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Invariant measures for a class of stochastic evolution equations

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Invariant measures for a class of stochastic evolution equations *

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Abstract

We give a sufficient condition for the existence of an invariant measure for a stochastic evolution equation with noise driven by a Lévy process.

1 Introduction

We consider a stochastic evolution equation on a separable Hilbert space H given by

$$\begin{aligned} dX &= (AX + F(X))dt + B(X)dZ(t), \\ X(0) &= \eta, \end{aligned} \qquad (*)$$

where $\eta \in H$, A is a linear operator, F is a bounded mapping from H into H, Z takes values in a separable Hilbert space U and B is a bounded mapping from H into space of linear continuos operators from U into H.

We extend Theorem 6.3.2 from [1] which gives a sufficient condition for the existence of an invariant measure for $(*)$ in the case that Z is a Wiener process. We use methods used in the proof of Theorem 6.3.2 and derive a sufficient condition for the existence of an invariant measure in the general case when Z is a Lévy process. We also show that this condition in a form involving Lipschitz constants is weaker than an analogous condition given by Gaans in [3].

2 Preliminaries

We will consider processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $Z(t)$ be a Lévy process (i.e. a process with independent and stationary increments) taking values in a separable Hilbert space $(U, \|\cdot\|_U)$. Associated with $Z(t)$ are two measures on U: the measure of jumps of Z, denoted μ, and the so-called Lévy measure of Z, denoted ν, given by

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$$\mu\left([0,t],\Gamma\right)=\sum_{0<s\leqslant t}\mathbf{1}_{\Gamma}\left(Z(s)-Z\left(s^{-}\right)\right),$$
$$t\nu\left(\Gamma\right)=\mathbb{E}\left(\mu\left([0,t],\Gamma\right)\right),$$

where Γ is a Borel subset of U such that $\overline{\Gamma}\subset U\setminus\{0\}$. It turns out that $\nu\left(\{0\}\right)=0$ and $\int\limits_U\min\left(\|y\|_U^2,1\right)\nu\left(dy\right)<\infty$. $Z(t)$ can be represented as

$$Z(t)=at+W(t)+\int\limits_0^t\int\limits_{\|y\|_U\leqslant 1}y\left(\mu\left(dy,ds\right)-\nu\left(dy\right)ds\right)+\int\limits_0^t\int\limits_{\|y\|_U>1}y\mu\left(dy,ds\right),$$

where $a\in U$, W is a Wiener process taking values in U, with covariance operator Q. We consider another separable Hilbert space $(H,\|\cdot\|)$. Let $L(H)$ denote the space of linear continuous operators from H into H, and let $L(U,H)$ denote the space of linear continuos operators from U into H. We consider a stochastic equation on H of the form

$$\begin{aligned}dX&=(AX+F(X))dt+B(X)dZ(t),\\X(0)&=\eta,\end{aligned}\qquad(*)$$

where $\eta\in H$, A is a linear operator, with dense domain, which in general may be unbounded, F is a bounded mapping from H into H and B is a bounded mapping from H into $L(U,H)$. We introduce the following conditions:

(i) $c:=\int\limits_U\|y\|_U^2\,\nu(dy)<\infty$,

(ii) A is the infinitesimal generator of a strongly continuous semigroup on H,

(iii) there exists $L_F>0$ such that

$$\|F(x)-F(y)\|\leqslant L_F\|x-y\|,$$

(iv) there exists $L_B>0$ such that

$$\|B(x)-B(y)\|_{L(U,H)}\leqslant L_B\|x-y\|.$$

Condition (i) implies the existence of $\int\limits_{\|y\|_U>1}y\nu(dy)$. Indeed, we have

$$\int\limits_{\|y\|_U>1}\|y\|_U\,\nu(dy)\leqslant\int\limits_{\|y\|_U>1}\|y\|_U^2\,\nu(dy)\leqslant\int\limits_U\|y\|_U^2\,\nu(dy)<\infty.$$

