HJM condition for models with Lévy noise

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Abstract

HJM model driven by Lévy process is considered. Necessary and sufficient conditions for the market prices of bonds being local martingales are given. HJM type conditions are derived as well.

0.1 Introduction

Let $P(t,\theta)$, $0 \le t \le \theta$ be the market price at moment t of a bond paying 1 at the maturity time θ. The forward rate curve is a function $f(t,\theta)$ defined for $\theta \ge t$ and such that

$$P(t,\theta) = e^{-\int_t^\theta f(t,s)ds}. \tag{1}$$

Heath, Jarrow and Morton [6] proposed to model the forward curves as Itô processes

$$df(t,\theta) = \alpha(t,\theta)dt + < \sigma(t,\theta),\ dW(t) >, \qquad 0 \le t \le \theta, \tag{2}$$

with W d-dimensional standard Wiener process. For each $\theta > 0$ the processes $\alpha(t,\theta)$, $\sigma(t,\theta)$, $t \le \theta$, are assumed to be adapted processes with respect to a given filtration $(\mathcal{F}_t)$. We consider a generalization of this model by taking, instead of the process W a Lévy proces Z with values in a separable Hilbert space U i.e.

$$df(t,\theta) = \alpha(t,\theta)dt + < \sigma(t,\theta),\ dZ(t) > . \tag{3}$$

In books by Bertoin [1] and Sato [9] one can find excellent discussions of Lévy processes in $\mathbb{R}^d$. Models of similar types have already been studied (see [2], [5]).

It is of special interest to find conditions on the forward rate process under which the discounted bond price processes $\hat{P}(\cdot,\theta)$, $\theta \in [0,T]$ are local martingales (see [4] or [8]).

The aim of our paper is to give *necessary and sufficient* conditions, in terms of characteristics of the Lévy proces Z, implying that $\hat{P}$ is a local martingale and to derive the HJM type condition. It will turn out that model implies existence of exponential moments of the noise processes and this is one of our main contributions.

In a similar way, see [7], we can obtain conditions on existence of exponential moments for models

$$df(t,\theta) = \alpha(t,\theta)dt + \sigma(t,\theta)\, dW(t) + \int_{|y|\le 1} \sigma_0(t,\theta,y)(\mu(dt,dy) - dt\nu(dy)) + \int_{|y|>1} \sigma_1(t,\theta,y)\mu(dt,dy)$$

with μ a Poissonian random measure with intensity ν.

In [2] sufficient, but not necessary, conditions are given and for models with the compensated jumps part only (Proposition 5.3 and Assumption 5.1). Eberlein and Raible [5] postulate a very specific form of the forward curve, see section 2,

$$df(t,\theta) = \frac{\partial J}{\partial x}(\sigma(t,\theta))\frac{\partial \sigma}{\partial \theta}(t,\theta)dt - \frac{\partial \sigma}{\partial \theta}(t,\theta)dZ_t, \tag{4}$$

where σ is a smooth, bounded and deterministic function and $J(x), x \in \mathbb{R}$, is the Lévy exponent of a 1-dimensional Lévy proces Z with Lévy measure ν having exponential moments:

$$\int_{|y|>1} e^{cy}\nu(dy) < \infty \tag{5}$$

for $c \in (-\gamma, \gamma)$, γ is a positive number. We derive formula (4) without requiring that σ is deterministic and with n-dimensional Lévy proces Z (see Theorem 5) and show that (5) is, to some extent, necessary.

0.2 Forward rate function

We assume that the basic probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ is complete. Let Z be a Lévy process in a separable Hilbert space U i.e. cadlag process with stationary independent increments having values in U. Let $\mathcal{F}_t^0 = \sigma(Z(s); s \le t)$ be σ-fields generated by $Z(t), t \ge 0$ and $\mathcal{F}_t$ be the completion of $\mathcal{F}_t^0$ by all sets of $\boldsymbol{P}$ probability zero. It is known that this filtration is right continuous, so it satisfies the "usual conditions". By μ we denote the measure associated to jumps of Z i.e for $\Gamma \in \mathcal{B}(U), \overline{\Gamma} \subset U \setminus \{0\}$

$$\mu([0,t],\Gamma) = \sum_{0<s\le t} \mathbf{1}_\Gamma(\Delta Z(s)).$$

A measure ν such that

$$E(\mu([0,t],\Gamma)) = t\nu(\Gamma)$$

is called a Lévy measure of the process Z

Through the paper we denote the inner product in U by $< \cdot , \cdot >$ and the norm in U by $| \cdot |$.

