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MANIFOLDS WITH A UNIQUE EMBEDDING

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Abstract. Let X be a smooth, compact manifold of dimension k. We show that any two smooth embeddings $X \to \mathbb{R}^n$ are tamely equivalent provided $n \geq 2k+2$. Moreover, if additionally X is a real analytic manifold, then we show that any two real analytic embeddings $X \to \mathbb{R}^n$ are real analytically equivalent.

We extend this result to some interesting sub-categories of the category of real smooth manifold. In particular we will prove that if X, Y are Nash $k-$dimensional submanifolds of $\mathbb{R}^n$ (where $n \geq 2k+2$) and $\phi : X \to Y$ is a diffeomorphism (Nash isomorphism, real analytic isomorphism), then ϕ can be extended to a tame diffeomorphism (tame Nash isomorphism, tame real analytic isomorphism) $\Phi : \mathbb{R}^n \to \mathbb{R}^n$.

We prove also that if X, Y are $k-$dimensional smooth algebraic subvarieties of $\mathbb{C}^n$ (where $n \geq 2k+2$), and $\phi : X \to Y$ is a biholomorphism (or polynomial isomorphism), then ϕ can be extended to a global tame biholomorphism (polynomial automorphism) $\Phi : \mathbb{C}^n \to \mathbb{C}^n$.

Finally we prove: if X, Y are $k-$dimensional closed semi-algebraic subsets of $\mathbb{R}^n$ (where $n \geq 2k+2$), and $\phi : X \to Y$ is a semi-algebraic homeomorphism, that ϕ can be extended to a global tame semi-algebraic homeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$. This last result is a semi-algebraic counterpart of a classical result of Herman Gluck [6], on extension of homeomorphisms of compact polyhedrons.

1. Introduction

Let us recall that diffeomorphism is said to be a *triangle* diffeomorphism, if it is of the form

$$\Phi : \mathbb{R}^n \ni (x_1, ..., x_n) \to (x_1, ..., x_{n-1}, x_n + p_n(x_1, ..., x_{n-1})) \in \mathbb{R}^n,$$

when $p_n(x_1, ..., x_{n-1})$ is a smooth function. A diffeomorphism F, which can be obtained as a composition of triangle diffeomorphisms and linear automorphisms with determinants equal to 1 is called *tame.* Of course a tame diffeomorphism is diffeotopic to the identity and *it preserves the volume.*

Let X be a smooth manifold. We say that two embeddings $f, g : X \to \mathbb{R}^n$ are *equivalent,* if there is a diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $g = \Phi \circ f$. If additionally Φ is a tame diffeomorphism, we say that f, g are *tamely equivalent.* If every two embeddings of X into $\mathbb{R}^n$ are equivalent (tamely equivalent) we say that X has a *unique (tamely unique) embedding* into $\mathbb{R}^n$.

For example if $X = \mathbf{S}^1$ is a circle, then X has infinitely many non-equivalent embeddings into $\mathbb{R}^3$ (every knot gives a one non-standard embedding). It is interesting to find sufficient conditions for a manifold X to have a unique or tamely unique embedding into $\mathbb{R}^n$. It can be deduced from the Whitney paper [13] that if X is a compact $k-$dimensional smooth manifold, then any two smooth embeddings $f, g : X \to \mathbb{R}^n$, where $n \geq 2k+2$ are equivalent. In this note we improve this result and we show that in fact in this case any two smooth

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embeddings $f, g : X \to \mathbb{R}^n$ are tamely equivalent. Moreover, we show (which seems to be a quite new result), that if X is *a compact real analytic manifold* and f, g are *real analytic embeddings*, then we can find Φ as *a tame real analytic isomorphism* $\Phi : \mathbb{R}^n \to \mathbb{R}^n$.

Of course the same question can be posed for a larger class of categories. In particular in [7], [8] and [10] this problem was solved for category of smooth complex algebraic affine varieties (where morphisms are polynomial mappings). The second aim of this paper is to generalize (and simplify) these results to the case of some other interesting categories: *pseudo-algebraic categories* (see Definition 4.3 and Definition 4.11). The examples of pseudo-algebraic categories are e.g.: the category of Nash (i.e. analytic and semi-algebraic) submanifolds of $\mathbb{R}^n$ with Nash (i.e., analytic and semi-algebraic) mappings as morphisms, the category of Nash submanifolds of $\mathbb{R}^n$ with smooth mappings as morphisms, the category of Nash submanifolds of $\mathbb{R}^n$ with real analytic mappings as morphisms, the category of smooth complex affine subvarieties of $\mathbb{C}^n$ with holomorphic (or polynomial, or smooth) mappings as morphisms.