So there exists $b:=\int\limits_{\|y\|_U>1}y\nu(dy)\in U$. Then

$$\begin{aligned}Z(t)&=at+W(t)+\int\limits_0^t\int\limits_{\|y\|_U\leqslant 1}y\left(\mu\left(dy,ds\right)-\nu\left(dy\right)ds\right)+\int\limits_0^t\int\limits_{\|y\|_U>1}y\mu\left(dy,ds\right)\\&\quad-\int\limits_0^t\int\limits_{\|y\|_U>1}y\nu(dy)ds+bt\\&=(a+b)t+W(t)+\iint\limits_{0\,U}^{t}y\left(\mu\left(dy,ds\right)-\nu\left(dy\right)ds\right).\end{aligned}$$

So $\mathbb{E}\, Z(1) = a + b$, and $\operatorname{Var} Z(1) = \operatorname{Var} W(1) + \int_0^1 \int_U \|y\|^2 \, \nu\,(dy)\, ds = \operatorname{Tr} Q + c$.

For process $Z(t), t \geqslant 0$, let $\overline{Z}(t), t \in \mathbb{R}$, denote process defined by

$$\overline{Z}(t) = \begin{cases} Z(t) & t \geqslant 0, \\ Z_2(-t) & t < 0, \end{cases} \tag{2.1}$$

where $(Z_2(t))_{t \geqslant 0}$ is a Lévy process with the same distribution as $(Z(t))_{t \geqslant 0}$ and independent of $(Z(t))_{t \geqslant 0}$.

3 Sufficient condition for the existence of an invariant measure

Theorem 3.1. *Assume that Z, A, F and B satisfy conditions* (i), (ii), (iii), (iv) *and let $A_n = nA(n - A)^{-1}, n \in \mathbb{N}$, be the sequence of Yosida approximations of A. If there exists $N, \omega > 0$ such that for every x, y in H and $n > N$*

$$2 \langle A_n(x-y) + F(x) - F(y) + (B(x) - B(y))\, \mathbb{E}\, Z(1), x - y \rangle + \operatorname{Var} Z(1) \, \|B(x) - B(y)\|^2_{L(U,H)} \leqslant -\omega \, \|x - y\|^2 , \tag{A}$$

then there exists an invariant measure for the equation

$$\begin{aligned} dX &= (AX + F(X))dt + B(X)dZ(t), \\ X(0) &= \eta. \end{aligned} \tag{$*$}$$

First we prove two lemmas.

Lemma 3.2. *Assume that Z satisfies condition* (i) *and $\mathbb{E}\, Z(1) = 0$. Let $\overline{Z}(t), t \in \mathbb{R}$, be defined by (2.1). If $dY(t) = \alpha(t)dt + \beta(t)d\overline{Z}(t)$, where $\alpha(t) \in H$ and $\beta(t) \in L(U,H)$ for $t \in \mathbb{R}$, then*

$$\frac{d}{dt} \mathbb{E} \, \|Y(t)\|^2 \leqslant \mathbb{E} \left(2 \langle Y(t), \alpha(t) \rangle + \operatorname{Var} Z(1) \, \|\beta(t)\|^2_{L(U,H)} \right).$$

Proof of Lemma 3.2. Applying Itô's lemma to the function $\varphi(x) = \|x\|^2$, we obtain

$$\|Y(t)\|^2 = \|Y(t_0)\|^2 + \int_{t_0}^t 2 \left\langle Y\left(s^-\right), dY(s) \right\rangle + \int_{t_0}^t \operatorname{Tr}\left(\beta(s) Q \left(\beta(s)\right)^*\right) ds + \int_{t_0}^t \int_U \psi(s,y) \mu_Y(dy, ds),$$

where $\psi(s,y) = \varphi\left(Y\left(s^-\right) + y\right) - \varphi\left(Y\left(s^-\right)\right) - D\varphi\left(Y\left(s^-\right)\right) y = \|y\|^2$ and μ_Y denotes the measure of jumps of Y. Hence

$$\begin{aligned} \|Y(t)\|^2 = \|Y(t_0)\|^2 &+ \int_{t_0}^t 2 \left\langle Y\left(s^-\right), dY(s) \right\rangle \\ &+ \int_{t_0}^t \operatorname{Tr}\left(\beta(s) Q \left(\beta(s)\right)^*\right) ds + \int_{t_0}^t \int_U \|\beta(s) y\|^2 \, \mu\,(dy, ds), \end{aligned} \tag{3.1}$$