The characteristic function of $Z(t)$ has a form (Lévy-Khintchine formula)

$$Ee^{i\lambda Z(t)} = e^{t\psi(\lambda)},$$

where

$$\psi(\lambda) = i < a, \lambda > -\frac{1}{2} < Q\lambda, \lambda > + \int_U (e^{i\langle \lambda, x\rangle} - 1 - i < \lambda, x > \mathbf{1}_{\{|x|\leq 1\}})\nu(dx),$$

and $a \in U$, Q is a symmetric non negative nuclear operator on U, ν is a measure on U with $\nu(\{0\}) = 0$ and

$$\int_U (|x|^2 \wedge 1)\, \nu(dx) < \infty. \tag{6}$$

Moreover Z has a decomposition

$$Z(t) = at + W(t) + \int_0^t \int_{|y|\leq 1} y(\mu(ds, dy) - ds\nu(dy)) + \int_0^t \int_{|y|>1} y\mu(ds, dy), \tag{7}$$

where W is a Wiener process having values in U with covariance operator Q.

Let $r(t)$, $t \geq 0$ be the short rate process: if at moment 0 one puts into the bank account 1 then at moment t one has

$$B_t = e^{\int_0^t r(\sigma)d\sigma}.$$

It is convenient to assume that once a bond has matured its money equivalent goes to the bank account. Thus $P(t, \theta)$, the market price at moment t of a bond paying 1 at the maturity time θ, is defined also for $t \geq \theta$ by the formula

$$P(t, \theta) = e^{\int_\theta^t r(\sigma)d\sigma}. \tag{8}$$

The forward rate curve function $f(t, \theta)$ defined by (1) is usually interpreted as the anticipated short rate at time θ as seen by the market at time t.

We consider a generalized Heath, Jarrow and Morton model (2) taking a Lévy proces Z in U instead of $W = (W_1, \ldots, W_d)$ i.e.

$$df(t, \theta) = \alpha(t, \theta)dt + < \sigma(t, \theta),\ dZ(t) > . \tag{9}$$

For simplicity of notation we sometimes used another form of equation (9), namely

$$df(t) = \alpha(t)dt + \widetilde{\sigma}(t)dZ(t), \tag{10}$$

where $\widetilde{\sigma}(t) : U \to L^2[0, T]$ is such that

$$(\widetilde{\sigma}(t)u)(\theta) = < \sigma(t, \theta), u > .$$

For each θ the processes $\alpha(t,\theta)$, $\sigma(t,\theta)$, $t \leq \theta$ are assumed to be adapted processes with respect to a given filtration $(\mathcal{F}_t)$ and such that integrals in (9), therefore also in (10), are well defined.
For $\theta < t$ we put

$$\alpha(t,\theta) = \sigma(t,\theta) = 0. \tag{11}$$

It follows from (9) that for $t \leq \theta$,

$$f(t,\theta) = f(0,\theta) + \int_0^t \alpha(s,\theta)ds + \int_0^t < \sigma(s,\theta),\ dZ(s) >$$

and by (11) that for $t > \theta$

$$f(t,\theta) = f(0,\theta) + \int_0^\theta \alpha(s,\theta)ds + \int_0^\theta < \sigma(s,\theta),\ dZ(s) > .$$

Consequently for each $\theta > 0$, $f(t,\theta)$, $t > \theta$, is a process constant in t and should be identified with the short rate:

$$r(\theta) = f(0,\theta) + \int_0^\theta \alpha(s,\theta)ds + \int_0^\theta < \sigma(s,\theta),\ dZ(s) > . \tag{12}$$

From now on we assume (9) and (11) and that the short rate is given by (12).

Let us recall that *HJM postulate* is the requirement that the discounted bond price processes $\hat{P}(\cdot,\theta)$, $\theta \in [0,T]$:

$$\hat{P}(t,\theta) = P(t,\theta)/B_t = e^{-\int_t^\theta f(t,s)ds} e^{-\int_0^t f(t,s)ds} = e^{-\int_0^\theta f(t,s)ds}$$

are local martingales.

We are looking for conditions on the forward rate process defined by (9) under which the HJM postulate is satisfied.

