Approximate solution of a Cauchy problem for the Helmholtz equation

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Revised version

Abstract

A problem of reconstruction of the radiation field in a domain $\Omega \subset \mathbb{R}^3$ from experimental data given on a part of boundary is considered. For the model problem described by a Cauchy problem for the Helmholtz equation, an approximate method based on regularization in the frequency space is analyzed. Convergence and stability are proved under a suitable choice of regularization parameter. Numerical implementation of the method is discussed.

1 Introduction

In optoelectronics, the determination of a radiation field surrounding a source of radiation (e.g. a light emitting diode) is a problem of frequent occurrence. As a rule, experimental determination of the whole radiation field is not possible. Practically, we are able to measure the electromagnetic field only on some subset of physical space (e.g. on some surfaces). So, the problem arises how to reconstruct the radiation field from such experimental data (see for instance [1], [13]).

In this article, we shall consider only stationary processes (for definitions see [3]). In such a case, the vectors of electric induction $\mathbf{D}(\mathbf{r},t)$ and of magnetic induction $\mathbf{B}(\mathbf{r},t)$ have the form

$$\mathbf{D}(\mathbf{r},t) = e^{i\omega t}\mathbf{D}(\mathbf{r}), \qquad \mathbf{B}(\mathbf{r},t) = e^{i\omega t}\mathbf{B}(\mathbf{r}),$$

where ω is a constant frequency, and the Maxwell's equations lead to the Helmholtz equations for the vectors $\mathbf{D}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$:

$$\Delta\mathbf{D}(\mathbf{r}) + k^2\mathbf{D}(\mathbf{r}) = 0, \ \ \Delta\mathbf{B}(\mathbf{r}) + k^2\mathbf{B}(\mathbf{r}) = 0 \quad \text{for } \mathbf{r} \in \Omega \subset \mathbb{R}^3. \qquad (1.1)$$

Here

$$k^2 = \left(\frac{\omega}{c}\right)^2,$$

and the domain Ω depends on the considered problem [10].

In case when boundary conditions for the fields $\mathbf{D}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are linear, we can formulate boundary value problems for each component of electromagnetic field separately.

The physical problem considered in this paper is connected with the notion of light beams. A good example of such a beam is a Gaussian beam. In electrodynamics, a Gaussian beam is a beam of light whose electric field intensity is a Gaussian function as a function of distance r from the axis of the beam (see Figures 1.1 and 1.2):

$$D(r) = D_0 \exp\left(\frac{-r^2}{\rho^2}\right). \qquad (1.2)$$

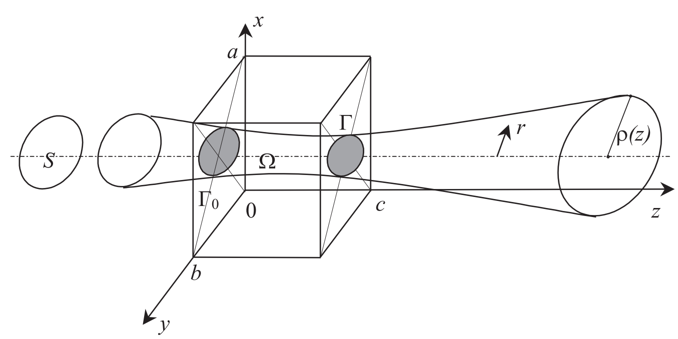

Figure 1.1: Geometry of the Initial Problem. Shown is the typical shape of a laser beam (here is presented the Gaussian beam as an example). $\rho(z)$ is the radius at which the electric field amplitude drops to $(1/e)D_0$. Proportions between the radius of beam and the dimensions of the cuboid are not conserved. In reality, this radius should be much smaller.

The Gaussian beam is a good approximation of the real beam of light generated by many lasers. So, formulating corresponding boundary problems, we can assume that the electric field (and, as a result, also a magnetic field) practically vanishes far from the axis of the beam.

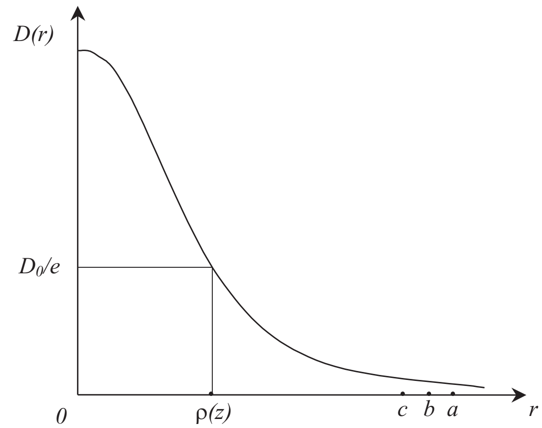

Figure 1.2: Radial distribution of the electric field amplitude in the Gaussian beam. The radius of beam should be much smaller than any of the edges a, b, c.

When considering more general models of laser beams (not necessary Gaussian beams), in analogy to the model presented above, we can formulate the following

Initial Problem. Let us consider a physical system presented in Figure 1.1. Some sources of the electromagnetic field are situated outside a cuboid. These sources generate the electromagnetic field in all the space $\mathbb{R}^3$. However, taking into account that fields practically vanish far from the axis, we may take an assumption that the electromagnetic field vanishes on the side-faces of the cuboid. Therefore, each component of this approximate field in the bounded domain $\Omega = (0,a) \times (0,b) \times (0,c)$ is a solution of the Helmholtz equation

$$\Delta u(\mathbf{r}) + k^2 u(\mathbf{r}) = 0 \qquad \text{in } \Omega, \tag{1.3}$$

and satisfies the boundary conditions

$$u(\mathbf{r}) = 0 \qquad \text{on side-faces of } \Omega. \tag{1.4}$$

The related inverse problem consists in reconstruction of a solution u of (1.3), (1.4) in Ω from values of u and its normal derivative on the boundary Γ

$$u(\mathbf{r}) = g(\mathbf{r}), \qquad \partial_z u(\mathbf{r}) = h(\mathbf{r}) \qquad \text{on } \Gamma. \tag{1.5}$$

In practice we have to solve the above problem with perturbed data g_δ and h_δ.

In this paper we consider the physical situation similar to that presented above, but the mathematical model is slightly modified:

Model Problem. Let us consider a certain modification of Initial Problem. Let Γ_0 and Γ be two parallel surfaces in the space $\mathbb{R}^3$:

$$\Gamma_0 = \{\mathbf{r} \in \mathbb{R}^3 : z = 0\}, \qquad \Gamma = \{\mathbf{r} \in \mathbb{R}^3 : z = d\}, \tag{1.6}$$

for $\mathbf{r} = (x, y, z)$, and let Ω_0 denote the half-a-space as shown in Figure 1.3.

