$$

and $g_n = f_n$ and $g_n = f_{n,\delta}$, respectively. Let us define

$$q_\alpha(\sigma) := \frac{\sigma^2+\alpha}{\sigma} \quad \text{for } \sigma\in(\sigma_k,\sigma_1).$$

The condition number for the operator $B_{n,\alpha}$ is given by

$$\kappa(B_{n,\alpha}) = \frac{\max_{1\le j\le n} q_\alpha(\sigma_j)}{\min_{1\le j\le n} q_\alpha(\sigma_j)}. \tag{3.6}$$

Thus, the relative error inequality (3.1) is satisfied with $\kappa_{n,\alpha} \ge \kappa(B_{n,\alpha})$.

The function $q_\alpha(\sigma)$ attains its minimum at $\sigma = \sqrt{\alpha}$, thus

$$\min_{1\le j\le n} q_\alpha(\sigma_j) \ge \min_{\sigma\in[\sigma_k,\sigma_1]} q_\alpha(\sigma) = \begin{cases} q_\alpha(\sigma_k) & \text{if } \alpha \in (0,\sigma_k^2), \\ 2\sqrt{\alpha} & \text{if } \alpha \in [\sigma_k^2,\sigma_1^2], \\ q_\alpha(\sigma_1) & \text{if } \alpha \in (\sigma_1^2,\infty). \end{cases} \tag{3.7}$$

On the other hand the maximum of $q_\alpha(\sigma)$ on $[\sigma_k,\sigma_1]$ is attained at σ_k or σ_1:

$$\max_{1\le j\le n} q_\alpha(\sigma_j) = \begin{cases} q_\alpha(\sigma_1) & \text{if } \alpha \le \sigma_k\sigma_1, \\ q_\alpha(\sigma_k) & \text{if } \alpha \ge \sigma_k\sigma_1. \end{cases} \tag{3.8}$$

The lemma follows now from (3.6), (3.7) (3.8). □

Let us consider $\kappa_{n,\alpha}$ as a function of α: $\kappa_n(\alpha)$. It is easy to see that $\kappa_n(\alpha)$ is a decreasing function on the interval $(0,\sigma_k\sigma_1)$ and increasing one on the interval $(\sigma_k\sigma_1,\infty)$. So, its minimum is attained at $\alpha = \sigma_k\sigma_1$ and is equal to

$$\kappa_n^{opt} = \frac{1}{2}\left(\sqrt{\frac{\sigma_1}{\sigma_k}} + \sqrt{\frac{\sigma_k}{\sigma_1}}\right). \tag{3.9}$$

Since for $\alpha = \sigma_1^2$ or $\alpha = \sigma_k^2$

$$\kappa_{n,\alpha} = \frac{1}{2}\left(\frac{\sigma_1}{\sigma_k} + \frac{\sigma_k}{\sigma_1}\right), \tag{3.10}$$

we get the following conclusion:

Theorem 3.2. *The optimal choice of regularization parameter α with respect to sensivity of discrete problem in X_n is*

$$\alpha_n = \sigma_{min}(A_n)\sigma_{max}(A_n) \tag{3.11}$$

and gives

$$\kappa_n^{opt} = \frac{1}{2}\left(\kappa(A_n)^{\frac{1}{2}} + \kappa(A_n)^{-\frac{1}{2}}\right). \tag{3.12}$$

Moreover, for any $\alpha \in (\sigma_k^2,\sigma_1^2)$

$$\kappa_{n,\alpha} \in \left[\kappa_n^{opt}, \frac{1}{2}\left(\kappa(A_n) + \kappa(A_n)^{-1}\right)\right], \tag{3.13}$$

and $\kappa_{n,\alpha} \to \kappa(A_n)$ as $\alpha \to 0$ or $\alpha \to \infty$.

4 Convergence

Now, instead the discrete equation (1.3) for fixed n, we consider the family $\{A_n\}$ of finite dimensional operators and corresponding equations. For any n, let u^{δ}_{n,α_n} denote the regularized solution corresponding to $\alpha_n = \sigma_{min}(A_n)\sigma_{max}(A_n)$. We are going to consider a convergence of u^{δ}_{n,α_n} to u^+ when $\delta \to 0$ and n depends on δ.

For our purposes it is now convenient to denote the singular system for A_n by $(\sigma_{n,j}, \varphi_{n,j}, \psi_{n,j})$ indicating dependence on n. Moreover, let $(\mu_j, \widetilde{\varphi}_j, \widetilde{\psi}_j)$ be the singular system for A.