as $\int_{t_0}^{t}\int_U \psi(s,y)\mu_Y(dy,ds) = \int_{t_0}^{t}\int_U \psi\left(s,\beta\left(s\right)y\right)\mu\left(dy,ds\right)$. We have

$$\left\langle Y\left(s^-\right),dY(s)\right\rangle = \left\langle Y\left(s^-\right),\alpha(s)\right\rangle ds + \left\langle Y\left(s^-\right),\beta(s)dW(s)\right\rangle + \left\langle Y\left(s^-\right),\beta(s)\int_U y\left(\mu\left(dy,ds\right)-\nu\left(dy\right)ds\right)\right\rangle,$$

so

$$\mathbb{E}\int_{t_0}^{t} 2\left\langle Y\left(s^-\right),dY(s)\right\rangle = \mathbb{E}\int_{t_0}^{t} 2\left\langle Y\left(s^-\right),\alpha(s)\right\rangle ds.$$

Let $B_s := \{Y(s)\neq Y\left(s^-\right)\}$. Then

$$\mathbb{E}\left\langle Y(s)-Y\left(s^-\right),\alpha(s)\right\rangle = \mathbb{E}\left(\mathbf{1}_{B_s}\left\langle Y(s)-Y\left(s^-\right),\alpha(s)\right\rangle\right)=0,$$

since $\mathbb{P}\left(B_s\right)=0$. Hence

$$\mathbb{E}\int_{t_0}^{t} 2\left\langle Y\left(s^-\right),dY(s)\right\rangle = \mathbb{E}\int_{t_0}^{t} 2\left\langle Y(s),\alpha(s)\right\rangle ds. \tag{3.2}$$

Given that $\mathrm{Tr}\left(\beta(s)Q\left(\beta(s)\right)^*\right)\leqslant \|\beta(s)\|^2_{L(U,H)}\,\mathrm{Tr}Q$, we have

$$\mathbb{E}\int_{t_0}^{t}\mathrm{Tr}\left(\beta(s)Q\left(\beta(s)\right)^*\right)ds \leqslant \mathrm{Tr}Q\,\mathbb{E}\int_{t_0}^{t}\|\beta(s)\|^2_{L(U,H)}\,ds. \tag{3.3}$$

On the other hand,

$$\begin{aligned}\mathbb{E}\int_{t_0}^{t}\int_U \|\beta(s)y\|^2\mu\left(dy,ds\right) &= \mathbb{E}\int_{t_0}^{t}\int_U \|\beta(s)y\|^2\nu\left(dy\right)ds\\ &\leqslant \mathbb{E}\int_{t_0}^{t}\int_U \|\beta(s)\|^2_{L(U,H)}\|y\|^2\nu\left(dy\right)ds\\ &= \int_U \|y\|^2_U\,\nu\left(dy\right)\mathbb{E}\int_{t_0}^{t}\|\beta(s)\|^2_{L(U,H)}\,ds.\end{aligned}$$

So,

$$\mathbb{E}\int_{t_0}^{t}\int_U \|\beta(s)y\|^2\mu\left(dy,ds\right)\leqslant c\,\mathbb{E}\int_{t_0}^{t}\|\beta(s)\|^2_{L(U,H)}\,ds. \tag{3.4}$$

Combining (3.1), (3.2), (3.3) and (3.4), we obtain

$$\mathbb{E}\|Y(t)\|^2\leqslant \mathbb{E}\|Y\left(t_0\right)\|^2+\mathbb{E}\int_{t_0}^{t}\left(2\left\langle Y(s),\alpha(s)\right\rangle+\left(\mathrm{Tr}Q+c\right)\|\beta(s)\|^2\right)ds,$$

from which, since $\mathrm{Var}\,Z(1)=\mathrm{Tr}Q+c$, we finally get

$$\frac{d}{dt}\mathbb{E}\|Y(t)\|^2\leqslant \mathbb{E}\left(2\left\langle Y(t),\alpha(t)\right\rangle+\mathrm{Var}\,Z(1)\,\|\beta(t)\|^2\right).$$

□

Lemma 3.3. *If B satisfies condition* (iv) *and there exist $N, \omega > 0$ such that for every x, y in H and $n > N$*

$$2\langle A_n(x-y) + F(x) - F(y), x-y\rangle + \operatorname{Var} Z(1)\,\|B(x)-B(y)\|^2_{L(U,H)} \leqslant -\omega\,\|x-y\|^2, \tag{A0}$$

then for some $C_1 > 0$

$$2\langle A_n x + F(x), x\rangle + \operatorname{Var} Z(1)\,\|B(x)\|^2_{L(U,H)} \leqslant -\frac{\omega}{2}\,\|x\|^2 + C_1$$

for every x in H and $n > N$.