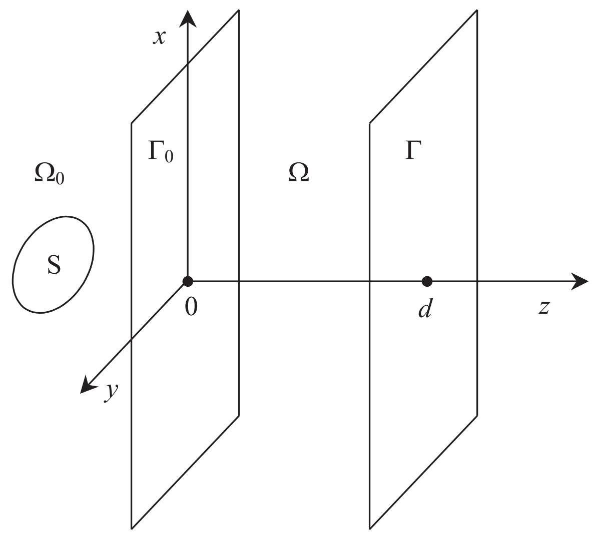

Figure 1.3: Geometry of Model Problem

Let Ω denote the part of the space $\mathbb{R}^3$ contained between two planes Γ_0 and Γ. The sources of the electromagnetic field are situated in Ω_0. Each component of electromagnetic field in Ω is a solution of the Helmholtz equation

$$\Delta u(\mathbf{r}) + k^2 u(\mathbf{r}) = 0 \qquad \text{in } \Omega. \tag{1.7}$$

We look for a solution satisfying the following conditions on a part of boundary:

$$u(\mathbf{r}) = g(\mathbf{r}) \qquad \partial_z u(\mathbf{r}) = h(\mathbf{r}) \qquad \text{for } \mathbf{r} \in \Gamma, \tag{1.8}$$

with the additional condition

$$\forall z \in (0, d) \quad u(\cdot, \cdot, z) \in L^2(\mathbb{R}^2), \tag{1.9}$$

for given functions g and h. The inverse problem formulated above is in fact a Cauchy problem for the Helmholtz equation. The problem is to reconstruct the field component u in Ω in the case when approximate values g_δ and h_δ (instead of g and h) on the boundary Γ are known. The values g_δ can be got directly from measurements of the field on the surface Γ. Approximate values of h_δ can be found by performing additional measurements on some surface slightly shifted with respect to Γ and by solving the relevant Dirichlet problem (for details see Section 3).

For reference we present below the related problem, when, in contrast to the Model Problem, the domain Ω is a bounded region.

Reference Problem. Let us consider a physical system presented in Figure 1.4.

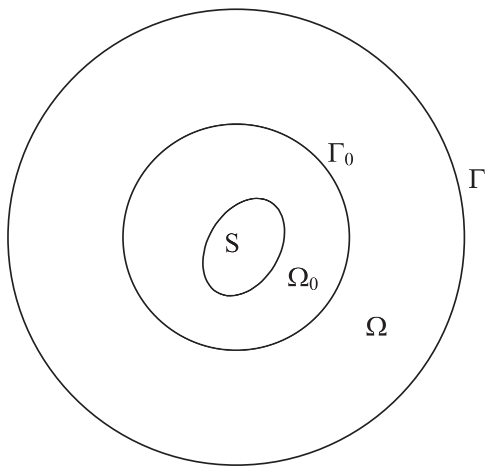

Figure 1.4: Geometry of Reference Problem

The sources of the electromagnetic field are situated in the domain Ω_0. These sources generate the electromagnetic field in all the space $\mathbb{R}^3$. Each component of the electromagnetic field satisfies the following boundary-value problem for the Helmholtz equation

$$\Delta u(\mathbf{r}) + k^2 u(\mathbf{r}) = 0 \text{ in } \mathbb{R}^3 \setminus \Omega_0 \tag{1.10}$$

with the boundary conditions

$$u(\mathbf{r}) = g(\mathbf{r}) \text{ on } \Gamma \tag{1.11}$$

and the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0 \tag{1.12}$$

where g is the exact value of the corresponding component of electromagnetic field on the boundary Γ and $r = \sqrt{x^2 + y^2 + z^2}$.

From measurements we get approximate values g_δ of this field on the boundary Γ with some measurement errors. The problem is to reconstruct the field component from given measured data g_δ.

The Reference Problem has been formulated only for comparison and will not be considered in the sequel.

The aim of this paper is to present an approximate method for solving in a stable way the Cauchy problem for Helmholtz equation in Ω (cf. Figure 1.3) with approximately given boundary values. Cauchy problems for elliptic equations are ill-posed [9], i.e. the solution does not depend continuously on the boundary data. Stability aspects of Cauchy problems were discussed for instance in [7], [12], [14], [2]. For the Helmholtz equation, an influence of the frequency k on the stability of Cauchy problems was described in [8]. Moreover, other ill-posed problems for the Helmholtz equation were extensively studied in literature, among others: inverse problem of determining the shape of a part of boundary [4], inverse problem of determination of sources [11], [5].

In this paper we discuss the nature of the ill-posedness of the considered problem and propose approximate method of solving, based on regularization in the frequency space. A similar technique was used in [6] for sidewise heat equation in the case of one dimensional space.

The paper is organized as follows. In Section 2 we consider the model problem and discuss its ill-posedness. The regularization method based on truncated Fourier transform is analyzed in Section 3 where its convergence and stability are proved under the suitable choice of regularization parameter. In Section 4 the numerical implementation of the method is discussed. Some numerical examples illustrating the proposed method are included.

2 Ill-posedness

Let us consider the Model Problem (1.7), (1.8), (1.9) described in Section 1. We have $\Omega = \mathbb{R}^2 \times (0, d) \subset \mathbb{R}^3$, $d > 0$. For simplicity, the first two variables will be denoted by $\boldsymbol{\rho} = (x, y)$. According to this notation $\Gamma_0 := \{(\boldsymbol{\rho}, 0), \boldsymbol{\rho} \in \mathbb{R}^2\} \subset \partial\Omega$, $\Gamma := \{(\boldsymbol{\rho}, d), \boldsymbol{\rho} \in \mathbb{R}^2\} \subset \partial\Omega$.

The problem under consideration can be written as follows:

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \Omega \\ u(\boldsymbol{\rho}, d) = g(\boldsymbol{\rho}) & \boldsymbol{\rho} \in \mathbb{R}^2, \\ \partial_z u(\boldsymbol{\rho}, d) = h(\boldsymbol{\rho}) & \boldsymbol{\rho} \in \mathbb{R}^2, \\ u(\cdot, z) \in L^2(\mathbb{R}^2) & z \in (0, d), \end{cases} \tag{2.1}$$

where $g,\ h \in L^2(\mathbb{R}^2)$ are given data. We assume that for these exact data the unique solution exists in $H^2(\Omega)$. We look for an approximate solution inside Ω in the case when the data are given approximately, i.e. when g_δ, $h_\delta \in L^2(\mathbb{R}^2)$ are used as the data and

$$\|g_\delta - g\|_{L^2(\mathbb{R}^2)} \leq \delta \qquad \|h_\delta - h\|_{L^2(\mathbb{R}^2)} \leq \delta. \tag{2.2}$$

It will be shown that this Cauchy problem is ill-posed, i.e. the solution does not depend continuously on the boundary data and small errors in the data can destroy the numerical solution. For numerical solving such a problem in a stable way, so-called regularization methods should be applied.