Let us define in the space X two auxiliary sequences $\{v^{\delta}_{\alpha_n}\}$, and $\{v^{\delta}_{n,\alpha_n}\}$ as follows:

$$v^{\delta}_{\alpha_n} = \sum_{j=1}^{\infty} \frac{\mu_j}{\mu_j^2 + \alpha_n} (f_\delta, \widetilde{\psi}_j)\widetilde{\varphi}_j, \tag{4.1}$$

$$v^{\delta}_{n,\alpha_n} = \sum_{j=1}^{n} \frac{\mu_j}{\mu_j^2 + \alpha_n} (f_\delta, \widetilde{\psi}_j)\widetilde{\varphi}_j. \tag{4.2}$$

It is easy to see that $v^{\delta}_{\alpha_n}$ is a regularized solution of (1.1) with perturbed right hand side f_δ obtained via the Tikhonov method with the regularization parameter equals α_n. Due to the known convergence results for the Tikhonov method (cf. [4], [14]) we have

REMARK 4.1. *If $n = n(\delta)$ is such that*

$$\lim_{\delta\to 0} n(\delta) = \infty \text{ and } \lim_{\delta\to 0} \frac{\delta}{\sqrt{\sigma_{min}(A_{n(\delta)})}} = 0 \tag{4.3}$$

then

$$\lim_{\delta\to 0} v^{\delta}_{\alpha_n} = u^+.$$

4.1 Method of truncated SVD with regularization

Let us consider the case when

$$A_n := Q_n A|_{X_n} \text{ and } f_n := Q_n f, \tag{4.4}$$

where Q_n is the orthogonal projector of X onto $Y_n := span\{\widetilde{\psi}_1, \cdots, \widetilde{\psi}_n\}$, and $X_n := span\{\widetilde{\varphi}_1, \cdots, \widetilde{\varphi}_n\} = A^* Y_n$. Then $\sigma_{n,j} = \mu_j$, $\varphi_{n,j} = \widetilde{\varphi}_j$, $\psi_{n,j} = \widetilde{\psi}_j$.

The least squares solution of $A_n u_n = f_{n,\delta}$ with $f_{n,\delta} = Q_n f_\delta$ is denoted by u_n^δ. The resulting method is the truncated singular value expansion. This method itself has regularization properties. Namely, if $u^+ \in R(A^*)$ then, according to Propositions 3.28, 3.29 in [4],

$$\|u_n^\delta - u^+\| = O\left(\mu_{n+1} + \frac{\delta}{\mu_n}\right). \tag{4.5}$$

Now, let us apply to the above truncated SVD method, the Tikhonow regularization with α_n chosen as (3.11), i.e. $\alpha_n = \mu_1\mu_n$. Then $u_{n,\alpha_n}^\delta = v_{n\alpha_n}^\delta$ given by (4.2). We are going to establish convergence of $u_{n,\alpha_n}^\delta\|$ to u^+ as $\delta \to 0$ and n depends on proper way on δ.

For the case $\delta = 0$ Remark 4.1 gives

$$\|u^+ - v_{\alpha_n}\| \text{ as } n \to \infty. \tag{4.6}$$

Moreover, according to (4.1) and (4.2) for the exact f, we have

$$\|v_{\alpha_n} - u_{n,\alpha_n}\|^2 = \sum_{j=n+1}^{\infty} \left(\frac{\mu_j^2}{\mu_j^2 + \alpha_n}\right)^2 \frac{(f, \widetilde{\psi}_j)^2}{\mu_j^2}.$$

Since

$$\max_{n+1 \le j < \infty} \frac{\mu_j^2}{\mu_j^2 + \alpha_n} \le \frac{\mu_{n+1}^2}{\alpha_n} \le \frac{\mu_{n+1}}{\mu_1},$$

we get

$$\|v_{\alpha_n} - u_{n,\alpha_n}\| \le \frac{\mu_{n+1}}{\mu_1}\|u^+\|. \tag{4.7}$$

Finally, taking into account Lemma 3.1 and Theorem 3.2

$$\|u_{n,\alpha_n} - u_{n,\alpha_n}^\delta\| \le \delta\kappa_n^{opt}\frac{\|u_{n,\alpha}\|}{\|f_n\|} \le C\frac{\delta}{\sqrt{\mu}_n}. \tag{4.8}$$

From (4.6), (4.7), (4.23) if follows desired result:

Lemma 4.1. *If $n = n(\delta)$ is such that*

$$\lim_{\delta \to 0} n(\delta) = \infty \text{ and } \delta = o(\sqrt{\mu_n(\delta)}) \tag{4.9}$$

then

$$\lim_{\delta \to 0} \|u_{n,\alpha}^\delta - u^+\| = 0. \tag{4.10}$$

Remark 4.2. *If* $\{\mu_j\}_{j=1}^{\infty}$ *is the non increasing sequence of singular values of A repeated according to their multiplicities, then the following function*

$$\tilde{n}(\delta) := max\{n : \quad \mu_n \geq \delta\} \tag{4.11}$$

is an example of $n(\delta)$ *satisfying the assumption (4.9).*

Indeed, if $\delta \to 0$ then $\tilde{n}(\delta) \to \infty$. Moreover,

$$\frac{\delta}{\sqrt{\mu_{\tilde{n}}}} \leq \sqrt{\delta} \to 0 \text{ as } \delta \to 0.$$

4.2 Convergence conditions

Let us take into account the discrete equation (1.3) with the following additional assumptions on space approximation:

$$r_n p_n v_n = v_n, \text{ and } (p_n v_n, z)_X = (v_n, r_n z)_{X_n} \tag{4.12}$$

$\forall v_n \in X_n$ and $\forall z \in X$. From this assumption it follows that $p_n r_n$ is orthogonal projection and $p_n^* = r_n$, $r_n^* = p_n$.

Now, we want to establish conditions on A_n under which the relations similar to (4.9) guarantee convergence of regularized discrete solutions.

In order to prove convergence of u_{n,α_n}^{δ} to u^+, we will consider norm behavior of each element appearing in the right hand side of the equality below:

$$u^+ - p_h u_{n,\alpha_n}^{\delta} = (u^+ - v_{\alpha_n}) + (v_{\alpha_n} - p_h u_{n,\alpha_n}) + \\ + p_h(u_{n,\alpha_n} - u_{n,\alpha_n}^{\delta}). \tag{4.13}$$

Lemma 4.2. *Let the assumption (4.12) be satisfied. If*

$$\|A^*A - p_n A_n^* A_n r_n\| = o(\sigma_{min}(A_n)), \tag{4.14}$$

then

$$\|v_{\alpha_n} - p_n u_{n,\alpha_n}\| \to 0 \text{ as } n \to \infty. \tag{4.15}$$

Proof. Let us denote $B_n := p_n A_n r_n$. From the assumption (4.12) it follows that the prolongation of regularized solution $u_{n,\alpha}$ satisfying the equation (1.7) for the exact data and an arbitrary positive α has the form

$$p_n u_{n,\alpha_n} = (B_n^* B_n + \alpha I)^{-1} B_n^* f_n.$$

Since A^*A commutes with $(A^*A+\alpha I)^{-1}$ as well as $B_n^*B_n$ commutes with $(B_n^*B_n+\alpha I)^{-1}$, we can verify that for $z \in X$

$$(4.16)\quad (B_n^*B_n+\alpha I)^{-1}B_n^*B_n(A^*A - B_n^*B_n)(A^*A+\alpha I)^{-1}z =$$

$$(B_n^*B_n+\alpha I)^{-1}B_n^*B_n z - B_n^*B_n(A^*A+\alpha I)^{-1}z.$$

If $z := p_n u_n$ where u_n is the generalized solution of (1.3) for the exact data, then $B_n z = p_n r_n f$ and $p_n u_{n,\alpha} = (B_n^*B_n+\alpha I)^{-1}B_n^* p_n r_n f$ and from (4.16)

$$(4.17)\qquad p_n u_{n,\alpha} = B_n^*B_n(A^*A+\alpha I)^{-1}z +$$

$$+ (B_n^*B_n+\alpha I)^{-1}B_n^*B_n(A^*A - B_n^*B_n)(A^*A+\alpha I)^{-1}z.$$

On the other hand

$$(4.18)\qquad v_\alpha - B_n^*B_n(A^*A+\alpha I)^{-1}z =$$

$$= (A^*A - B_n^*B_n)(A^*A+\alpha I)^{-1}z + (A^*A+\alpha I)^{-1}A^*A(u^+ - p_n u_n).$$