In particular we prove that if X, Y are Nash $k-$dimensional submanifolds of $\mathbb{R}^n$ (where $n \geq 2k + 2$) and $\phi : X \to Y$ is a diffeomorphism (Nash isomorphism, real-analytic isomorphism), then ϕ can be extended to a tame diffeomorphism (Nash isomorphism, real-analytic isomorphism) $\Phi : \mathbb{R}^n \to \mathbb{R}^n$.

We also prove that if X, Y are $k-$dimensional smooth algebraic subvarieties of $\mathbb{C}^n$ (where $n \geq 2k + 2$), and $\phi : X \to Y$ is a biholomorphism, then ϕ can be extended to a global tame biholomorphism $\Phi : \mathbb{C}^n \to \mathbb{C}^n$.

Finally we show this theorem for a category of closed semi-algebraic sets with continuous semi-algebraic mappings as morphisms. More precisely, we show: if X, Y are $k-$dimensional closed semi-algebraic subsets of $\mathbb{R}^n$ (where $n \geq 2k + 2$), and $\phi : X \to Y$ is a semi-algebraic homeomorphism, then ϕ can be extended to a global tame semi-algebraic homeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ (in particular X and Y are homeotopic). This theorem is a semi-algebraic counterpart of the classical paper of Herman Gluck on extension of homeomorphisms of polyhedrons (see [6]).

We give also examples of $k = n + 1$ dimensional Nash manifolds $X_k \subset \mathbb{R}^{2n}$, (where n is any even number different from $2, 4, 8$) which has at least two different embedding into $\mathbb{R}^{2n}$. This shows that our results can not be much improved for large n. Note also that for $k = 1$ and $n = 3$ our result (about Nash manifolds) is optimal.

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2. Preliminaries

We start with the following basic definition:

Definition 2.1. *Let X, Y be smooth manifolds and let $f : X \to Y$ be a smooth morphism. We say that the mapping f is an embedding if*

1) $f(X)$ is a closed submanifold of Y,

2) the mapping $f : X \to f(X)$ is a diffeomorphism.

Let Y be a smooth manifold. We will denote by $\mathbf{C}(Y)$ the algebra of all smooth functions on Y. If $f : X \to Y$ is a smooth morphism of smooth manifolds, then we have the natural homomorphism $f^* : \mathbf{C}(Y) \ni h \to h \circ f \in \mathbf{C}(X)$.

If X is a closed submanifold of Y, then we consider the ideal $\mathbf{I}(X) = \{f \in \mathbf{C}(Y) : f|X = 0\}$. Using a partition of unity it is easy to see that every function on X is a restriction of some smooth function on Y, and consequently we have

$$\mathbf{C}(X) \cong \mathbf{C}(Y)/\mathbf{I}(X).$$

In particular the mapping i^* induced by the inclusion $i : X \to Y$ is an epimorphism. In fact we have the following more general fact:

Proposition 2.2. *Let X, Y be smooth manifolds and $f : X \to Y$ be a smooth morphism. The following conditions are equivalent:*

1) f is an embedding,

2) the induced mapping $f^* : \mathbf{C}(Y) \to \mathbf{C}(X)$ is an epimorphism,

3) the mapping f is proper, injective and for every $x \in X$ the mapping $d_x f : T_x X \to T_{f(x)} Y$ is a monomorphism.

Proof. 1) $\implies$ 2) It follows from remarks above.

2) $\implies$ 3) We can assume that X is embedded in some $\mathbb{R}^N$ (as a closed submanifold). Let $x_1, ..., x_N$ be coordinates on $\mathbb{R}^N$. By the assumption we can find smooth functions $H_i \in \mathbf{C}(Y)$ such that $x_i = H_i \circ f$ (on X). Put $H = (H_1, ..., H_N)$. We have *identity* $= H \circ f$. This easily implies that the mapping f is injective and proper. Moreover, after computing derivatives of both sides we have

$$identity = d_{f(x)} H \circ d_x f,$$

which easily implies that $d_x f$ is a monomorphism.

