Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

C. R. Acad. Sci. Paris, Ser. I 338 (2004) 113–116

COMPTES RENDUS MATHEMATIQUE

Algebra/Topology

# The Chern–Galois character

Tomasz Brzeziński [a], Piotr M. Hajac [b,c]

[a] Department of Mathematics, University of Wales Swansea, Singleton Park, Swansea SA2 8PP, UK
[b] Instytut Matematyczny, Polska Akademia Nauk, ul. Śniadeckich 8, Warszawa, 00-956 Poland
[c] Katedra Metod Matematycznych Fizyki, Uniwersytet Warszawski ul. Hoża 74, Warszawa, 00-682 Poland

Received 31 July 2003; accepted after revision 12 November 2003

Presented by Alain Connes

**Abstract**

Following the idea of Galois-type extensions and entwining structures, we define the notion of a principal extension of noncommutative algebras. We show that modules associated to such extensions via finite-dimensional corepresentations are finitely generated projective, and determine an explicit formula for the Chern character applied to the modules so obtained. ***To cite this article: T. Brzeziński, P.M. Hajac, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***


**Résumé**

**Le caractère de Chern–Galois.** Nous nous inspirons des extensions de type Galois et des structures enlacées pour définir la notion d'extension principale d'algèbres non commutatives. Nous montrons que les modules associés à de telles extensions au travers de coreprésentations de dimension finie sont projectifs et de type fini, et nous déterminons une formule explicite pour le caractère de Chern appliqué aux modules ainsi obtenus. ***Pour citer cet article : T. Brzeziński, P.M. Hajac, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***


## 1. Introduction

The aim of this paper is twofold. First we need to determine a class of Galois-type extensions that are sufficiently general to accommodate interesting examples and sufficiently specific to derive a number of desired properties. This leads to the concept of principal extensions. They play the role of algebraic analogues of principal bundles. To any such extension one can associate modules much as vector bundles are associated to principal bundles. For finite-dimensional corepresentations these modules are always finitely generated projective (see Theorem 3.1), and thus fit the formalism of the Chern–Connes pairing between $K$-theory and cyclic cohomology [6]. On the other hand, just as the commutative faithfully flat Hopf–Galois extensions with bijective antipodes coincide with affine group scheme torsors of algebraic geometry, the principal extensions are precisely the (noncommutative) faithfully flat Hopf–Galois extensions with bijective antipodes whenever the defining coaction is an algebra homomorphism

*E-mail address:* T.Brzezinski@swansea.ac.uk (T. Brzeziński).
*URL:* http://www.fuw.edu.pl/~pmh (P.M. Hajac).

1631-073X/$ – see front matter 
doi:10.1016/j.crma.2003.11.009

(cf. Theorem 2.5 and [12]). A very interesting and non-trivial example of a principal extension encoding a noncommutative version of the instanton fibration $\mathrm{SU}(2) \to S^7 \to S^4$ was recently constructed in [1].

The second and main outcome of this work is the construction of and an explicit formula for the Chern–Galois character for a given principal extension. This is a homomorphism of Abelian groups that assigns the homology class of an even cyclic cycle to the isomorphism class of a finite-dimensional corepresentation. This construction is in analogy with the Chern–Weil formalism for principal bundles and bridges the coalgebra-Galois extension [13,4,3] and $K$-theoretic formalisms (cf. [8] for the Hopf–Galois version). In particular, with the help of finitely summable Fredholm modules, it allows one to apply the analytic tool of the noncommutative index formula [6] to compute the $K_0$-invariants of line bundles over generic Podleś spheres [9], which are among prime examples going beyond the Hopf–Galois framework.

Except for the last formula, we work over a general field $k$. We use the usual notations $\Delta c = c_{(1)} \otimes c_{(2)} \in C \otimes C$, $((\mathrm{id} \otimes \Delta) \circ \Delta)(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)} \in C \otimes C \otimes C$, etc., $\Delta_V(v) = v_{(0)} \otimes v_{(1)} \in V \otimes C$ (summation understood) for the coproduct of a coalgebra $C$, its iterations and a right $C$-coaction on $V$, respectively. We denote the counit of $C$ by $\varepsilon$. For an algebra $A$, ${}_A\mathrm{Hom}(V, W)$ stands for the space of left $A$-linear maps. Similarly, for a coalgebra $C$, we write $\mathrm{Hom}^C(V, W)$ for the space of right $C$-colinear maps.

## 2. Principal extensions and strong connections

The concept of a faithfully flat Hopf–Galois extension with a bijective antipode is a cornerstone of Hopf–Galois theory. The following notion of a *principal extension* generalizes this key concept in such a way that it encompasses interesting examples escaping Hopf–Galois theory, yet still enjoys a number of crucial properties of the aforementioned class of Hopf–Galois extensions. It is an elaboration of the Galois-type extension [3, Definition 2.3] (see the condition (1) below), which evolved from [13, p. 182], [4] and other papers.