It is known (cf. [4],[7]) that

$$\|(B_n^*B_n+\alpha I)^{-1}B_n^*B_n\| \le 1,$$

$$\|(A^*A+\alpha I)^{-1}A^*A\| \le 1$$

$$\|(A^*A+\alpha I)^{-1}\| \le \frac{1}{\alpha}.$$

Thus

$$(4.19)\qquad \|p_n u_{n,\alpha} - v_\alpha\| \le$$

$$\le \frac{1}{\alpha}\|A^*A - B_n^*B_n\|\|p_n u_n\| + \|(u^+ - p_n u_n)\|.$$

Let $\alpha = \alpha_n$. Taking into account the assumption (4.14) and convergence of $p_n u_n$ to u^+ as $n \to \infty$ for the exact data and, in consequence, uniformly boundedness of $p_n u_n$ we get the desired conclusion. □

THEOREM 4.3. *Let the assumptions of Lemma 4.2 be satisfied. If*

$$\alpha_n = \sigma_{min}(A_n)\sigma_{max}(A_n)$$

and $n = n(\delta)$ *is such that*

$$\lim_{\delta \to 0} n(\delta) = \infty \text{ and } \delta = o(\sqrt{\sigma_{min}(A_n)}) \tag{4.20}$$

and u^{δ}_{n,α_n} *is regularized discrete solution satisfying (1.7) then*

$$\|u^+ - p_n u^{\delta}_{n,\alpha_n}\| \to 0 \text{ as } \delta \to 0. \tag{4.21}$$

PROOF. From Remark 4.1 for $\delta = 0$ we have

$$\|u^+ - v_{\alpha_n}\| \to 0 \text{ as } n \to \infty. \tag{4.22}$$

Moreover, due to Lemma 3.1

$$\|u_{n,\alpha_n} - u^{\delta}_{n,\alpha_n}\| \le \delta \kappa_n^{opt} \frac{\|u_{n,\alpha}\|}{\|f_n\|}. \tag{4.23}$$

From (3.9)

$$\kappa_{n,}^{opt} < \sqrt{\frac{\sigma_{max}(A_n)}{\sigma_{min}(A_n)}}.$$

Finally observe that

$$\|u_{n,\alpha_n}\| < \|u_n\|. \tag{4.24}$$

Indeed, by (3.3) we get

$$\|u_{n,\alpha}\|^2 = \sum_{j=1}^{n} \left(\frac{\sigma_{n,j}(f_n, \psi_{n,j})}{\sigma_{n,j}^2 + \alpha} \right)^2 =$$

$$= \sum_{j=1}^{n} \left(\frac{(f_n, \psi_{n,j})}{\sigma_{n,j}} \right) \left(\frac{\sigma_{n,j}(f_n, \psi_{n,j})}{\sigma_{n,j}^2 + \alpha} \right) \left(\frac{\sigma_{n,j}^2}{\sigma_{n,j}^2 + \alpha} \right) \le \|u_n\| \|u_{n,\alpha}\|,$$

since $\left(\frac{\sigma_{n,j}^2}{\sigma_{n,j}^2 + \alpha} \right) < 1$ for $\alpha > 0$, which proves (4.24). By our approximation assumptions, $\|u_n\|$ and $\frac{1}{\|f_n\|}$ are uniformly bounded.

Now, the theorem follows from (4.13) and Lemma 4.2. □

REMARK 4.3. *The function*

$$\tilde{n}(\delta) := max\{n : \quad \sigma_{min}(A_n) \ge \delta\} \tag{4.25}$$

is an example of $n(\delta)$ *satisfying the assumption (4.20) (cf. Remark 4.2).*

5 Conclusion

A number of a-priori and a-posteriori parameter choice rules is known (cf. [4]) which all depend on the given data error bound δ. In real world examples such noise bound information is not always available or reliable. Often it is necessary to consider alternative heuristic parameter choice rules basing on the actual performance of the regularization method under consideration and not depending on δ explicitly. It must be emphasized, however, that parameter choice rules not depending on δ cannot provide a convergent regularization method in the usual sense, i.e. cannot provide convergence of $u^{\delta}_{n,\alpha}$ to u^+ as $\delta \to 0$ and $n \to \infty$ independently (cf [4], [8], [9], [15]).

In this paper we have shown that the method of Tikhonow regularization applied to some finite dimensional approximation of (1.1) with regularization parameter depending on A_n but not depending on δ can give a sequence of regularized discrete solutions $\{u^{\delta}_{n,\alpha_n}\}$ convergent to u^+ under the assumption that n depends in a proper way on δ and $\delta \to 0$.

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