In order to simplify an analysis of the ill-posed Cauchy problem (2.1) we can make the additional assumption that $h = 0$. Let us observe that the solution of the general problem (2.1) is the sum $u = v + \widetilde{u}$ of the solution $v \in H^2(\Omega)$ of the problem

$$\begin{cases} \Delta v + k^2 v = 0, & \text{in } \Omega \\ v|_{\Gamma_0} = 0, & \\ \partial_z v|_{\Gamma} = h, & \\ v(\cdot, z) \in L^2(\mathbb{R}^2) & z \in (0, d), \end{cases} \tag{2.3}$$

and the solution $\widetilde{u} \in H^2(\Omega)$ of the problem

$$\begin{cases} \Delta \widetilde{u} + k^2 \widetilde{u} = 0, & \text{in } \Omega \\ \widetilde{u}|_{\Gamma} = g - v|_{\Gamma}, & \\ \partial_z \widetilde{u}|_{\Gamma} = 0, & \\ \widetilde{u}(\cdot, z) \in L^2(\mathbb{R}^2) & z \in (0, d). \end{cases} \tag{2.4}$$

Lemma 2.1 *If $k < \frac{\pi}{2d}$, then the solution of the problem (2.3) is continuously dependent in L^2 norm on the data h*

$$\|v\|_{L^2(\Omega)} \leq C \|h\|_{L^2(\mathbb{R}^2)}.$$

PROOF: Since $v(\cdot, z) \in H^2(\mathbb{R}^2)$ for $z \in (0, d)$ and $h \in L^2(\mathbb{R}^2)$, we can apply to them the Fourier transform with respect to variables $\boldsymbol{\rho} \in \mathbb{R}^2$

$$\widehat{v}(\boldsymbol{\xi}, z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} v(\boldsymbol{\rho}, z) e^{-i\boldsymbol{\xi} \cdot \boldsymbol{\rho}} d\boldsymbol{\rho},$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $\boldsymbol{\xi} \cdot \boldsymbol{\rho} = \xi_1 x + \xi_2 y$. We have

$$\widehat{\frac{\partial^2 v}{\partial x^2}}(\boldsymbol{\xi}, z) = -\xi_1^2 \widehat{v}(\boldsymbol{\xi}, z), \ \widehat{\frac{\partial^2 v}{\partial y^2}}(\boldsymbol{\xi}, z) = -\xi_2^2 \widehat{v}(\boldsymbol{\xi}, z).$$

Thus, the problem (2.3) can now be formulated in the frequency space as follows:

$$\begin{cases} \widehat{v}_{zz}(\boldsymbol{\xi}, z) = (|\boldsymbol{\xi}|^2 - k^2)\, \widehat{v}(\boldsymbol{\xi}, z), & \boldsymbol{\xi} \in \mathbb{R}^2, \ z \in (0, d) \\ \widehat{v}(\boldsymbol{\xi}, 0) = 0 & \boldsymbol{\xi} \in \mathbb{R}^2, \\ \partial_z \widehat{v}(\boldsymbol{\xi}, d) = \widehat{h}(\boldsymbol{\xi}) & \boldsymbol{\xi} \in \mathbb{R}^2. \end{cases} \tag{2.5}$$

Looking for the solution of the form

$$w(\cdot, z) = \alpha_1(\cdot) e^{z\sqrt{\eta(\cdot)}} + \alpha_2(\cdot) e^{-z\sqrt{\eta(\cdot)}},$$

where $\alpha_i \in L^2(\mathbb{R}^2)$, $i = 1, 2$ and

$$\eta(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2 - k^2,$$

we find from the condition on Γ_0 that $\alpha_1 = -\alpha_2$. Thus,

$$w(\cdot, z) = 2\alpha_1(\cdot) \sinh(z\sqrt{\eta(\cdot)}) \tag{2.6}$$

and

$$\frac{\partial w}{\partial z} = 2\alpha_1(\cdot)\sqrt{\eta(\cdot)} \cosh(z\sqrt{\eta(\cdot)}).$$

From the condition on Γ we get

$$w(\cdot, z) = \widehat{h}(\cdot) \frac{\sinh(z\sqrt{\eta(\cdot)})}{\sqrt{\eta(\cdot)}\cosh(d\sqrt{\eta(\cdot)})}.$$

If $|\boldsymbol{\xi}| \longrightarrow k$, then $w(\boldsymbol{\xi}, z) \longrightarrow \widehat{h}(\boldsymbol{\xi}) z$. For $|\boldsymbol{\xi}| > k$ and $z \in (0, d)$, $\sinh(z\sqrt{\eta(\boldsymbol{\xi})}) \leq \sinh(d\sqrt{\eta(\boldsymbol{\xi})})$ and the function $\frac{1}{x}\tanh(x)$ is decreasing for $x \geq 0$, thus

$$|w(\cdot, z)| \leq d|\widehat{h}(\cdot)|. \tag{2.7}$$

For $|\boldsymbol{\xi}| < k$, $0 < z\sqrt{-\eta(\boldsymbol{\xi})} = z\sqrt{k^2 - |\boldsymbol{\xi}|^2} \leq dk < \frac{\pi}{2}$, thus

$$w(\cdot, z) = \widehat{h}(\cdot) \frac{\sin(z\sqrt{-\eta(\cdot)})}{\sqrt{-\eta(\cdot)}\cos(d\sqrt{-\eta(\cdot)})} \leq \widehat{h}(\cdot) \frac{\tan(d\sqrt{-\eta(\cdot)})}{\sqrt{-\eta(\cdot)}}.$$

Since the function $\frac{\tan(x)}{x}$ is increasing for $0 < x < \frac{\pi}{2}$, it follows that for $|\boldsymbol{\xi}| < k$

$$|w(\cdot, z)| \leq |\widehat{h}(\cdot)| \frac{1}{k} \tan(dk). \tag{2.8}$$

Therefore, from (2.7) and (2.8) we have

$$|w(\cdot, z)| \leq C|\widehat{h}(\cdot)|,$$

for $C := \max\{d, \frac{1}{k}\tan(dk)\}$, and

$$\|v\|_{L^2(\Omega)} = \left(\int_0^d \|w(\cdot, z)\|^2_{L^2(\mathbb{R}^2)} dz\right)^{\frac{1}{2}} \leq \sqrt{d}C\|h\|_{L^2(\mathbb{R}^2)},$$

which ends the proof. ∎

Remark 2.2 *The assumption $dk < \frac{\pi}{2}$ means that we consider the inverse problem (2.1) in a domain Ω which depends on the parameter k appearing in the Helmholtz equation.*