Proof of Lemma 3.3. Let $\lambda := \operatorname{Var} Z(1)$. By (A0) with $y = 0$, we have

$$\begin{aligned}
-\omega\,\|x\|^2 &\geqslant 2\langle A_n x + F(x) - F(0), x\rangle + \lambda\,\|B(x)-B(0)\|^2_{L(U,H)}\\
&\geqslant 2\langle A_n x + F(x), x\rangle - 2\,\|F(0)\|\,\|x\| + \lambda\left(\|B(x)\|_{L(U,H)} - \|B(0)\|_{L(U,H)}\right)^2\\
&= 2\langle A_n x + F(x), x\rangle - 2\,\|F(0)\|\,\|x\|\\
&\quad + \lambda\Big(\,\|B(x)\|^2_{L(U,H)} - \|B(0)\|^2_{L(U,H)}\\
&\qquad - 2\,\|B(0)\|_{L(U,H)}\,(\|B(x)\|_{L(U,H)} - \|B(0)\|_{L(U,H)})\Big),
\end{aligned}$$

so

$$\begin{aligned}
2\langle A_n x + F(x), x\rangle + \lambda\,\|B(x)\|^2_{L(U,H)} &\leqslant -\omega\,\|x\|^2 + 2\,\|F(0)\|\,\|x\| + \lambda\,\|B(0)\|^2_{L(U,H)}\\
&\quad + 2\lambda\,\|B(0)\|_{L(U,H)}\left(\|B(x)\|_{L(U,H)} - \|B(0)\|_{L(U,H)}\right)\\
&\leqslant -\omega\,\|x\|^2 + 2\,\|F(0)\|\,\|x\| + \lambda\,\|B(0)\|^2_{L(U,H)}\\
&\quad + 2\lambda\,\|B(0)\|_{L(U,H)}\,L_B\,\|x\|\\
&= -\frac{\omega}{2}\,\|x\|^2 - \frac{\omega}{2}\,\|x\|^2 + \lambda\,\|B(0)\|^2_{L(U,H)}\\
&\quad + 2\left(\|F(0)\| + \lambda\,\|B(0)\|_{L(U,H)}\,L_B\right)\|x\|.
\end{aligned}$$

Since $ar^2 + br + c \leqslant -\frac{b^2-4ac}{4a} = c - \frac{1}{a}\frac{b^2}{4}$, we have

$$2\langle A_n x + F(x), x\rangle + \lambda\,\|B(x)\|^2_{L(U,H)} \leqslant -\frac{\omega}{2}\,\|x\|^2 + C_1,$$

where

$$C_1 = \lambda\,\|B(0)\|^2_{L(U,H)} + \frac{2}{\omega}\left(\|F(0)\| + \lambda\,\|B(0)\|_{L(U,H)}\,L_B\right)^2.$$

□

Proof of Theorem 3.1. I. First assume that $\mathbb{E}\,Z(1) = 0$.

Let $\overline{Z}(t), t \in \mathbb{R}$, be defined by (2.1) and let $A_n = nA(n-A)^{-1}, n \in \mathbb{N}$, be the sequence of Yosida approximations of A. Denote by $X_n(t, s, \eta)$ the solution of the equation

$$\begin{aligned}
dX_n &= (A_n X_n + F(X_n))dt + B(X_n)d\overline{Z}(t),\\
X_n(s) &= \eta,
\end{aligned}$$

and by $X(t,s,\eta)$ the solution of the equation

$$\begin{aligned} dX &= (AX + F(X))dt + B(X)d\overline{Z}(t), \\ X(s) &= \eta. \end{aligned} \tag{$*$}$$