**Definition 2.1.** Let $C$ be a coalgebra and $P$ an algebra and a right $C$-comodule via $\Delta_P : P \to P \otimes C$. Put $B = P^{coC} := \{b \in P \mid \Delta_P(bp) = b\Delta_P(p),\ \forall p \in P\}$. We say that the inclusion $B \subseteq P$ is a *$C$-extension*. A $C$-extension $B \subseteq P$ is called *principal* if and only if

(1) $can_R : P \otimes_B P \to P \otimes C,\ p \otimes_B p' \mapsto p\Delta_P(p')$ is bijective (Galois or freeness condition);
(2) $\psi : C \otimes P \to P \otimes C,\ c \otimes p \mapsto can(can^{-1}(1 \otimes c)p)$ is bijective (invertibility of the canonical entwining);
(3) there is a group-like element $e \in C$ such that $\Delta_P(p) = \psi(e \otimes p)$, for all $p \in P$ (co-augmentation);
(4) $P$ is $C$-equivariantly projective as a left $B$-module (existence of a strong connection).

The meaning of the last condition in Definition 2.1 is as follows. Let $X$ be a left $B$-module and a right $C$-comodule such that the coaction is $B$-linear. We say that $X$ is a *$C$-equivariantly projective $B$-module* if and only if for every $B$-linear $C$-colinear epimorphism $\pi : M \to N$ that is split as a $C$-comodule map, and for any $B$-linear $C$-colinear homomorphism $f : X \to N$, there exists a $B$-linear $C$-colinear map $g : X \to M$ such that $\pi \circ g = f$. For the trivial $C$ we recover the usual concept of projectivity. Much as for the trivial $C$, one can show that equivariant projectivity is equivalent to the existence of a $B$-linear $C$-colinear splitting of the multiplication map $m : B \otimes X \to X$. If we take $A = \mathrm{Hom}(C, k)^{op}$ to be the opposite of the convolution algebra of $C$, then such a splitting is the same as a $(B, A)$-bimodule splitting of $m$. Now one can reverse the argument and prove that the existence of such a $(B, A)$-bimodule splitting is equivalent to $A$-equivariant projectivity defined analogously as $C$-equivariant projectivity. If $A$ is a commutative ring and $B$ is an algebra over $A$, then we obtain the familiar concept of relative projectivity [5, p. 197]). On the other hand, as explained in [8, p. 314], a $(B, A)$-bimodule splitting of $m$ can be interpreted as a Cuntz–Quillen type connection [7]. The unitalized version of such connections corresponds to strong connections. More precisely, if $B \subseteq P$ is a principal $C$-extension, a *strong connection* is a unital left $B$-linear right $C$-colinear splitting of the multiplication map $B \otimes P \to P$ [8, Remark 2.11]. The following lemma allows us to conclude that principal extensions always admit strong connections.

**Lemma 2.2.** *Let $B \subseteq P$ be a $C$-extension satisfying conditions* (1) *and* (3) *in Definition* 2.1. *Then $P$ is $C$-equivariantly projective as a left $B$-module if and only if there exists a strong connection.*

The right-to-left part of the assertion is immediate from the discussion preceding the lemma. It is the proof of the existence of a strong connection (unital splitting) that requires some work. Next, note that the conditions (2)–(3) of Definition 2.1 allow us to give a symmetric formulation of a strong connection. To begin with, one can define a left coaction ${}_P\Delta : P \to C \otimes P$, ${}_P\Delta(p) = \psi^{-1}(p \otimes e)$, and prove that $can_L : P \otimes_B P \to C \otimes P$, $p \otimes_B p' \mapsto {}_P\Delta(p)p'$, is bijective. One can also show that $can_L^{-1} \circ (\mathrm{id} \otimes 1) = can_R^{-1} \circ (1 \otimes \mathrm{id})$, so that the concept of the translation map $\tau := can_R^{-1} \circ (1 \otimes \mathrm{id})$, $\tau(c) =: c^{[1]} \otimes_B c^{[2]}$ (summation suppressed), is left-right symmetric. This leads to:

**Lemma 2.3.** *Let $B \subseteq P$ be a principal $C$-extension, and let $\pi_B : P \otimes P \to P \otimes_B P$ be the canonical surjection. Then the formulae $s \mapsto (\ell : c \mapsto c^{[1]} s(c^{[2]}))$, $\ell \mapsto (s : p \mapsto p_{(0)} \ell(p_{(1)}))$ define mutually inverse maps between the space of strong connections and linear maps $\ell : C \to P \otimes P$ such that $\pi_B \circ \ell = \tau$, $(\mathrm{id} \otimes \Delta_P) \circ \ell = (\ell \otimes \mathrm{id}) \circ \Delta$, $({}_P\Delta \otimes \mathrm{id}) \circ \ell = (\mathrm{id} \otimes \ell) \circ \Delta$, $\ell(e) = 1 \otimes 1$.*