From Lemma 2.1 it follows that for $kd < \frac{\pi}{2}$ the problem (2.1) can be reduced to (2.4), which by the assumption about (2.1) has the unique solution for the exact data $\widetilde{g} = g - v|_{\Gamma}$, where v is the solution of well posed problem (2.3) for the exact h. Therefore, for simplicity, we will assume later that in the considered problem (2.1)

$$h(\boldsymbol{\rho}) \equiv 0. \tag{2.9}$$

Applying the Fourier transform with respect to variables $\boldsymbol{\rho} \in \mathbb{R}^2$, we transform the problem (2.1),(2.9) to the following one in the frequency space:

$$\begin{cases} \widehat{u}_{zz}(\boldsymbol{\xi}, z) = (|\boldsymbol{\xi}|^2 - k^2)\,\widehat{u}(\boldsymbol{\xi}, z), & \boldsymbol{\xi} \in \mathbb{R}^2,\ z \in (0, d) \\ \widehat{u}(\boldsymbol{\xi}, d) = \widehat{g}(\boldsymbol{\xi}) & \boldsymbol{\xi} \in \mathbb{R}^2, \\ \partial_z \widehat{u}(\boldsymbol{\xi}, d) = 0 & \boldsymbol{\xi} \in \mathbb{R}^2. \end{cases} \tag{2.10}$$

If u is the solution of (2.1), then its Fourier transform $\widehat{u}$ is the solution of (2.10) and has the following form:

$$\widehat{u}(\boldsymbol{\xi}, z) = \widehat{g}(\boldsymbol{\xi}) \cosh((d - z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}). \tag{2.11}$$

If $|\boldsymbol{\xi}| > k$, then $|\boldsymbol{\xi}|^2 - k^2 > 0$ and

$$|\cosh\left((d - z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}\right)|^2 > \frac{1}{4}e^{2(d-z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}} > \frac{e^{2|\boldsymbol{\xi}|(d-z)}}{4e^{2k(d-z)}}.$$

So, for any fixed $z \in (0, d)$

$$\|u(\cdot, z)\|^2_{L^2(\mathbb{R}^2)} = \|\widehat{u}(\cdot, z)\|^2_{L^2(\mathbb{R}^2)} \geq c\int_{|\boldsymbol{\xi}|>k} |\widehat{g}(\boldsymbol{\xi})|^2 e^{2|\boldsymbol{\xi}|(d-z)} d\boldsymbol{\xi}, \tag{2.12}$$

with $c = 0,25e^{-2kd}$. Hence the boundednes of the norm of $u(\cdot, z)$ in $L^2(\mathbb{R}^2)$ implies the rapid decay of $\widehat{g}(\boldsymbol{\xi})$ when $|\boldsymbol{\xi}| \to \infty$. From this it follows that in the L^2-setting the considered problem is ill-posed in the Hadamard sense. Namely, let us assume that in the place of the exact data we have some measurements data g_δ with a small measurement error. We cannot expect the $\widehat{g}_\delta$ to have the same decay in frequency as the exact data $\widehat{g}$. So, the solution $u_\delta(\cdot, z)$ of the problem (2.1) with the boundary condition $u|_\Gamma = g_\delta$ does not in general exist and even if it exists, it is not continuously depending on g_δ in L^2 norm.

One of the method to stabilize the considered Cauchy problem consists in cutting off high frequencies.

3 Regularization in the frequency space

Let us introduce the following family of bounded subsets of $\mathbb{R}^2$:

$$S_\alpha = \{\boldsymbol{\xi} \in \mathbb{R}^2 : |\boldsymbol{\xi}|^2 \le \alpha\},$$

parameterized by the parameter $\alpha \in \mathbb{R}^+$. Let for $\delta \ge 0$

$$\widehat{g}^\delta_\alpha(\boldsymbol{\xi}) := \begin{cases} \widehat{g_\delta}(\boldsymbol{\xi}) & \text{for } \boldsymbol{\xi} \in S_\alpha, \\ 0 & \text{for } \boldsymbol{\xi} \in \mathbb{R}^2 \setminus S_\alpha \end{cases} \tag{3.1}$$

and $\widehat{g}_\alpha := \widehat{g}^0_\alpha$. Then $\widehat{u}^\delta_\alpha$ given by the formula

$$\widehat{u}^\delta_\alpha(\boldsymbol{\xi}, z) = \widehat{g}^\delta_\alpha(\boldsymbol{\xi}) \cosh\left((d-z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}\right) \tag{3.2}$$

is the solution of the problem (2.10) with the condition $\widehat{u}^\delta_\alpha(\boldsymbol{\xi}, d) = \widehat{g}^\delta_\alpha(\boldsymbol{\xi})$. Let $u^\delta_\alpha(\boldsymbol{\rho}, z)$ be the inverse Fourier transform (with respect to the first two variables) of $\widehat{u}^\delta_\alpha(\boldsymbol{\xi}, z)$

$$u^\delta_\alpha(\boldsymbol{\rho}, z) = \frac{1}{2\pi} \int_{R^2} \widehat{g}^\delta_\alpha(\boldsymbol{\xi}) \cosh\left((d-z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}\right) e^{i\boldsymbol{\xi}\cdot\rho} d\boldsymbol{\xi}. \tag{3.3}$$

This function will be considered as a regularized solution to the problem (2.1) (with $h \equiv 0$) where α is the parameter of regularization which should depend on the error bound δ. The regularized solution for the exact data will be denoted by $u_\alpha(\boldsymbol{\rho}, z)$.

Lemma 3.1 *If u is the exact solution of (2.1) for $h \equiv 0$ and u_α is the function defined above, then for $z \in [0, d]$*

$$\|u(\cdot, z) - u_\alpha(\cdot, z)\|_{L^2(\mathbb{R}^2)} \longrightarrow 0 \text{ as } \alpha \longrightarrow \infty.$$

Moreover, if $\alpha > k^2 - \left(\frac{\pi}{2d}\right)^2$ *and* M_0 *is a given constant such that*

$$\|u(\cdot,0)\|_{L^2(\mathbb{R}^2)} \le M_0,$$

then for $z \in (0,d]$

$$\|u(\cdot,z) - u_\alpha(\cdot,z)\|_{L^2(\mathbb{R}^2)} \le M_0 |e^{-z\kappa_\alpha}||1 + e^{-2(d-z)\kappa_\alpha}|, \tag{3.4}$$

where

$$\kappa_\alpha := \sqrt{\alpha - k^2}. \tag{3.5}$$

Proof: From (2.11) and (3.2) it follows that

$$\|u(\cdot,z) - u_\alpha(\cdot,z)\|^2_{L^2(\mathbb{R}^2)} = \int_{|\boldsymbol{\xi}|^2>\alpha} |\widehat{u}(\boldsymbol{\xi},z)|^2 d\boldsymbol{\xi} \longrightarrow 0 \text{ as } \alpha \to \infty. \tag{3.6}$$

In order to obtain the estimation (3.4) let us observe that if $|\boldsymbol{\xi}|^2 \ge k^2$, then $\zeta := \sqrt{|\boldsymbol{\xi}|^2 - k^2} \in \mathbb{R}$ and thus $\cosh \zeta d \ge 1$. Moreover, if $|\boldsymbol{\xi}|^2 < k^2$, then $\zeta = i\sqrt{k^2 - |\boldsymbol{\xi}|^2}$ and

$$\cosh \zeta d = 0 \Longleftrightarrow |\boldsymbol{\xi}|^2 = k^2 - \left(\frac{2l+1}{2}\right)^2 \frac{\pi^2}{d^2}.$$