$X_n(t,s,\eta)$ converges in $L_2(\Omega)$ to $X(t,s,\eta)$. Fix $s \in \mathbb{R}$ and let $X_n(t) = X_n(t,s,\eta)$. Then $dX_n = (A_n X_n + F(X_n))\, dt + B(X_n) d\overline{Z}(t)$. We apply Lemma 3.2 with

$$\begin{aligned} Y(t) &= X_n(t), \\ \alpha(t) &= A_n X_n(t) + F(X_n(t)), \\ \beta(t) &= B(X_n(t)). \end{aligned}$$

By Lemma 3.3,

$$2\langle Y(t), \alpha(t)\rangle + \operatorname{Var} Z(1) \|\beta(t)\|^2_{L(U,H)} \leqslant -\frac{\omega}{2} \|X_n(t)\|^2 + C_1,$$

from which

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \|X_n(t)\|^2 &\leqslant \mathbb{E} \left(2\langle Y(t), \alpha(t)\rangle + \operatorname{Var} Z(1) \|\beta(t)\|^2_{L(U,H)} \right) \\ &\leqslant -\frac{\omega}{2} \mathbb{E} \|X_n(t)\|^2 + C_1. \end{aligned}$$

By Gronwall's lemma,

$$\mathbb{E} \|X_n(t)\|^2 \leqslant \frac{2C_1}{\omega} + \mathbb{E} \|X_n(s)\|^2,$$

so for every $s \in \mathbb{R}$ and every $t \geqslant s$

$$\mathbb{E} \|X_n(t,s,\eta)\|^2 \leqslant \frac{2C_1}{\omega} + \|\eta\|^2. \tag{3.5}$$

Now fix $\delta > \gamma > 0$ and let $U_n(t) = X_n(t, -\gamma, \eta)$, $V_n(t) = X_n(t, -\delta, \eta)$. Then

$$d(U_n - V_n) = (A(U_n - V_n) + F(U_n) - F(V_n))\, dt + (B(U_n) - B(V_n))\, d\overline{Z}(t).$$

We apply Lemma 3.2 with

$$\begin{aligned} Y(t) &= U_n(t) - V_n(t), \\ \alpha(t) &= A_n (U_n(t) - V_n(t)) + F(U_n(t)) - F(V_n(t)), \\ \beta(t) &= B(U_n(t)) - B(V_n(t)). \end{aligned}$$

By (A),

$$2\langle Y(t), \alpha(t)\rangle + \operatorname{Var} Z(1) \|\beta(t)\|^2_{L(U,H)} \leqslant -\omega \|U_n(t) - V_n(t)\|^2,$$

from which

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \|U_n(t) - V_n(t)\|^2 &\leqslant \mathbb{E} \left(2\langle Y(t), \alpha(t)\rangle + \operatorname{Var} Z(1) \|\beta(t)\|^2_{L(U,H)} \right) \\ &\leqslant -\omega \mathbb{E} \|U_n(t) - V_n(t)\|^2. \end{aligned}$$

By Gronwall's lemma, for every $t \geqslant s$

$$\mathbb{E} \|U_n(t) - V_n(t)\|^2 \leqslant e^{-\omega(t-s)} \mathbb{E} \|U_n(s) - V_n(s)\|^2.$$

Letting $t = 0$ and $s = -\gamma$, we obtain

$$\begin{aligned}\mathbb{E}\,\|X_n(0,-\gamma,\eta) - X_n(0,-\delta,\eta)\|^2 &\leqslant e^{-\omega\gamma}\,\mathbb{E}\,\|\eta - X_n(-\gamma,-\delta,\eta)\|^2 \\ &\leqslant e^{-\omega\gamma}\left(2\,\|\eta\|^2 + 2\,\mathbb{E}\,\|X_n(-\gamma,-\delta,\eta)\|^2\right).\end{aligned}$$

Now, recalling (3.5),

$$\mathbb{E}\,\|X_n(0,-\gamma,\eta) - X_n(0,-\delta,\eta)\|^2 \leqslant e^{-\omega\gamma}\left(4\,\|\eta\|^2 + \frac{4C_1}{\omega}\right).$$

Since $X_n(t,s,\eta)$ converges in $L_2(\Omega)$ to $X(t,s,\eta)$,

$$\mathbb{E}\,\|X(0,-\gamma,\eta) - X(0,-\delta,\eta)\|^2 \leqslant e^{-\omega\gamma}\left(4\,\|\eta\|^2 + \frac{4C_1}{\omega}\right).$$