To avoid multiplying terminology, such unital bicolinear liftings of the translation map are also called strong connections. Among other consequences of the principality of an extension is its coflatness. Recall first that, for any right $C$-comodule $V$ with a coaction $\Delta_V$ and a left $C$-comodule $W$ with a coaction ${}_W\Delta$, the *cotensor product* is defined as $V \Box_C W := \mathrm{Ker}(\mathrm{id} \otimes_W \Delta - \Delta_V \otimes \mathrm{id}) \subseteq V \otimes W$. A right (resp. left) $C$-comodule $M$ is said to be *coflat* if the functor $M \Box_C -$ (resp. $- \Box_C M$) is exact. Next, recall that there is a general concept of an entwining structure $(A, C, \psi)$, where $A$ is an algebra, $C$ a coalgebra, and $\psi : C \otimes A \to A \otimes C$ is a linear map satisfying certain axioms [4, Definition 2.1]. With these definitions, we obtain:

**Lemma 2.4.** *Let $(A, C, \psi)$ be an entwining structure such that $\psi$ is bijective. Assume also that there exists a group-like $e \in C$ such that $A$ is a right $C$-comodule via $\psi \circ (e \otimes \mathrm{id})$ and a left $C$-comodule via $\psi^{-1} \circ (\mathrm{id} \otimes e)$. Then $A$ is coflat as a right* (*resp. left*) *$C$-comodule if and only if there exists $j_R \in \mathrm{Hom}^C(C, A)$* (*resp. $j_L \in {}^C\mathrm{Hom}(C, A)$*) *such that $j_R(e) = 1$* (*resp. $j_L(e) = 1$*). (*Here $C$ is a $C$-comodule via the coproduct.*)

The axioms (1)–(3) of a principal extension guarantee that $(P, C, \psi)$ is an entwining structure satisfying the assumptions of the above Lemma [3, Theorem 2.7]. Moreover, with the help of Lemma 2.3, it can be shown that maps $j_L$ and $j_R$ as in Lemma 2.4 can be constructed for any principal $C$-extension. Combining together the results described in this section, one can prove the following:

**Theorem 2.5.** *Let $B \subseteq P$ be a principal $C$-extension. Then*: (1) *There exists a strong connection.* (2) *$P$ is a projective left and right $B$-module.* (3) *$B$ is a direct summand of $P$ as a left and right $B$-module.* (4) *$P$ is a faithfully flat left and right $B$-module.* (5) *$P$ is a coflat left and right $C$-comodule.*

## 3. Associated projective modules and the Chern–Galois character

If $B \subseteq P$ is a principal $C$-extension and $\varphi : V_\varphi \to V_\varphi \otimes C$ is a finite-dimensional corepresentation, then, using the technology from the previous section, one can produce a short proof that the left $B$-module $\mathrm{Hom}^C(V_\varphi, P)$ of all colinear maps from $V_\varphi$ to $P$ is finitely generated projective. We call such modules associated modules, as they play the role of sections of vector bundles associated to principal bundles (cf. [2, Theorem 5.4]). The main result of this paper is an explicit formula for an idempotent representing an associated module. By virtue of Theorem 2.5, we already know that there exists a strong connection $\ell$ and a unital left $B$-linear map $\sigma_L : P \to B$. Thus we can state (cf. [11, Corollary 2.4]):

**Theorem 3.1.** *Let $\ell$ be a strong connection on a principal $C$-extension $B \subseteq P$ and $\varphi : V_\varphi \to V_\varphi \otimes C$ be a finite-dimensional corepresentation. Let $\{p_\mu\}_\mu$ be a basis of $P$, $\{p^\mu\}_\mu$ its dual, $r_\mu := (p^\mu \otimes \mathrm{id}) \circ \ell$, and $\{e_i\}_i$ be a basis of $V_\varphi$, $\varphi(e_j) =: \sum_{i=1}^{\dim V_\varphi} e_i \otimes e_{ij}$. Take any $\sigma_L \in {}_B\mathrm{Hom}(P, B)$ such that $\sigma_L(1) = 1$, and set $E_{(\mu,i)(\nu,j)} := \sigma_L(r_\mu(e_{ij})p_\nu)$, $E := (E_{(\mu,i)(\nu,j)}) \in M_N(B)$. Then $E$ is an idempotent matrix and $B^N E$ is isomorphic to $\mathrm{Hom}^C(V_\varphi, P)$ as a left $B$-module.*

