Symplectic packing of a torus

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Abstract

We show a construction of a symplectic packing of a four-dimensional ball into a torus.

0 Introduction

A symplectic manifold is a real $\mathcal{C}^{\infty}$-manifold of dimension $2n$ with a nondegenerate, closed differential 2-form ω. The simplest example is $\mathbb{R}^{2n}$ with the form $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$. Another examples are given by smooth, complex, projective algebraic surfaces. If S is any such a surface, embedded by an ample line bundle L, the form $\omega = \omega_L$ is given by the first Chern class of L, after the identification of $H^2(S,\mathbb{Z})$ with $H^2_{DR}(S)$.
One of the classical problems in symplectic geometry is a problem of symplectic embedding of a ball (or a disjoint sum of balls) into a symplectic manifold. Let $(B(a),\omega_0) \subset \mathbb{R}^{2n}$ be a ball of the volume (!) a and let (M,ω) be any symplectic manifold. If $\phi : \bigsqcup_{i=1}^{s} B(a_i) \longrightarrow M$ is a diffeomorphism on the image, then ϕ is a symplectic embedding if $\phi^*\omega = \omega_0$.
The famous problem: how big a ball can be symplectically embedded into the set $(\{x_1^2 + y_1^2 \leq \varepsilon\} \times \mathbb{R}^{2n-2}, \omega_0)$ was solved by Gromov in [5]. He proved that the radius of the ball must be less than ε. In the same paper, Gromov proved that only a half of a 4-dimensional ball can be symplectically packed by two equal balls.
Let us now consider the case when our symplectic manifold S is a smooth projective algebraic surface with an ample line bundle L. The problem of symplectic packing of such a manifold with a disjoint sum of equal balls has a connection with the problem of finding Seshadri constants of the line bundle L. Let us remind that a Seshadri constant of L in the points $x_1, ..., x_r$ of S is defined as

$$\varepsilon(S, L, x_1, ..., x_r) := \inf_C \{ \frac{LC}{\sum_{i=1}^{r} \mathrm{mult}_{x_i} C} \},$$

where C is a (reduced and irreducible) curve on S (cf [4]). We will write $\varepsilon(L, r)$ for $\varepsilon(S, L, x_1, ..., x_r)$ with $x_1, ..., x_r$ generic on a given S.
Let us also recall the definition of a packing number.

$$v_r := \sup \frac{\mathrm{vol}\phi_a(\bigsqcup_{i=1}^r B(a), \omega_0)}{\mathrm{vol}(M, \omega_L)},$$

where supremum is taken over all such a that ϕ_a is a symplectic embedding. When $v_r = 1$ we say that there exists a full packing of the manifold with r balls.

Analogously, we may define a 'holomorphic packing number'

$$v_r^{hol} := \sup \frac{\mathrm{vol}\phi_a(\bigsqcup_{i=1}^r B(a), \omega_0)}{\mathrm{vol}(M, \omega_L)},$$

where supremum is taken over all such a that ϕ_a is a symplectic and holomorphic embedding.

The connection between the Seshadri constants and symplectic packing was first stated in [10] and then in [1], [2], [8] and others.

Lazarsfeld in [8] proved that $\varepsilon(L, 1) \geq \sqrt{v_1^{hol} L^2}$. In a similar way one can prove that $\varepsilon(L, r) \geq \sqrt{v_r^{hol} \frac{L^2}{r}}$.

On the other hand, Biran and Cieliebak in [3] [Theorem G], proved (as a corollary to the construction of symplectic blowing down, see [10]) that $\sqrt{v_r \frac{L^2}{r}} \geq \varepsilon(L, r)$.

The problem arises when the above inequalities are equalities. Take the case of $\mathbb{P}^2$ with $L = \mathcal{O}_{\mathbb{P}^2}(1)$, for example. For $r = 1, ..., 9$ we have $\varepsilon(L, r) = 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{5}, \frac{2}{5}, \frac{3}{8}, \frac{6}{17}, \frac{1}{3}$ respectively. In the same range of r, we have (cf [2]): $v_r = 1, \frac{1}{2}, \frac{3}{4}, 1, \frac{20}{25}, \frac{24}{25}, \frac{63}{64}, \frac{288}{289}, 1$, so $\varepsilon(L, r) = v_r$ here. For $r \geq 10$ we know by the results of Biran, [2, 1], that $v_r = 1$, whereas $\varepsilon(L, r)$ is still unknown (unless r is a square of a natural number, when $\varepsilon(L, r) = \frac{1}{\sqrt{r}}$, cf eg [7]).