It follows that $(X(0,-\gamma,\eta))_\gamma$ is a Cauchy sequence in $L_2(\Omega)$, so there exists random variable $\mathcal{X} \in L_2(\Omega)$ such that $X(0,-\gamma,\eta)$ converges to $\mathcal{X}$ in $L_2(\Omega)$, which implies that $X(0,-\gamma,\eta)$ converges to $\mathcal{X}$ also in law

$$\mathcal{L}\left(X(0,-\gamma,\eta)\right) \longrightarrow_{\gamma\to\infty} \mathcal{L}\left(\mathcal{X}\right).$$

Since $\mathcal{L}\left(X(0,-\gamma,\eta)\right) = \mathcal{L}\left(X(\gamma,0,\eta)\right)$, we also have

$$\mathcal{L}\left(X(\gamma,0,\eta)\right) \longrightarrow_{\gamma\to\infty} \mathcal{L}\left(\mathcal{X}\right).$$

Therefore $\mathcal{L}(\mathcal{X})$ is an invariant measure for equation $(*)$.

II. Now let $m := \mathbb{E}\,Z(1)$ be any in U. Equation $(*)$ can be written as

$$\begin{aligned} dX &= (AX + \tilde{F}(X))dt + B(X)d\tilde{Z}(t), \\ X(0) &= \eta, \end{aligned} \tag{$**$}$$

where

$$\begin{aligned}\tilde{Z}(t) &= Z(t) - mt, \\ \tilde{F}(x) &= F(x) + B(x)m.\end{aligned}$$

It suffices to prove that there exists an invariant measure for $(**)$. We have $\mathbb{E}\,\tilde{Z}(1) = 0$, $\operatorname{Var}\tilde{Z}(1) = \operatorname{Var} Z(1)$, hence

$$\begin{aligned}2\,\langle\, A(x-y) + \tilde{F}(x) - \tilde{F}(y), x-y\,\rangle + \operatorname{Var}\tilde{Z}(1)\,\|B(x)-B(y)\|^2_{L(U,H)} \\ = 2\,\langle A(x-y) + F(x) - F(y) + (B(x)-B(y))\,m, x-y\rangle \\ + \operatorname{Var} Z(1)\,\|B(x)-B(y)\|^2_{L(U,H)}\,,\end{aligned}$$

so the existence of an invariant measure for $(**)$ follows from step I.

□

4 Sufficient condition in terms of Lipschitz constants

Theorem 4.1. *Assume that* Z*,* A*,* F *and* B *satisfy conditions* (i)*,* (ii)*,* (iii)*,* (iv)*. Let* $S(t)_{t\geqslant 0}$ *be the semigroup generated by* A *and assume that there exists* $\alpha > 0$ *such that for every* $t \geqslant 0$

$$\|S(t)\|_{L(H)} \leqslant e^{-\alpha t}.$$

If

$$-2\alpha + 2L_F + 2L_B \|\mathbb{E}\, Z(1)\|_U + \operatorname{Var} Z(1) {L_B}^2 < 0, \tag{B}$$

then there exists an invariant measure for the equation

$$\begin{aligned} dX &= (AX + F(X))dt + B(X)dZ(t), \\ X(0) &= \eta. \end{aligned} \tag{$*$}$$

Proof of Theorem 4.1. We shall prove that condition (B) implies condition (A) so the result will follow from the previous theorem. If (B) is fullfilled, then there exists $N > 0$ such that for $n > N$

$$-\frac{2\alpha n}{n+\alpha} + 2L_F + 2L_B \|\mathbb{E}\, Z(1)\|_U + \operatorname{Var} Z(1) {L_B}^2 < 0.$$