3) $\implies$ 1) It is well known from differential geometry. □

3. Smooth and analytic compact case

In this section we will prove our first main result. To do this we need a series of lemmas:

Lemma 3.1. *Let X be a submanifold of $\mathbb{R}^n$. of dimension k. Assume that the projection $\pi : X \ni (x_1, ..., x_n) \to (x_1, ..., x_l, 0, ..., 0) \in \mathbb{R}^l \times \{0\}$ is an embedding. Then, there exists a tame diffeomorphism $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Pi|_X = \pi$.*

Proof. Let $X' := \pi(X)$, it is a closed submanifold of $\mathbb{R}^n$. Consider the mapping $\pi : X \to X' \subset \mathbb{R}^n$. It is an embedding, so the mapping $\pi^* : \mathbf{C}(\mathbb{R}^n) \to \mathbf{C}(X)$ is an epimorphism. In particular for every $k > l$ there exists a function $p_k \in \mathbf{C}(\mathbb{R}^n)$ such that $x_k = p_k(x_1, ..., x_l)$ (on X). Consider the mapping

$$\Pi(x_1, ..., x_n) = (x_1, ..., x_l, x_{l+1} - p_{l+1}(x_1, ..., x_l), ..., x_n - p_n(x_1, ..., x_l)).$$

The mapping Π is a tame diffeomorphism of $\mathbb{R}^n$ and

$$\Pi|_X = \pi.$$

□

The next Lemma is a smooth variant of a Bertini Theorem:

Lemma 3.2. *Let X be a smooth manifold. Let $f : X \to \mathbb{P}^m$ be a smooth morphism. Then there is a subset $E \subset \mathbb{P}^{m*}$ of measure 0 such that: if π is a projective hyperplane and $\pi \notin E$, then $f^{-1}(\pi)$ is a smooth submanifold of X.*

Proof. First assume that $f : X \to \mathbb{R}^m$. Hence $f = (f_1, ..., f_m)$ and f_i are smooth functions. Consider the mapping

$$\Psi : X \times \mathbb{R}^m \ni (x, (\lambda_1, ..., \lambda_m)) \to (\sum_{i=1}^{n} \lambda_i f_i(x), (\lambda_1, ..., \lambda_m)) \in \mathbb{R} \times \mathbb{R}^m.$$

Now our conclusion follows from the Sard Theorem (see [12]).

To prove the general case let $H_1, ..., H_{m+1} \subset \mathbb{P}^m$ be hyperplanes in general position. By the previous result a preimage of a general hyperplane is smooth in each open subset $U_i = X \setminus f^{-1}(H_i)$. Since sets $\{U_i\}$ cover X the Lemma follows. □

Next Lemma is a variant of the Whitney Embedding Theorem:

Lemma 3.3. *Let X be compact submanifold of $\mathbb{R}^n$ of dimension k. If $n > 2k + 1$, then there exists a system of coordinates $(x_1, ..., x_{2k+1}, x_{2k+2}, ..., x_n)$ such that the projection $\pi : X \ni (x_1, ..., x_{2k+1}, x_{2k+2}, ..., x_n) \to (x_1, ..., x_{2k+1}, 0, ..., 0) \in \mathbb{R}^{2k+1} \times \{0\}$ is an embedding.*

Proof. Let us denote by π_∞ the hyperplane at infinity of $\mathbb{R}^n$. Thus $\pi_\infty \cong \mathbb{P}^{n-1}$ is a real projective space of dimension $n - 1 > 2k$. For a non-zero vector $v \in \mathbb{R}^n$ let $[v]$ denote an appropriate point in $\mathbb{P}^{n-1}$.

Let $\Delta = \{(x, y) \in X \times X : x = y\}$ and let TX denote a tangent bundle of X. Set $TX' = TX \setminus X \times \{0\}$. Consider two mappings

$$A : X \times X \setminus \Delta \ni (x, y) \to [x - y] \in \pi_\infty$$

and

$$B : TX' \ni (x, v) \to [v] \in \pi_\infty.$$

Since A, B are smooth mappings and manifolds $X \times X \setminus \Delta$ and TX are of dimension $2k$, we have by the Sard Theorem (see [12]) that $\pi_\infty \setminus (A(X \times X \setminus \Delta) \cup B(TX')) \neq \emptyset$. Let $P \in \pi_\infty \setminus (A(X \times X \setminus \Delta) \cup B(TX'))$ and let $H \subset \mathbb{R}^{n-1}$ be a hyperplane, which does not contain the point P (at infinity). Thus the projection $S : X \ni x \to Px \cap H \in H \cong \mathbb{R}^{n-1}$ is an embedding. Now we can apply the mathematical induction. □