Nagata's conjecture says that $\varepsilon(L, r) = \frac{1}{\sqrt{r}}$ for all $r > 9$, (cf eg [7, 12, 15]), so conjecturally $\varepsilon(L, r) = \sqrt{v_r \frac{L^2}{r}}$ for $\mathbb{P}^2$ with $L = \mathcal{O}_{\mathbb{P}^2}(1)$.

However, it may happen that $\varepsilon < \sqrt{v_r \frac{L^2}{r}}$. This fact is already stated (not explicitly) in [1]. Namely, let S be any abelian surface with the polarization of type $(1, 1)$, ie $\omega_L = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Such surfaces are all symplectomorphic. In case $S = E \times E$ where E is an elliptic curve, $\varepsilon(S, L, 1) = 1 = \frac{L.E}{\mathrm{mult}_x E}$.

In case S is generic ($\mathrm{Num}S \cong \mathbb{Z}$), $\varepsilon(S, L, 1) = \frac{4}{3}$ (cf [16]). Thus, for all S as above, $v_1 \geq \frac{8}{9}$, as they are all symplectomorphic. This means that for $S = E \times E$ we have $1 = \varepsilon(S, L, 1) < \sqrt{v_1 L^2} = \sqrt{2\frac{8}{9}} = \frac{4}{3}$.

The problem of its own is to find explicit constructions of the symplectic embeddings. Such construction are given for example in the papers of Karshon, [6], Traynor, [17, 9] and Schlenk, [13, 14].
In this paper we present an explicit construction of a packing of one four-dimensional ball into a torus.
First, we show the details of the construction of the full packing of the ball $B(2-4\varepsilon)$ into the torus $E \times E$ with the polarization of type $(1,2)$ (here $v_1 = 1$).
Then, we sketch an analogous construction for the embedding of the ball (slightly less than) $B(\frac{8}{9})$ into $E \times E$ with $(1,1)$ polarization.
The ideas of the construction are based mainly on the paper [13] of Schlenk.
For the convenience of the reader and to keep the homogeneity of the notation, we write down here some paragraphs, instead of sending the reader to [13].

1 Notation and basic facts

We work in $\mathbb{R}^4$ with the symplectic coordinates denoted by (x_1, y_1, x_2, y_2). The standard symplectic form in $\mathbb{R}^4$ is $\omega_0 := dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. The pair (x_i, y_i) may be also denoted by $z_i,\ i = 1, 2$.
Following [13], for a set U in the z_1-plane and for a positive, continuous function $h : U \ni z_1 \longrightarrow \mathbb{R}_+$ denote by $\mathcal{F}(U, h)$ the set

$$\mathcal{F}(U, h) := \{(z_1, z_2) \mid \pi|z_2|^2 < h(z_1), z_1 \in U\}.$$

Let $R(a)$ be a rectangle $(-\gamma, a-\gamma) \times [0, 1]$. For $N \in \mathbb{N}$, let R_N be a set $([0, a-\gamma) \cup (N + a - \gamma, N + a]) \times [0, 1]$ with the points with coordinates $(0, y_1)$ and $(N + a, y_1)$ identified. For the simplicity of the notation, let us write $R_l := [0, a-\gamma) \times [0, 1]$ and $R_r := (N + a - \gamma, N + a] \times [0, 1]$. By $D(a)$ we denote the disc

$$\{(x_1, y_1) \mid \pi(x_1^2 + y_1^2) < a\}.$$

For a set $U \subset \mathbb{R}^2$, $|U|$ denotes the area of U.

In [13] Schlenk introduced the following notion:

Definition 1 *A family of loops $\mathcal{L}$ in a simply connected domain $U \subset \mathbb{R}^2$ is called admissible if there is a diffeomorphism $\beta : D(|U|) \setminus \{0\} \longrightarrow U \setminus \{p\}$, for a point $p \in U$, such that*
1. concentric circles are mapped to elements of $\mathcal{L}$
2. in a neighbourhood of 0, β is a translation.