Thus $\cosh(\zeta d) \ne 0$ for $|\boldsymbol{\xi}|^2 > k^2 - \left(\frac{\pi}{2d}\right)^2$. So, taking into account (2.11) and that $\widehat{u}_\alpha(\boldsymbol{\xi},z) = 0$ for $|\boldsymbol{\xi}|^2 > \alpha$, we can write the integral in (3.6) for $\alpha > k^2 - \left(\frac{\pi}{2d}\right)^2$ as follows:

$$\int_{|\boldsymbol{\xi}|^2>\alpha} |\widehat{u}(\boldsymbol{\xi},z) - \widehat{u}_\alpha(\boldsymbol{\xi},z)||\widehat{g}(\boldsymbol{\xi}) \cosh(d)\sqrt{|\boldsymbol{\xi}|^2 - k^2})\frac{\cosh((d-z)\sqrt{|\boldsymbol{\xi}|^2 - k^2})}{\cosh(d\sqrt{|\boldsymbol{\xi}|^2 - k^2})} d\boldsymbol{\xi},$$

which is bounded by

$$C_\alpha(z,d) \int_{|\boldsymbol{\xi}|^2>\alpha} |\widehat{u}(\boldsymbol{\xi},z) - \widehat{u}_\alpha(\boldsymbol{\xi},z)||\widehat{u}(\boldsymbol{\xi},0)| d\boldsymbol{\xi},$$

where for $z \in (0,d)$

$$C_\alpha(z,d) := \sup_{|\boldsymbol{\xi}|^2>\alpha} \left| \frac{\cosh((d-z)\sqrt{|\boldsymbol{\xi}|^2 - k^2})}{\cosh(d\sqrt{|\boldsymbol{\xi}|^2 - k^2})} \right|. \tag{3.7}$$

Thus

$$\|u(\cdot,z) - u_\alpha(\cdot,z)\|_{L^2(\mathbb{R}^2)} \le C_\alpha(z,d)\|u(\cdot,0)\|_{L^2(\mathbb{R}^2)} \le M_0 C_\alpha(z,d) \tag{3.8}$$

due to the a-priori assumption on u. From (3.7) it follows that

$$C_\alpha(z,d) = \sup_{|\boldsymbol{\xi}|^2>\alpha} \left| \frac{e^{(d-z)\zeta} + e^{-(d-z)\zeta}}{e^{d\zeta} + e^{-d\zeta}} \right| = \sup_{|\boldsymbol{\xi}|^2>\alpha} e^{-z\zeta} \left| \frac{1+e^{-2(d-z)\zeta}}{1+e^{-2d\zeta}} \right|.$$

Thus

$$C_\alpha(z,d) \le \left| e^{-z\kappa_\alpha} \right| \left| 1 + e^{-2(d-z)\kappa_\alpha} \right|,$$

which together with (3.8) gives (3.4). ∎

Remark 3.2 *Under the assumption of Lemma 2.1 the relation* $\alpha > k^2 - \left(\frac{\pi}{2d}\right)^2$ *is fulfilled for any* $\alpha \ge 0$.

Lemma 3.3 *Let* u_α *and* u_α^δ *be regularized solutions defined by (3.3) with data* g *and* g_δ*, respectively. If* $\|g - g_\delta\| \le \delta$*, then for* $z \in [0,d]$

$$\|u_\alpha(\cdot,z) - u_\alpha^\delta(\cdot,z)\|_{L^2(\mathbb{R}^2)} \le \left(e^{(d-z)\kappa_\alpha} + e^{-(d-z)\kappa_\alpha} \right) \frac{\delta}{2}, \tag{3.9}$$

where κ_α *is defined by (3.5).*

Proof: Due to (3.2) we have

$$\|u_\alpha(\cdot,z) - u_\alpha^\delta(\cdot,z)\|^2_{L^2(\mathbb{R}^2)} = \|\widehat{u}_\alpha(\cdot,z) - \widehat{u}_\alpha^\delta(\cdot,z)\|^2_{L^2(\mathbb{R}^2)} \le$$

$$\le \int_{\boldsymbol{\xi}\in S_\alpha} |\widehat{g}(\boldsymbol{\xi}) - \widehat{g}_\delta(\boldsymbol{\xi})|^2 \left| \cosh((d-z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}) \right|^2 d\boldsymbol{\xi} \le$$

$$\le \delta^2 \sup_{\boldsymbol{\xi}\in S_\alpha} \left| \cosh((d-z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}) \right|^2.$$

Let us observe that for $|\boldsymbol{\xi}|^2 < k^2$

$$\left| \cosh((d-z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}) \right| = \left| \cos((d-z)\sqrt{k^2 - |\boldsymbol{\xi}|^2}) \right| \le 1.$$

On the other hand, for $|\boldsymbol{\xi}|^2 \ge k^2$ the function $\cosh(\sqrt{|\boldsymbol{\xi}|^2 - k^2}z)$ is real and increasing, so its supremum is attained at $|\boldsymbol{\xi}|^2 = \alpha$ and is equal to $\frac{1}{2}(e^{(d-z)\kappa_\alpha} + e^{-(d-z)\kappa_\alpha})$. Thus we get the desired result. ∎

The main result of this section can be formulated as follows:

Proposition 3.4 *Let u be the exact solution of (2.1) with $h \equiv 0$ and let u_α^δ be the regularized solution (3.3) with noisy data g_δ. Let us assume that $\|u(\cdot, 0)\|_{L^2(\mathbb{R}^2)} \leq M_0$ for a-priori known constant M_0 and $\|g - g_\delta\| \leq \delta$ for $\delta \leq 2M_0$. If $\alpha = \alpha(\delta)$ is such that*

$$\kappa_{\alpha(\delta)} = -\frac{1}{d} \ln \frac{\delta}{2M_0}, \tag{3.10}$$

then for $z \in [0, d]$

$$\|u(\cdot, z) - u_\alpha^\delta(\cdot, z)\|_{L^2(\mathbb{R}^2)} \longrightarrow 0 \text{ as } \delta \longrightarrow 0 \tag{3.11}$$

and for $z \in (0, d]$

$$\|u(\cdot, z) - u_\alpha^\delta(\cdot, z)\|_{L^2(\mathbb{R}^2)} \leq \delta + 2M_0^{\frac{d-z}{d}} \delta^{\frac{z}{d}}. \tag{3.12}$$

PROOF: If $\alpha = \alpha(\delta)$ satisfies (3.10), then $\kappa_\alpha \in \mathbb{R}^+$ and, due to Lemmas 3.1 and 3.3, we have

$$\|u(\cdot, z) - u_\alpha^\delta(\cdot, z)\|_{L^2(\mathbb{R}^2)} \leq \|u(\cdot, z) - u_\alpha(\cdot, z)\|_{L^2(\mathbb{R}^2)} + \|u_\alpha(\cdot, z) - u_\alpha^\delta(\cdot, z)\|_{L^2(\mathbb{R}^2)} \leq$$

$$\leq \left(M_0 e^{-d\kappa_\alpha} + \frac{\delta}{2} \right) \left(e^{(d-z)\kappa_\alpha} + e^{-(d-z)\kappa_\alpha} \right).$$