We will assume that for given T, the integrals in the definition of f exist in the sense of the Hilbert space $H = L^2(0,T)$. Denote the scalar product in H by $\langle \cdot,\cdot \rangle$ and the characteristic function of the interval $[0,\theta]$ by $\mathbf{1}_{[0,\theta]}$. One should distinguish between the scalar product in U denoted by $< \cdot,\cdot >$ and the scalar product in L^2 denoted by $\langle \cdot,\cdot \rangle$.
If

$$F^\theta(x) = e^{-\langle x, \mathbf{1}_{[0,\theta]} \rangle}, \quad x \in H = L^2(0,T),$$

then

$$\hat{P}(t,\theta) = e^{-\langle f(t), \mathbf{1}_{[0,\theta]} \rangle} = F^\theta(f(t)), \quad t \geq 0$$

so the process $M^\theta(t) = \hat{P}(t,\theta)$, $t \geq 0$ is a semi-martingale and one can find its decomposition using Itô's formula (see e.g. [3]).

Denote $g_\theta = \mathbf{1}_{[0,\theta]}$. Then we have

$$M^\theta(t) = e^{-\langle f(t), \mathbf{1}_{[0,\theta]} \rangle} = e^{-\langle f(t), g_\theta \rangle}.$$

Since the dynamics of forward rate is given by (10) we see that

$$d\langle g_\theta, f(t)\rangle = \langle g_\theta, \alpha(t)\rangle dt + < \widetilde{\sigma}^*(t)g_\theta,\ dZ(t) > . \tag{13}$$

The following assumptions are used through this paper:

$(H1)$ For each $\theta \in [0,T]$ the processes $\widetilde{\sigma}^*(t)g_\theta,\ t \in [0,T]$, are locally bounded.

$(H2)$ for some θ and some s there exists $r > 0$ such that $B(0,r)$, the ball with center in zero and radius r, is contained in $supp\ (\widetilde{\sigma}^*(s)g_\theta)$,

By $supp\ (X)$ we denote the support of the distribution of the random variable X.

Theorem 1 *a) Assume (H2). If HJM postulate is satisfied, then*

$$\psi(s) = \int_{|y|>1} e^{<c,y>}\nu(dy) < \infty \tag{14}$$

for c in dense subset of $B(0,r)$. Moreover, ψ is lower semicontinuous.

b) If HJM postulate is satisfied, then for each s and θ

$$\langle g_\theta, \alpha(s)\rangle + < \widetilde{\sigma}^*(s)g_\theta, a > + \frac{1}{2} < Q\widetilde{\sigma}^*(s)g_\theta, \widetilde{\sigma}^*(s)g_\theta > +$$
$$\int_U \Big[e^{-<\widetilde{\sigma}^*(s)g_\theta,y>} - 1 - \mathbf{1}_{\{|y|\le 1\}}(y)\ < \widetilde{\sigma}^*(s)g_\theta, y > \Big]\nu(dy) \equiv 0. \tag{15}$$

Proof. Using (7) we see that $X^\theta(t) = \langle g_\theta, f(t)\rangle$ satisfies

$$\begin{aligned} dX^\theta(t) &= \langle g_\theta, \alpha(t)\rangle dt + < \widetilde{\sigma}^*(t)g_\theta, dZ(t) > \\ &= \langle g_\theta, \alpha(t)\rangle dt + < \widetilde{\sigma}^*(t)g_\theta,\ adt + dW(t) + \\ &\int_U \mathbf{1}_{\{|y|\le 1\}}(y)y(\mu(dt,dy) - dt\nu(dy)) + \int_U \mathbf{1}_{\{|y|>1\}}(y)\ y\mu(dt,dy)\ > \end{aligned}$$

Since $M^\theta(t) = \varphi(X^\theta(t))$, applying Itô's formula to function $\varphi(x) = e^{-x}$ and process X^θ we obtain

$$\begin{aligned} M^\theta(t) = N^\theta(t) + &\int_0^t \varphi_x(X^\theta(s-))\langle g_\theta, \alpha(s)\rangle ds + \\ &+ \int_0^t \varphi_x(X^\theta(s-)) < \widetilde{\sigma}^*(s)g_\theta, a > ds \\ &+ \int_0^t \int_U \varphi(X^\theta(s-))\mathbf{1}_{\{|y|>1\}}(y) < \widetilde{\sigma}^*(s)g_\theta, y > \mu(ds,dy) + \\ &+ \frac{1}{2}\int_0^t \varphi_{xx}(X^\theta(s-)) < Q\widetilde{\sigma}^*(s)g_\theta, \widetilde{\sigma}^*(s)g_\theta > ds + \end{aligned} \tag{16}$$