Our main tool will be the fact, proved in [13] (Lemma 2.5).

Lemma 2 *Let U and V be bounded and simply connected diffeomorphic domains in $\mathbb{R}^2$ and let $\mathcal{L}_\mathcal{U}, \mathcal{L}_\mathcal{V}$ be admissible families of loops in U and V respectively. Then there is a symplectomorphism $U \longrightarrow V$, mapping loops to loops.*

2 The elements of the construction

Our goal is to construct a symplectic embedding of the ball $B(2-4\varepsilon)$ into the torus $\mathbb{T}_2 := ([0,1] \times [0,1] \times [0,1] \times [0,2], \omega_0)$.
The idea of the construction is to map symplectically a 'long trapezoid' (defined below) into the torus. Then we embed the ball into this 'long trapezoid'. The map of the trapezoid restricted to the image of the ball gives the required symplectic embedding.
Throughout we assume that $\varepsilon << 1$.

2.1 Element 1: Long trapezoid

We define the 'long trapezoid' $\mathcal{T}$ as $\mathcal{F}(U,h)$, where $U := (0, 1-\varepsilon) \times [0,1] \cup [1-\varepsilon, 1+\varepsilon+N] \times [0, \varepsilon'] \cup (1+\varepsilon+N, 2+N) \times [0,1]$, and

$$h(z_1) = h(x_1) = \begin{cases} -x_1 + 2 - 3\varepsilon & \text{if } x_1 \leq 1 - 2\varepsilon \\ \delta & \text{if } x_1 \in (1-\varepsilon, N+1+\varepsilon+\delta) \\ x_1 - N - 1 - \varepsilon & \text{if } x_1 \geq N+1+\varepsilon+\delta, \end{cases}$$

and is continuous and goes linearly on the interval $(1-2\varepsilon, 1-\varepsilon)$.
For our construction we assume that $\delta < \frac{\varepsilon}{7}$ and that $N = \lceil \frac{\varepsilon}{6} \rceil$.
As for ε', we will say precisely how small it must be in Element 6 of the construction.

2.2 Element 2: The fibers of the trapezoid

We need to change the shape of the fibers of $\mathcal{T}$. We must change the circles into rectangles with smooth corners, of width (almost) one, ie $x_2 \in (0,1)$. Moreover, we require that $(0,0) \in D(2-4\varepsilon)$ goes to the point $(\frac{1}{2}, 1-\varepsilon)$. Such a construction is described in [13] [Lemma 2.8]. We provide a detailed description of this construction in Element 6 below. Thus, here we skip the details, but just draw a picture, see Figure 4.

2.3 Element 3: The cut off function and the lifting of fibers

Let us describe the procedure of lifting the fibres, following [13]. Let $c_j : \mathbb{R} \longrightarrow [0, 1-2\varepsilon'], j = 0...N-1$ be smooth functions as in the Figure 5. We assume that c_j is nonzero on the interval $(j - \frac{\varepsilon}{2}, j + \frac{\varepsilon}{2})$. Define $I_j(t) := \int_j^t c_j(s)ds$. Assume moreover that $I_j(j+\varepsilon) = \frac{\varepsilon}{6}$. Then define a symplectic mapping:

$$\phi_j(x_1, y_1, x_2, y_2) := (x_1, y_1 + c_j(x_1)x_2, x_2, y_2 + I_j(x_1)).$$

The maps ϕ_j are symplectic as their derivatives satisfy $d\phi_j^T J_0 d\phi_j = J_0$ (cf [11], Chapter I).
If $x_1 \leq j - \frac{\varepsilon}{2}$ then $\phi_j = \text{Id}$, if $x_1 \geq j + \frac{\varepsilon}{2}$ then $\phi_j(x_1, y_1, x_2, y_2) := (x_1, y_1, x_2, y_2 + \frac{\varepsilon}{6})$. Thus, applying ϕ_j results in lifting the fibres by $\frac{\varepsilon}{6}$ along y_2-axis.
Observe, that the projection of $\phi_j(\mathcal{T})$ to the (x_1, y_1)-plane is the union of U with the set $A_j := \{(x_1, y_1) : 0 \leq y_1 \leq c_j(x_1)\}$. Important for our construction is the fact that $A_j \subset [j-\varepsilon, j+\varepsilon] \times [0,1]$.
Consider now the set $\phi_{N-1}(...(\phi_1(\mathcal{T})...)) =: \widetilde{\mathcal{T}}$. Observe, that ϕ_j lifts only the fibres over the points with $x_1 > j - \frac{\varepsilon}{2}$. Each ϕ_j lifts the fibres by $\frac{\varepsilon}{6}$ and we do it $N-1$ times, (ie $j = 1...N-1$). This means that finally we have the fibres moved up along y_2 by more than $1 - \frac{\varepsilon}{6}$.