From (3.10) for $\alpha = \alpha(\delta)$

$$e^{-d\kappa_\alpha} = \frac{\delta}{2M_0},$$

thus

$$\|u(\cdot, z) - u_\alpha^\delta(\cdot, z)\|_{L^2(\mathbb{R}^2)} \leq \delta(e^{(d-z)\kappa_\alpha} + e^{-(d-z)\kappa_\alpha}).$$

Since

$$e^{(d-z)\kappa_\alpha} = \left(\frac{\delta}{2M_0} \right)^{-\frac{d-z}{d}},$$

so, taking into account that $\delta < 2M_0$ and $0 \leq \frac{(d-z)}{d} \leq 1$, we obtain

$$\|u(\cdot, z) - u_\alpha^\delta(\cdot, z)\|_{L^2(\mathbb{R}^2)} \leq \delta \left(\left(\frac{\delta}{2M_0} \right)^{-\frac{(d-z)}{d}} + \left(\frac{\delta}{2M_0} \right)^{\frac{(d-z)}{d}} \right) \leq$$

$$\leq \delta \left(1 + 2M_0^{\frac{(d-z)}{d}} \delta^{-\frac{(d-z)}{d}} \right),$$

which completes the proof of Proposition 3.4. ∎

Remark 3.5 *The assumption (3.10) gives an explicite formula for the regularization parameter α dependent on the data error δ:*

$$\alpha(\delta) = k^2 + \frac{1}{d^2}(\ln\frac{\delta}{2M_0})^2. \tag{3.13}$$

Remark 3.6 *From Proposition 3.4 it follows that the method gives convergent approximation of $u(\boldsymbol{\xi}, z)$, and for $z \in (0, d)$ the error estimation is of the order $O(\delta^{\frac{z}{d}})$.*

So far we have assumed that $h = 0$, i.e. that the normal derivative is zero on the surface Γ. A general problem for $k < \frac{\pi}{2d}$ can always be reduced to this case, as explained in Section 2. However, in such a case, for computation we should take a new function $\widetilde{g_\delta} = g_\delta - v_\delta|_\Gamma$ as a boundary conditions on Γ, where v_δ is the solution of the boundary value problem (2.3) with an approximately given normal derivative h_δ on Γ.

In practice, measurements of the field on the boundary Γ are available, while those of the normal derivative $\partial_z u$ are not. Approximate values of normal derivatives on Γ can be found by performing additional measurements of the field on a surface $\Gamma_1 = \{(\boldsymbol{\rho}, d_1), \boldsymbol{\rho} \in \mathbb{R}^2\}$ for sufficiently small $d_1 - d > 0$, and solving the following Dirichlet problem:

$$\begin{cases} \Delta u(\mathbf{r}) + k^2 u(\mathbf{r}) = 0 \quad \text{in } \Omega_1 = \{(\boldsymbol{\rho}, z) : \boldsymbol{\rho} \in \mathbb{R}^2, z \in (d, d_1)\}, \\ u|_\Gamma = g, \\ u|_{\Gamma_1} = f, \end{cases} \tag{3.14}$$

in $H^2(\Omega_1)$ with approximately given boundary values g and f. The trace of $\frac{\partial u}{\partial z}$ on Γ will be the desired approximation of h. The problem is well posed in the following sense:

Lemma 3.7 *Let $(d_1 - d)k < \frac{\pi}{2}$. If $g \in H^1(\mathbb{R}^2)$ and $f \in L^2(\mathbb{R}^2)$, then*

$$\|\frac{\partial u}{\partial z}(\cdot, d)\| \leq C \left(\|g\|_{H^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)}\right).$$

PROOF: For simplicity let us put $d = 0$ and $d_1 = q$ and let us transform the problem (3.14) to the following one in the frequency space (cf. the proof of Lemma 2.1):

$$\begin{cases} \widehat{w}_{zz}(\boldsymbol{\xi}, z) = \eta(\boldsymbol{\xi})\widehat{w}(\boldsymbol{\xi}, z), & \boldsymbol{\xi} \in \mathbb{R}^2, \ z \in (0, q) \\ \widehat{w}(\boldsymbol{\xi}, 0) = \widehat{g} & \boldsymbol{\xi} \in \mathbb{R}^2, \\ \widehat{w}(\boldsymbol{\xi}, q) = \widehat{f}(\boldsymbol{\xi}) & \boldsymbol{\xi} \in \mathbb{R}^2, \end{cases} \tag{3.15}$$

where $\eta(\boldsymbol{\xi}) := |\boldsymbol{\xi}|^2 - k^2$, $w(\boldsymbol{\xi}, z) = \widehat{u}(\boldsymbol{\xi}, z)$ and u is the solution of (3.14). Let

$$\phi(\cdot, z) := \frac{\sinh(z\sqrt{\eta(\cdot)})}{\sinh(q\sqrt{\eta(\cdot)})}.$$

Then

$$w(\cdot, z) = \widehat{g}(\cdot)\phi(\cdot, q-z) + \widehat{f}(\cdot)\phi(\cdot, z),$$

and

$$\frac{\partial w(\cdot, z)}{\partial z} = -\widehat{g}(\cdot)\phi'_z(\cdot, q-z) + \widehat{f}(\cdot, q-z)\phi'_z(\cdot, z),$$

where

$$\phi'_z(\cdot, z) = \sqrt{\eta(\cdot)}\frac{\cosh(z\sqrt{\eta(\cdot)})}{\sinh(q\sqrt{\eta(\cdot)})}.$$

From this we have

$$\frac{\partial w(\cdot, 0)}{\partial z} = \sqrt{\eta(\cdot)}\left[-\widehat{g}(\cdot)\tanh(q\sqrt{\eta(\cdot)}) + \widehat{f}(\cdot)\frac{1}{\sinh(q\sqrt{\eta(\cdot)})}\right]. \tag{3.16}$$

For $|\boldsymbol{\xi}| > k$ the function $\eta(\boldsymbol{\xi})$ is positive. Therefore, $\tanh(q\sqrt{\eta(\cdot)}) < 1$ and, since the function $\frac{x}{\sinh(x)}$ is decreasing for $x \in (0, \infty)$,

$$\frac{\sqrt{\eta(\cdot)}}{\sinh(q\sqrt{\eta(\cdot)})} \le \frac{1}{q}.$$

Thus

$$|w'_z(\boldsymbol{\xi}, 0)| \le |\boldsymbol{\xi}||\widehat{g}(\boldsymbol{\xi})| + q^{-1}|\widehat{f}(\boldsymbol{\xi})| \qquad \text{for } |\boldsymbol{\xi}| > k. \tag{3.17}$$

On the other hand, for $|\boldsymbol{\xi}| < k$, the formula (3.16) takes the form

$$w'_z(\cdot, 0) = \sqrt{-\eta(\cdot)}\left[-\widehat{g}(\cdot)\tan(q\sqrt{-\eta(\cdot)}) + \widehat{f}(\cdot)\frac{1}{\sin(q\sqrt{-\eta(\cdot)})}\right].$$