$$+\int_0^t\int_U\Big[\varphi\big(X^\theta(s-)+<\widetilde{\sigma}^*(s)g_\theta,y>\big)-$$

$$\varphi(X^\theta(s-))-\varphi_x(X^\theta(s-))<\widetilde{\sigma}^*(s)g_\theta,y>\Big]\mu(ds,dy)$$

$$=\ N^\theta(t)+\int_0^t\varphi_x(X^\theta(s-))\langle g_\theta,\alpha(s)\rangle ds+$$

$$\int_0^t\varphi_x(X^\theta(s-))<\widetilde{\sigma}^*(s)g_\theta,a>ds$$

$$+\frac{1}{2}\int_0^t\varphi_{xx}(X^\theta(s-))<Q\widetilde{\sigma}^*(s)g_\theta,\widetilde{\sigma}^*(s)g_\theta>ds+$$

$$+\ \int_0^t\int_U\Big[\varphi\big(X^\theta(s-)+<\widetilde{\sigma}^*(s)g_\theta,y>\big)-\varphi(X^\theta(s-))-$$

$$\mathbf{1}_{\{|y|\le 1\}}(y)\ \varphi_x(X^\theta(s-))<\widetilde{\sigma}^*(s)g_\theta,y>\Big]\mu(ds,dy),$$

where N^θ is a local martingale.

Now we give the proof of part a). If M^θ is a local martingale with localizing sequence of stopping times ϑ_n, then

$$\int_0^t\int_U\mathbf{1}_{[0,\vartheta_n]}(s)\Big[\varphi\big(X^\theta(s-)+<\widetilde{\sigma}^*(s)g_\theta,y>\big)-$$

$$\varphi(X^\theta(s-))-\mathbf{1}_{\{|y|\le 1\}}(y)\ \varphi_x(X^\theta(s-))<\widetilde{\sigma}^*(s)g_\theta,y>\Big]\mu(ds,dy)$$

is well defined and

$$E\int_0^t\int_U\mathbf{1}_{[0,\vartheta_n]}(s)\Big[\varphi\big(X^\theta(s-)+<\widetilde{\sigma}^*(s)g_\theta,y>\big)- \tag{17}$$

$$\varphi(X^\theta(s-))-\mathbf{1}_{\{|y|\le 1\}}(y)\ \varphi_x(X^\theta(s-))<\widetilde{\sigma}^*(s)g_\theta,y>\Big]\mu(ds,dy)<\infty.$$

Since the process under integral in (17) is predictable, then

$$E\int_0^t\int_U\mathbf{1}_{[0,\vartheta_n]}(s)\Big|\varphi\big(X^\theta(s-)+\widetilde{\sigma}^*(s)g_\theta,y>\big)- \tag{18}$$

$$\varphi(X^\theta(s-))-\mathbf{1}_{\{|y|\le 1\}}(y)\ \varphi_x(X^\theta(s-))<\widetilde{\sigma}^*(s)g_\theta,y>\Big|ds\nu(dy)<\infty.$$

Since $\varphi(x)=e^{-x}$, we obtain

$$E\Big[\int_0^t\int_U\mathbf{1}_{[0,\vartheta_n]}(s)\mathbf{1}_{\{|y|>1\}}(y)\Big|\varphi\big(X^\theta(s-)+<\widetilde{\sigma}^*(s)g_\theta,y>\big)-$$

$$\varphi(X^\theta(s-))\Big|ds\nu(dy)\Big]$$

$$=E\Big[\int_0^t\mathbf{1}_{[0,\vartheta_n]}(s)e^{-X^\theta(s-)}\Big(\int_{\{|y|>1\}}\Big|e^{-<\widetilde{\sigma}^*(s)g_\theta,y>}-1\Big|\nu(dy)\Big)ds\Big]<\infty.$$

Hence for P-almost all $\omega \in \Omega$

$$\int_0^{\vartheta_n} \int_{\{|y|>1\}} e^{-<\widetilde{\sigma}^*(s,\omega)g_\theta, y>} \nu(dy)ds < \infty. \tag{19}$$

Since $\vartheta_n \uparrow \infty$ as $n \to \infty$ for almost all s and P-almost all ω,

$$\int_{\{|y|>1\}} e^{-<\widetilde{\sigma}^*(s,\omega)g_\theta, y>} \nu(dy) < \infty,$$

which together with assumption (H2) implies (14).
To prove lower semi-continuity of ψ we use Fatou lemma. Let $\liminf_{c_n \to c} \psi(c_n) = \gamma$. If $\gamma < \infty$, then

$$\liminf_{c_n \to c} \int_{|y|>1} e^{<c_n, y>} \nu(dy) \geq \int_{|y|>1} \liminf_{c_n \to c} e^{<c_n, y>} \nu(dy) = \psi(c). \tag{20}$$

If $\gamma = \infty$, then (20) is obvious. Therefore, ψ is lower semi-continuous.