2.4 Element 4: $\widetilde{\mathcal{T}}$ into $\mathbb{T}_2$.

We map $\widetilde{\mathcal{T}}$ into $\mathbb{T}_2$ by dividing $\widetilde{\mathcal{T}}$ modulo $\mathbb{Z}^4$ (ie by gluing the points with integer x_1-coordinates). This map is not injective, but we will see below, that it will be injective, when restricted to the image of the ball in $\mathcal{T}$.

The next two elements present the construction of the embedding of the ball into $\mathcal{T}$.

2.5 Element 5: A rectangle $R(2-4\varepsilon)$ into R_N .

The rectangle $R(2-4\varepsilon) := (-1+2\varepsilon, 1-2\varepsilon) \times [0,1]$ and the set $R_N = ([0, 1-2\varepsilon) \cup (N+1+2\varepsilon, N+2]) \times [0,1]$ with the points $(0, y_1)$ and $(N+2, y_1)$ identified, are symplectomorphic. Indeed, define $\gamma : R(2-4\varepsilon) \longrightarrow R_N$ such that

$$\gamma(x_1, y_1) = \begin{cases} (x_1, y_1) \text{ if } x_1 \geq 0 \\ (x_1 + N + 2, y_1) \text{ if } x_1 \leq 0. \end{cases}$$

Then γ is an area and orientation preserving diffeomorphism.

2.6 Element 6: A disc into R_N

We follow closely the idea of Lemma 2.8 from [13]. If you are familiar with the paper, you may skip reading this section.
We will construct a symplectomorphism α of $D(2-4\varepsilon)$ into $R(2-4\varepsilon)$. Having α, we define a symplectomorphism

$$\Gamma : D(2-4\varepsilon) \longrightarrow R_N$$

as $\gamma \circ \alpha = \Gamma$.
Moreover, we require that Γ (so α) satisfies some additional conditions. Let $x_1 := x_1(\Gamma(z_1))$ denote the real part of the image of $z_1 \in D(2-4\varepsilon)$ by Γ. We want that

$$x_1 \leq \pi |z_1|^2 + \varepsilon$$

if $x_1 \in R_l$ and that

$$N - x_1 \leq \pi |z_1|^2 - 3 + 3\varepsilon$$

if $x_1 \in R_r$.
α is constructed as follows:
Step I. A translation of a small disc.
We translate the disc of the radius $\frac{\varepsilon}{8}$ to point $(\frac{\varepsilon}{4}, \frac{1}{2}) \in R_N$.
Step II. Rectangular loops filling (almost all) the rectangle R_l.
We define rectangular loops by giving the coordinates of the lower-left and upper-right corners. We define a first part of a family of rectangular loops starting from L_0 with the corners $(\frac{\varepsilon}{4}, \frac{\varepsilon}{8})$ and $(\frac{3\varepsilon}{4}, 1-\frac{\varepsilon}{8})$. The last of these loops, L_1, has coordinates $(\varepsilon_2+\varepsilon_3, \varepsilon_2+\varepsilon_3)$ and $(1-\varepsilon-\varepsilon_2-\varepsilon_3, 1-\varepsilon_2-\varepsilon_3)$, where $\varepsilon_2, \varepsilon_3$ are such, that the area enclosed by L_1 is bigger than $1-2.5\varepsilon$. The coordinates of the corners within this family change linearly.
Step III. Rectangular loops filling R_l and then R_r.
Let us now define the next part of our family of loops. The first loop, L_2, has corners $(\varepsilon_2-\varepsilon_3, \varepsilon_2-\varepsilon_3)$ and $(1-\varepsilon-\varepsilon_2+\varepsilon_3, 1-\varepsilon_2+\varepsilon_3)$, the last is the whole R_N. Again we assume that the coordinates of the corners within the family change linearly. More precisely, the lower-left corner moves linearly from $(\varepsilon_2-\varepsilon_3, \varepsilon_2-\varepsilon_3)$ to the point $(0, s_0(\varepsilon_2-\varepsilon_3)$, where $s_0 = \frac{\varepsilon_2-\varepsilon_3}{1-2\varepsilon+\varepsilon_2-\varepsilon_3}$ and then from the point $(N+2, s_0(\varepsilon_2-\varepsilon_3)$ to $(N-1+2\varepsilon, 0)$.
Step IV. Smooth loops.
We change rectangular loops into smooth ones, by smoothing the corners in such the way that the area enclosed by the smooth loop differs from the area enclosed by the rectangular one by less than $\frac{\varepsilon}{4}$.
Step V. Complete.