It is an immediate corollary of this theorem that a left $B$-module $\mathrm{Hom}^C(V_\varphi, P)$ is always finitely generated projective. On the other hand, take all the isomorphism classes of finite-dimensional corepresentations of $C$ and view them as a semi-group via the direct sum. Denote by $R_f(C)$ the Grothendieck group of this semi-group. It is now straightforward to verify that the assignment $[\varphi] \mapsto [\mathrm{Hom}^C(V_\varphi, P)]$ defines a homomorphism of Abelian groups $R_f(C) \to K_0(B)$. Combining this homomorphism with the Chern character $ch_{2n} : K_0(B) \to HC_{2n}(B)$, $n \in \mathbb{N}$, (see [10, p. 264]) yields a homomorphism $chg_{2n} : R_f(C) \to HC_{2n}(B)$. We call the collection of homomorphisms $chg_{2n}$, $n \in \mathbb{N}$, *the Chern–Galois character*. The main point of this work is that we can use Theorem 3.1 to determine an explicit formula for the Chern–Galois character.

**Corollary 3.2.** *With the assumptions and notation as in Theorem* 3.1, *define the character of $\varphi$ as $c_\varphi := \sum_{i=1}^{\dim V_\varphi} e_{ii}$ and, for any $c \in C$, put $\ell(c) =: c^{\langle 1\rangle} \otimes c^{\langle 2\rangle}$ (summation understood). If $\mathbb{Q} \subseteq k$, then the Chern–Galois character has the following explicit form*:

$$\forall\, n \in \mathbb{N}:\ chg_{2n}\big([\varphi]\big) = (-1)^n \big[ {c_{\varphi\,(2n+1)}}^{\langle 2\rangle} {c_{\varphi\,(1)}}^{\langle 1\rangle} \otimes {c_{\varphi\,(1)}}^{\langle 2\rangle} {c_{\varphi\,(2)}}^{\langle 1\rangle} \otimes \cdots \otimes {c_{\varphi\,(2n)}}^{\langle 2\rangle} {c_{\varphi\,(2n+1)}}^{\langle 1\rangle} \big].$$

Note that, since any strong connection yields an idempotent representing the same module and the Chern character does not dependent on the choice of a representing idempotent, the formula for the Chern–Galois character is manifestly independent of the choice of a strong connection appearing on the right-hand side.


## Acknowledgements

T. Brzeziński thanks the EPSRC for an Advanced Research Fellowship. P.M. Hajac thanks the European Commission for the Marie Curie fellowship HPMF-CT-2000-00523 and the KBN for the grant 2 P03A 013 24. Both authors are very grateful to R. Taillefer for the French translation and to R. Matthes for his helpful comments on the manuscript. A preliminary version of the full account of this work can be found at http://www.fuw.edu.pl/~pmh.


## References


[1] F. Bonechi, L. Dąbrowski, N. Ciccoli, M. Tarlini, Bijectivity of the canonical map for the noncommutative instanton bundle, J. Geom. Phys., in press.
[2] T. Brzeziński, On modules associated to coalgebra Galois extensions, J. Algebra 215 (1999) 290–317.
[3] T. Brzeziński, P.M. Hajac, Coalgebra extensions and algebra coextensions of Galois type, Comm. Algebra 27 (1999) 1347–1367.
[4] T. Brzeziński, S. Majid, Coalgebra bundles, Comm. Math. Phys. 191 (1998) 467–492.
[5] H. Cartan, S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, NJ, 1956.
[6] A. Connes, Non-commutative differential geometry, Inst. Hautes Études Sci. Publ. Math. 62 (1985) 257–360.
[7] J. Cuntz, D. Quillen, Algebra extensions and nonsingularity, J. Amer. Math. Soc. 8 (1995) 251–289.
[8] L. Dąbrowski, H. Grosse, P.M. Hajac, Strong connections and Chern–Connes pairing in the Hopf–Galois theory, Comm. Math. Phys. 220 (2001) 301–331.
[9] P.M. Hajac, R. Matthes, W. Szymański, Chern numbers for two families of noncommutative Hopf fibrations, C. R. Acad. Sci. Paris, Ser. I 336 (2003) 925–930.
[10] J.-L. Loday, Cyclic Homology, Second edition, Springer-Verlag, Berlin, 1998.
[11] E.F. Müller, H.-J. Schneider, Quantum homogeneous spaces with faithfully flat module structures, Israel J. Math. 111 (1999) 157–190.
[12] P. Schauenburg, H.-J. Schneider, Galois-type extensions and Hopf algebras, in preparation.
[13] H.-J. Schneider, Principal homogeneous spaces for arbitrary Hopf algebras, Israel J. Math. 72 (1990) 167–195.