Taking into account that $q\sqrt{-\eta(\boldsymbol{\xi})} \le qk < \frac{\pi}{2}$ and $\frac{x}{\sin x}$ is increasing for $x \in (0, \frac{\pi}{2})$, we get

$$|w'_z(\boldsymbol{\xi}, 0)| \le k\tan(qk)|\widehat{g}(\boldsymbol{\xi})| + \frac{k}{\sin(qk)}|\widehat{f}(\boldsymbol{\xi})| \text{ for } |\boldsymbol{\xi}| < k. \tag{3.18}$$

So, it is easily seen that for certain constants C_1, C_2

$$\|w'_z(\cdot, 0\|^2_{L^2(\mathbb{R}^2)} \le C_1\int_{\mathbb{R}^2}(1 + |\boldsymbol{\xi}|)^2|\widehat{g}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} + C_2\int_{\mathbb{R}^2}|\widehat{f}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}, \tag{3.19}$$

which gives the desired conclusion. ∎

4 Numerical implementation

As it was mentioned in Section 1, in certain physical situations we can expect that the data function is almost zero outside a certain small domain $\Gamma_2 \subset \mathbb{R}^2$. For simplicity (but without lost of generality) let us assume that $\Gamma_2 = (0, 2\pi)^2$ and the support of the measured data g_δ is contained in Γ_2. In such a case $g_\delta|_{\Gamma_2} \in L^2(\Gamma_2)$ and it has in this space the following representation in the trigonometric orthonormal basis $\{\frac{1}{2\pi} e^{i\boldsymbol{m}\cdot\boldsymbol{\rho}}\}_{\boldsymbol{m}\in\mathbb{Z}^2}$

$$g_\delta(\boldsymbol{\rho}) = \frac{1}{2\pi} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \widehat{g}_{\delta,\boldsymbol{m}} e^{i\boldsymbol{m}\cdot\boldsymbol{\rho}}, \text{ for } \boldsymbol{\rho} \in \Gamma_2, \tag{4.1}$$

where $\mathbb{Z}$ denotes the integers and

$$\widehat{g}_{\delta,\boldsymbol{m}} = \frac{1}{2\pi} \int_{\Gamma_2} g_\delta(\boldsymbol{\rho}) e^{-i\boldsymbol{m}\cdot\boldsymbol{\rho}} d\boldsymbol{\rho} = \widehat{g_\delta}(\boldsymbol{m}),$$

i.e. $\widehat{g}_{\delta\boldsymbol{m}}$ is the Fourier transform of g_δ at the point $\boldsymbol{\xi} = \boldsymbol{m}$ and

$$\widehat{g_\delta}(\boldsymbol{\xi}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \delta(\boldsymbol{\xi} - \boldsymbol{m}) \widehat{g}_{\delta,\boldsymbol{m}},$$

where $\delta(\boldsymbol{\xi} - \boldsymbol{m})$ is the Dirac δ-function at $\boldsymbol{m}$. Let $\mathbb{Z}^2_\alpha = S_\alpha \cap \mathbb{Z}^2$. According to the definition (3.1), the function g^δ_α is defined by its Fourier transform

$$\widehat{g^\delta_\alpha}(\boldsymbol{\xi}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^2_\alpha} \delta(\boldsymbol{\xi} - \boldsymbol{m}) \widehat{g}_{\delta,\boldsymbol{m}}.$$

Due to (3.2)

$$\widehat{u^\delta_\alpha}(\boldsymbol{\xi}, z) = \widehat{g^\delta_\alpha}(\boldsymbol{\xi}) \cosh\left((d-z)\sqrt{|\boldsymbol{\xi}|^2 - k^2}\right),$$

thus applying the inverse Fourier transform we easily get

$$u^\delta_\alpha(\boldsymbol{\rho}, z) = \frac{1}{2\pi} \sum_{\boldsymbol{m}\in\mathbb{Z}^2_\alpha} \widehat{g}_{\delta,\boldsymbol{m}} cosh((d-z)\sqrt{|\boldsymbol{m}|^2 - k^2}) e^{i\boldsymbol{m}\cdot\boldsymbol{\rho}}. \tag{4.2}$$

If the assumptions of Proposition 3.4 are satisfied and the parameter α is chosen according to the relation (3.13), then the norm of difference between the exact solution and the regularized solution given above is bounded by $\delta + 2M_0^{\frac{d-z}{d}} \delta^{\frac{z}{d}}$ for any $z \in (0, d)$.

The above result yields to a simple numerical application of this method in the case when the data function g is almost 0 outside Γ_2 and $h \equiv 0$. Numerical experiments are presented in Section 4.1.

4.1 Numerical experiments

In this section we give some numerical examples demonstrating how the considered method works.

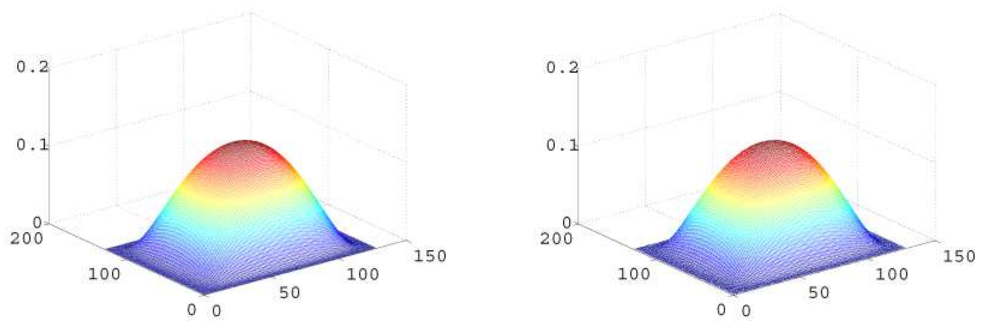

Figure 4.1: The exact data g (left) and the noisy data g_δ with $\delta = 10^{-3}$ (right)

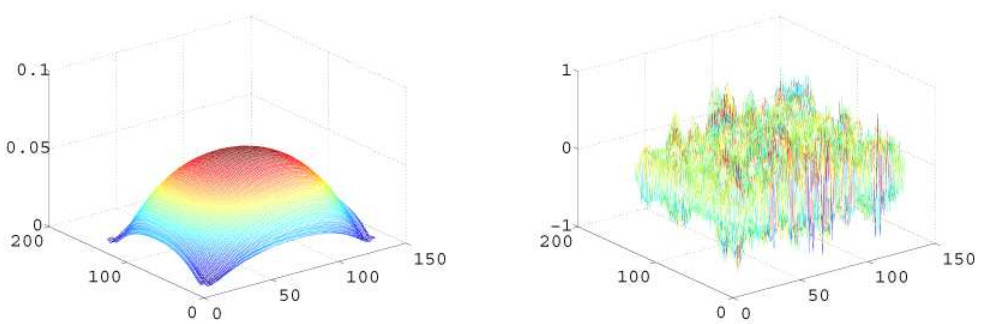

Figure 4.2: The solution $u(\cdot, 0.8)$ (left) and unregularized solution reconstructed from g_δ (right) for $z = 0.8$