We complete the loops to an admissible family between the circle and L_0 and between L_1 and L_2.
Step VI. The bounds $x_1 \leq \pi|z_1|^2 + \varepsilon$ and $3 - 3\varepsilon - x_1 \leq \pi|z_1|^2 + N$ are obtained by calculating the area bounded by the loops. Indeed, suppose that $x_1 \in P_l$. If moreover $\pi|z_1|^2 > 1 - 3\varepsilon$, then immediately $x_1 \leq \pi|z_1|^2 + \varepsilon$. If $\pi|z_1|^2 \leq 1 - 3\varepsilon$, then from our construction it follows, that x_1 is either on one of the loops (say L_t) enclosed by L_0 and L_1, or x_1 lies inside L_0. In the latter case $x_1 \leq \frac{3}{4}\varepsilon \leq \varepsilon + \pi|z_1|^2$. In the former case the area enclosed by L_t (which equals $\pi|z_1|^2$) is greater than $-\frac{\varepsilon}{4} + (x_1 - (1-s)\frac{\varepsilon_1}{4} + s\varepsilon_2 + s\varepsilon_3)(1 - \frac{\varepsilon}{4}) > x_1 - \frac{\varepsilon}{2}$. This gives the required inequality. If $\pi|z_1|^2 > 1 - 3\varepsilon$ and $x_1 \in P_r$, then x_1 lies on the loop of the area ($= \pi|z_1|^2$) greater than $-\frac{\varepsilon}{4} + 1 - 2.5\varepsilon + (N + 2 - x_1)(1 - \frac{\varepsilon}{4}) > 3 - 3\varepsilon + N - x_1$, (remember, that $N + 2 - x_1 < 1$).
(Look at the pictures. Figure 1 goes to Figure 2 by α and then goes by γ to Figure 3).
We may now define ε'. We choose ε' such, that for $y_1 \in (0, \varepsilon')$ and any $x_1 \in (0, 1 - 2\varepsilon)$ we have $\pi|\alpha^{-1}(x_1, y_1)|^2 = \pi|z_1|^2 > 2 - 4\varepsilon - \frac{\varepsilon}{5}$. If for this z_1 the pair $(z_1, z_2) \in B(2 - 4\varepsilon)$, then $\pi|z_2|^2 < \frac{\varepsilon}{5}$.

2.7 Element 7: A ball into $\mathcal{T}$

Define a symplectic embedding β from a ball $B(2 - 4\varepsilon)$ to $\mathcal{T}$ by

$$\beta(z_1, z_2) = (\Gamma(z_1), z_2).$$

The map is symplectic, but we have to check that it embeds the ball into $\mathcal{T}$. If $(z_1, z_2) \in B(2 - 4\varepsilon)$, then $\Gamma(z_1) \in R$ and

$$\pi|z_2|^2 < 2 - 4\varepsilon - \pi|z_1|^2 < h(z_1)$$

because such is the construction in Element 6.

2.8 B into $\mathbb{T}_2$

To pack $\mathbb{T}_2$ with one ball of the volume $2 - 4\varepsilon$, first pack the ball into the long trapezoid, $\mathcal{T}$, (Elements 6 and 7), then apply Elements 1,2,3 and 4 to the image of the ball in $\mathcal{T}$.