For the Yosida approximations A_n we have $\langle A_n x, x\rangle \leqslant -\frac{\alpha n}{n+\alpha} \|x\|^2$, since $\|S(t)\|^2_{L(H)} \leqslant e^{-\alpha t}$. Thus

$$\begin{aligned} 2\langle A_n(x-y), x-y\rangle &\leqslant -2\frac{\alpha n}{n+\alpha}\|x-y\|^2, \\ 2\langle F(x)-F(y), x-y\rangle &\leqslant 2L_F\|x-y\|^2, \\ 2\langle (B(x)-B(y))\,\mathbb{E}\, Z(1), x-y\rangle &\leqslant 2L_B\|\mathbb{E}\, Z(1)\|_U \|x-y\|^2, \\ \operatorname{Var} Z(1)\|B(x)-B(y)\|^2_{L(U,H)} &\leqslant \operatorname{Var} Z(1){L_B}^2\|x-y\|^2, \end{aligned}$$

whence

$$\begin{aligned} 2\langle\, & A_n(x-y) + F(x) - F(y) + (B(x)-B(y))\,\mathbb{E}\, Z(1), x-y\,\rangle \\ &\quad + \operatorname{Var} Z(1)\|B(x)-B(y)\|^2_{L(U,H)} \\ &\qquad \leqslant \left(-\frac{2\alpha n}{n+\alpha} + 2L_F + 2L_B\|\mathbb{E}\, Z(1)\|_U + \operatorname{Var} Z(1){L_B}^2\right)\|x-y\|^2. \end{aligned}$$

And condition (A) is fullfilled, since $-\frac{2\alpha n}{n+\alpha} + 2L_F + 2L_B\|\mathbb{E}\, Z(1)\|_U + \operatorname{Var} Z(1){L_B}^2 < -\omega$, for some $\omega > 0$ and $n > N$.

□

Remark

Gaans [3] proves the existence of an invariant measure for $(*)$ under the asumption that $\|S(t)\|^2_{L(H)} \leqslant Me^{-\alpha t}$ for some $\alpha, M > 0$, which is less restrictive condition than the condition $\|S(t)\|^2_{L(H)} \leqslant e^{-\alpha t}$. His sufficient condition for the existence of an invariant measure is

$$6M^2\left(\frac{{L_F}^2}{\alpha} + \operatorname{Var} Z(1){L_B}^2\right) < \alpha, \tag{GM}$$

under the assumption that $\mathbb{E}\, Z(1) = 0$. In the case $M = 1$, we get

$$6\left(\frac{{L_F}^2}{\alpha} + \operatorname{Var} Z(1){L_B}^2\right) < \alpha. \tag{G1}$$

Condition (B) in the case $\mathbb{E}\, Z(1) = 0$ is

$$-2\alpha + 2L_F + \operatorname{Var} Z(1){L_B}^2 < 0. \tag{B0}$$

If (G1) is fullfilled, then so is (B0). Indeed, (G1) is equivalent to

$$\frac{{L_F}^2}{\alpha} - \frac{\alpha}{6} + \operatorname{Var} Z(1){L_B}^2 < 0. \tag{G'1}$$

So it is enough to prove that

$$-2\alpha + 2L_F + \operatorname{Var} Z(1){L_B}^2 < \frac{{L_F}^2}{\alpha} - \frac{\alpha}{6} + \operatorname{Var} Z(1){L_B}^2.$$

We have

$$0 < 5\alpha^2 + 6\left(\alpha - L_F\right)^2 = 5\alpha^2 + 6\alpha^2 + 6{L_F}^2 - 12\alpha L_F = 6{L_F}^2 - \alpha^2 + 12\alpha^2 - 12\alpha L_F,$$

so

$$0 < \frac{{L_F}^2}{\alpha} - \frac{\alpha}{6} + 2\alpha - 2L_F.$$

References

[1] G. Da Prato & J. Zabczyk, *Ergodicity for infinite dimensional systems*, Cambridge University Press 1996.

[2] G. Da Prato & J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press 1992.

[3] O. van Gaans, *Invariant measures for stochastic evolution equations with Hilbert space valued Lévy noise*, Technical Report 05-08, Faculty of Mathematics and Computer Science, Friedrich Schiller Universtity, Jena 2005.

[4] J. Jakubowski & J. Zabczyk, *Exponential moments for HJM models with jumps*, Institute of Mathematics, Polish Academy of Sciences, Preprint 651 (2004), 1-11

[5] W. Rudin, *Functional Analysis*, McGraw-Hill, New York 1973.

[6] M. Tehranchi, *A note on invariant measures for HJM models*, Finance and Stochastics **9** (2005), 389-398