In our test problem $d = 1$ and $k = 4$ and the function

$$g(x, y) = \exp\left(-\frac{\pi^2}{\pi^2 - x^2}\right) \exp\left(-\frac{\pi^2}{\pi^2 - y^2}\right)$$

on $\Gamma_2 = (-\pi, \pi)^2$ is taken as an exact data function on Γ. A normally distributed noise of variance 10^{-3} was added to g giving g_δ (cf. Figure 4.1). Given is a matrix G_δ containing samples from g_δ on an equidistant grid $\{t_{s,l}\}_{s,l=0}^{n}$ of Γ_2 for $n = 128$. Using the fast Fourier transform (FFT) to the matrix G_δ we compute $n \times n$ matrix $\widehat{G_\delta}$ approximating the Fourier transform of g_δ. Next, the frequency components corresponding to $s > \sqrt{\alpha}$, $l > \sqrt{\alpha}$

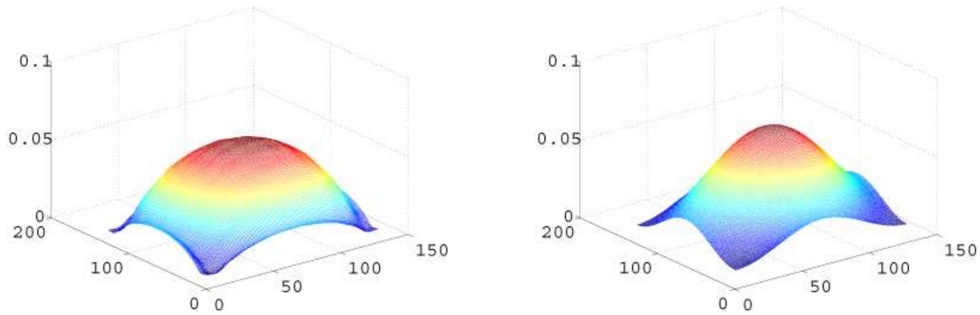

Figure 4.3: The regularized solutions at $z = 0.8$ for $\sqrt{\alpha} = 10$ (left) and $\sqrt{\alpha} = 2$ (right).

are explicitly set to 0. Finally, to this new matrix $\widehat{G_{\delta\alpha}}$ multiplied by the matrix $[cosh((d-z)\sqrt{|\boldsymbol{m}|^2 - k^2})]_{\boldsymbol{m}\in(0,n)\times(0,n)}$ the inverse FFT is applied. As a result we obtain approximate values of regularized solution $u_\alpha^\delta(\cdot, z)$ on the grid $\{t_{s,l}\}_{s,l=0}^n$. In Figure 4.2 we display for the fixed point $z = 0.8$ and $(x, y) \in \Gamma_2$ the solution $u(\cdot, z)$ reconstructed from the exact q for $\sqrt{\alpha} = 40$ and the reconstructed solution $u_\delta(\cdot, z)$ from the noisy data g_δ without regularization. We see that u_δ does not approximate the solution and some regularization procedure is necessary. In Figure 4.3 the regularized solutions defined by the regularization parameter $\sqrt{\alpha} = 10$ and $\sqrt{\alpha} = 2$ are presented. It can be observed, that for too small α the difference between the exact and regularized solution again increases. In the next two figures we show how the method works as the noise increases. The figures correspond to a noise of variance 0.005 and 0.01, respectively. In order to compare regularization results for several values of regularization parameter, we display the cross-section of the function plots by the plane $y = 0$. In Figure 4.4 the regularized solutions for $\sqrt{\alpha} = 20, 15, 10$ and a noise of variance 0.005 and the solution reconstructed from the exact g for $\sqrt{\alpha} = 40$ are displayed. Similarly, in Figure 4.5, for a noise of variance 0.01, regularized solutions for $\sqrt{\alpha} = 15, 7, 2$ are compared with $u(\cdot, 0.8)$. We see that, if the greater δ, the smaller the appropriate regularization parameter α.

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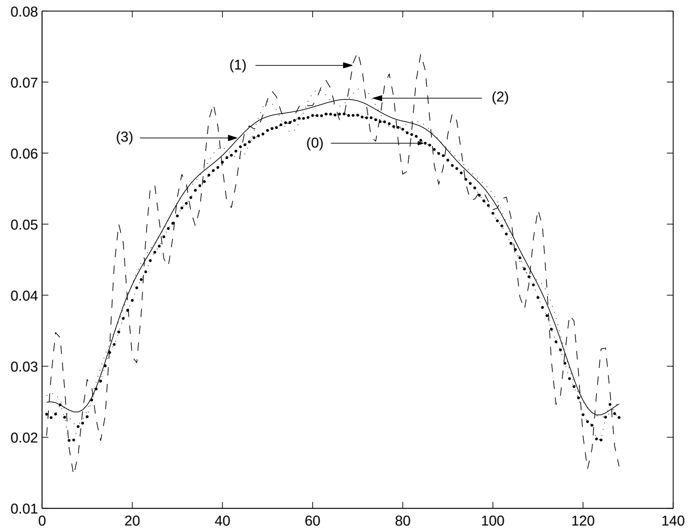

Figure 4.4: $\delta = 0.005$. The regularized solutions $u_\alpha^\delta(x, 0, 0.8)$ for $\sqrt{\alpha} = 20, 15, 10$ are denoted by (1), (2), (3), respectively. The solution $u_\alpha(x, 0, 0.8)$ for $\sqrt{\alpha} = 40$ is denoted by (0).

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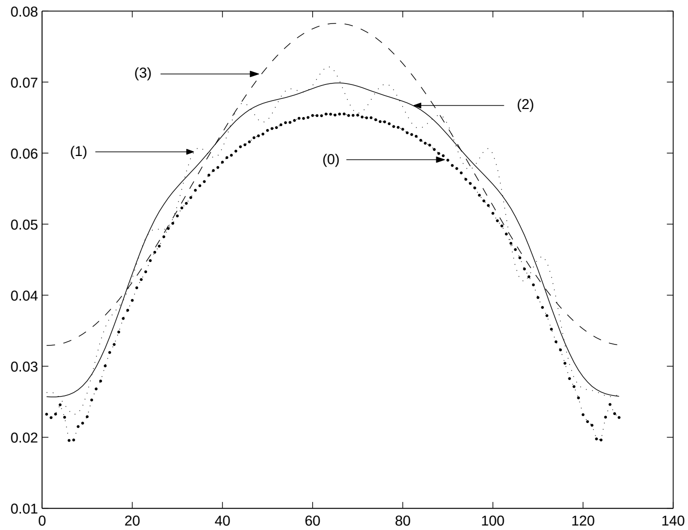

Figure 4.5: $\delta = 0.01$. The regularized solutions $u_\alpha^\delta(x, 0, 0.8)$ for $\sqrt{\alpha} = 15, 7, 2$ are denoted by (1), (2), (3), respectively. The solution $u_\alpha(x, 0, 0.8)$ for $\sqrt{\alpha} =$ 40 is denoted by (0).

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