The equation (15) (i.e. point b) of Theorem 1) is a simple consequence of (16), because M^θ is a local martingale and $\varphi(x) = -\varphi_x(x) = \varphi_{xx}(x) = e^{-x}$. ∎

Theorem 2 *Assume* (15) . *If for all* $c \in U$

$$\int_{|y|>1} e^{<c,y>} \nu(dy) < \infty, \tag{21}$$

then HJM postulate is satisfied.

Proof. We have

$$\int_U \Big[(e^{-<\widetilde{\sigma}^*(s)g_\theta, y>} - 1 + \mathbf{1}_{\{|y|\leq 1\}}(y) \ < \widetilde{\sigma}^*(s)g_\theta, y > \Big] \nu(dy) = I_1 + I_2, \tag{22}$$

where

$$I_1 = \int_{\{|y|\leq 1\}} \Big(e^{-<\widetilde{\sigma}^*(s)g_\theta, y>} - 1 + \ < \widetilde{\sigma}^*(s)g_\theta, y > \Big) \nu(dy),$$
$$I_2 = \int_{\{|y|>1\}} (e^{-<\widetilde{\sigma}^*(s)g_\theta, y>} - 1) \ \nu(dy).$$

In the neighborhood of zero

$$e^{<c,x>} - 1 - < c, x > \approx (< c, x >)^2,$$

so by (6)

$$I_1 \leq const \int_{\{|y|\leq 1\}} |y|^2 \nu(dy) < \infty.$$

$I_2 < \infty$ by assumption (21) and (6). Therefore

$$\int_0^t \int_U \Big[(e^{-<\widetilde{\sigma}^*(s)g_\theta, y>} - 1 + \mathbf{1}_{\{|y|\leq 1\}}(y) \ < \widetilde{\sigma}^*(s)g_\theta, y > \Big] \nu(dy)ds$$

is well defined. This, assumptions (15) and equation (16) imply that

$$M^\theta(t) = N^\theta(t) + \int_0^t \int_U \Big[\varphi\big(X^\theta(s-) + < \widetilde{\sigma}^*(s)g_\theta, y > \big) -$$

$$\varphi(X^\theta(s-)) - \mathbf{1}_{\{|y|\leq 1\}}(y)\varphi_x(X^\theta(s-)) < \widetilde{\sigma}^*(s)g_\theta, y > \Big] \big(\mu(ds, dy) - ds\nu(dy)\big) =$$

$$= N^\theta(t) + N_1^\theta(t),$$

where N_1^θ is a local martingale, so M^θ is a local martingale. ■

Remark 1 a) If $M^\theta = e^{-X^\theta}$ is a local martingale, then (14) implies existence of Laplace transform of driven Lévy process Z in some neighborhood of zero. If ν is a Lévy measure of the α stable symmetric process Z, then

$$\nu(dy) = c|y|^{-1-\alpha}dy \quad \text{and} \quad \forall u \neq 0 \int_{\{|y|>1\}} e^{<u,y>}|y|^{-1-\alpha}dy = \infty.$$

Therefore, as a consequence of Theorem 1 we obtain that α stable symmetric process Z can not be used for modelling forward rate.

b) If $U = \mathbb{R}^d$, then in Theorem 1a we prove in fact that (14) is satisfied for

$$c \in \bigcup_{\theta\in[0,T]} \bigcup_{s\in[0,T]} supp\ (\widetilde{\sigma}^*(s)g_\theta).$$

c) If the processes $\widetilde{\sigma}^*(s)g_\theta$ are bounded (e.g. σ is deterministic and bounded as in [5]), so there exists a constant K_θ such that $|\widetilde{\sigma}^*(s)g_\theta| \leq K_\theta < \infty$, then we can weaken assumptions of Theorem 2. It is enough to assume that (21) is satisfied for $|c| \leq K$, where $K = \sup_\theta K_\theta$ instead of assumption that (21) is satisfied for all c.