3 B into $\mathbb{T}_1$

In this section we sketch the idea how to pack a ball of the volume slightly less than $\frac{8}{9}$ into $\mathbb{T}_1$. As we do not give the details, we also skip, for simplicity of notation, the ε-argument. Thus, we work with $B(\frac{8}{9})$ instead of $B(\frac{8}{9} - \varepsilon)$ etc.

3.1 Long trapezoids

Let us define the trapezoid $\mathcal{T}_1$ as $\mathcal{F}(U, h)$, where U is a subset of z_1-plane, $U := [0,1]^2 \cup [0,N] \times [0,\varepsilon'] \cup [N - \frac{1}{3}, N] \times [0,1]$, for suitable N and ε'. The function $h(z_1) = h(x_1)$ is given in the Figure 6. Let us also define an analogous trapezoid, $\mathcal{T}_2$, with the base in z_2-plane and the fibres in z_1-plane. The base U and the function h are here like the ones for $\mathcal{T}_1$, the only difference is that we change z_1 with z_2 and, in the definition of h, y_2 replaces x_1. The fibres of $\mathcal{T}_2$ are now the rectangles in z_1-plane, with $y_1 \in [0,1]$ and with the area depending on x_1.

3.2 Fibres

We modify the fibres of the trapezoid to get, instead of discs, the rectangles looking as in the Figure 7.

3.3 Reversing the roles

Observe, that after the above changing the fibres, we may look at our ball $B(\frac{8}{9})$ as packed into the trapezoid $\mathcal{T}_2$.
Let us denote by P_1 the set $\{(z_1, z_2) \in \mathcal{T}_1 | z_1 \in (N - \frac{1}{3}, N)\}$ and by P_2 the set $\{(z_1, z_2) \in \mathcal{T}_2 | z_2 \in (N - \frac{1}{3}, N)\}$.

3.4 Moving up fibres

Now, we move the fibres of $\mathcal{T}_1$ up along y_2 by $\frac{2}{3}$, analogously as we did it in Element 3.
Then, we move the fibres of $\mathcal{T}_2$ up along x_1 (!) by $\frac{2}{3}$.

3.5 Gluing

After lifting the fibres, we glue the points with integer coordinates.

3.6 Result

After the lifting and gluing procedures, we see that P_1 goes to the set $\{(z_1, z_2) | x_1 \in (\frac{2}{3}, 1), y_1 \in (0,1), x_2 \in (0,1), y_2 \in (\frac{2}{3}, 1)$ and $x_1 > y_2\}$, where the last condition is given by the fact that the area of the fibre is less than $h(x_1)$. Analogously, P_2 goes to the set $\{(z_1, z_2) | x_1 \in (\frac{2}{3}, 1), y_1 \in (0,1), x_2 \in$

$(0,1), y_2 \in (\frac{2}{3}, 1)$ and $x_1 < y_2\}$. Now this last condition follows from the fact that the area of the fibre (ie almost x_1) must be less than $h(y_2)$.
The two sets are disjoint.

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Figure 1

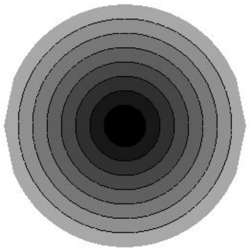

Figure 2

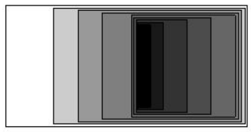

Figure 3

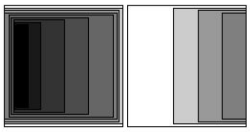

Figure 4

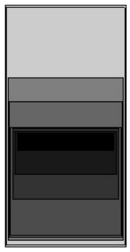

Figure 5

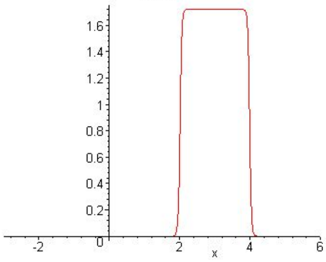

Figure 6

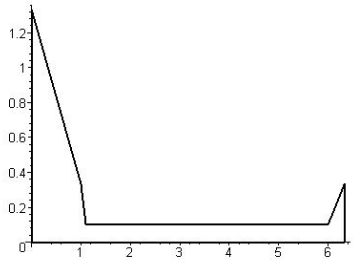

Figure 7