Proposition 3 *Let $U = \mathbb{R}^d$.*
a) If

$$\int_{|y|>1} e^{\gamma\ |y|}\nu(dy) < \infty \tag{23}$$

for $\gamma \in \mathbb{R}$, $|\gamma| \leq r$, then (14) *is satisfied for $c \in B(0, r)$.*

b) If the condition (14) *is satisfied for $c \in B(0, r)$, then the condition* (23) *is satisfied for $|\gamma| \leq \frac{r}{\sqrt{d}}$.*

Proof. a) Since $< c, y > \leq | < c, y > | \leq |c| \cdot |y|$ the condition (23) for $|\gamma| \leq r$ implies (14) for $c \in B(0, r)$.

b) Fix $|\gamma| \leq \frac{r}{\sqrt{d}}$.

Since

$$\sum_{j=1}^{d} |y_j| = \sum_{j=1}^{d} y_j sgn y_j,$$

so for arbitrary orthant $A_h = \{sgn\ y_1 = e_1, \ldots, sgn\ y_d = e_d\}$ for $h = (e_1, \ldots, e_d)$, $e_i \in \{-1, 1\}$, taking $c_h = \gamma h$ we obtain by (14)

$$\int_{A_h \cap \{|y|>1\}} e^{\gamma\ |y|} \nu(dy) \leq \int_{A_h \cap \{|y|>1\}} e^{<c_h, y>} \nu(dy) \leq$$

$$\int_{\{|y|>1\}} e^{<c_h, y>} \nu(dy) < \infty$$

because $\gamma |y| \leq < c_h, y >$ and $|c_h| \leq r$. Hence

$$\int_{|y|>1} e^{\gamma\ |y|} \nu(dy) \leq \sum_h \int_{\{|y|>1\}} e^{<c_h, y>} \nu(dy) < \infty. \quad \blacksquare$$

It is convenient to present HJM condition in terms of the function J:

$$J(u) = - < u, a > + \frac{1}{2} < Qu, u > \ + \hat{J}(u), \tag{24}$$

where

$$\hat{J}(u) = \int_{\{|y| \leq 1\}} \Big(e^{-<u,y>} - 1 + < u, y >) \nu(dy) \tag{25}$$

$$+ \int_{\{|y|>1\}} (e^{-<u,y>} - 1) \nu(dy).$$

Theorem 4 *Under the assumptions (H1):*
a) If the discounted bond price processes $\hat{P}(\cdot, \theta)$, $\theta \in [0, T]$ *satisfy HJM postulate, then HJM type condition*

$$\int_0^\theta \alpha(t, v) dv = J\Big(\int_0^\theta \sigma(t, v) dv \Big) \tag{26}$$

holds.

b) The HJM type condition (26) *and (H2) implies HJM postulate.*

Proof. a) By Theorem 1b) HJM postulate implies (15), all factors are well defined and by definition of J we obtain (26).

b) The HJM type condition (26) gives (15), so applying Theorem 2 completes the proof. $\blacksquare$

In the next theorem we find the dynamics of f in the case $U = \mathbb{R}^d$.

Theorem 5 *Let $U = \mathbb{R}^d$. Assume that exists a deterministic constant K such that*

$$\sum_{j=1}^{d} \langle g, \sigma_j(t) \rangle^2 \leq K < \infty \quad t, \theta \in [0, T],$$

and moreover for some $\varepsilon > 0$

$$\int_{|y| \geq 1} e^{<c,y>} \nu(dy) < \infty \quad \text{for } |c| \leq K(1 + \varepsilon). \tag{27}$$

Then the HJM type condition (26) *implies that the dynamics of f has a form*

$$df(t, \theta) = \sum_{j=1}^{d} \frac{\partial J}{\partial u_j} \Big(\int_0^\theta \sigma(t, v) dv \Big) \sigma_j(t, \theta) dt + \sum_{j=1}^{d} \sigma_j(t, \theta) dZ_j(t). \tag{28}$$

Proof. Using assumption (27) one can check differentiability of J. So, by (26) we have

$$\alpha(t, \theta) = \sum_{j=1}^{d} \frac{\partial J}{\partial u_j} \Big(\int_0^\theta \sigma(t, v) dv \Big) \sigma_j(t, \theta).$$

and (28) follows. ■

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