Lemma 3.4. *Let X be a compact manifold of dimension k. Assume that $X \subset \mathbb{R}^{2n}$, where $n \geq 2k + 1$. If mappings*

$$\pi_1 : X \ni (x_1, ..., x_n, y_1, ..., y_n) \to (y_1, ..., y_n) \in \mathbb{R}^n$$

and

$$\pi_2 : X \ni (x_1, ..., x_n, y_1, ..., y_n) \to (x_1, ..., x_n) \in \mathbb{R}^n$$

are embeddings, then there are affine coordinates $(X_1, ..., X_n)$ in $\mathbb{R}^n \times \{0\}$ and affine coordinates $(Y_1, ..., Y_n)$ in $\{0\} \times \mathbb{R}^n$ such that all projection

$$q_r : X \ni (X_1, ..., X_n, Y_1, ..., Y_n) \to (X_1, ..., X_r, Y_{r+1}, ..., Y_n) \in \mathbb{R}^n, \ r = 0, ..., n$$

are embeddings.

Proof. Let us denote by π_∞ the hyperplane at infinity of $\mathbb{R}^n \times \mathbb{R}^n$. Thus $\pi_\infty \cong \mathbb{P}^{2n-1}$ is a real projective space of dimension $2n - 1$.

Again let $\Delta = \{(x, y) \in X \times X : x = y\}$ be the diagonal and let TX denote a tangent bundle of X. Set $TX' = TX \setminus X \times \{0\}$. Consider two mappings

$$A : X \times X \setminus \Delta \ni (x, y) \to [x - y] \in \pi_\infty$$

and

$$B : TX' \ni (x, v) \to [v] \in \pi_\infty.$$

Denote $\Lambda := (A(X \times X \setminus \Delta) \cup B(TX')) \subset \pi_\infty$. Let $L = (L_1, ..., L_n) : \mathbb{R}^{2n} \to \mathbb{R}^n$ be a linear mapping. Set $S(L) := \{x \in \pi_\infty : L_i(x) = 0\ , i = 1, ..., n\}$. It is easy to see that the mapping $L|X$ is an injective immersion if and only if $\Lambda \cap S = \emptyset$.

Now we show that there are affine coordinates $(X_1, ..., X_n)$ in $\mathbb{R}^n \times \{0\}$ and affine coordinates $(Y_1, ..., Y_n)$ in $\{0\} \times \mathbb{R}^n$ such that all projection

$$q_i : X \ni (X_1, ..., X_n, Y_1, ..., Y_n) \to (X_1, ..., X_{n-i}, Y_{n-i+1}, ..., Y_n) \in \mathbb{R}^n,\ i = 0, ..., n$$

are embeddings. On π_∞ we have coordinates $(x : y)$. Since the projection $\pi_1|X$ is an embedding we have $\{(x : y) \in \pi_\infty : x_1 = 0, ..., x_n = 0\} \cap \Lambda = \emptyset$. Consequently, if we set $\psi : \pi_\infty \ni (x : y) - \to x \in \mathbb{P}^{n-1}$, then the mappings $g := \psi \circ A : X \times X \setminus \Delta \to \mathbb{P}^{n-1}$ and $k := \psi \circ B : TX \to \mathbb{P}^{n-1}$ are well defined and smooth.

By Lemma 3.2 this means that if $H = \{x \in \mathbb{P}^{n-1} : \sum_{i=1}^n c_i x_i = 0\}$ is a generic hyperplane, then $g^{-1}(H)$ and $k^{-1}(H)$ are smooth submanifolds of $X \times X \setminus \Delta$ and TX', of dimension at most $2k - 1$. Set $X_1 = \sum_{i=1}^n c_i x_i$.

Continuing in this fashion we see that we can choose n generic hyperplanes given by equations: $X_i = \sum_{k=1}^n a_{i,k} x_k,\ i = 1, ..., n$, such that $A^{-1}(\{X_1 = 0, ..., X_r = 0\})$ and $B^{-1}(\{X_1 = 0, ..., X_r = 0\})$ are smooth submanifolds of $X \times X \setminus \Delta$ and TX', of dimension at most $2k - r$. In particular we have dim $A^{-1}(\{X_1 = 0, ..., X_{n-1} = 0\}) \le 0$ and dim $B^{-1}(\{X_1 = 0, ..., X_{n-1} = 0\}) \le 0$.

Now in the same way we can choose a generic hyperplane given by the equation $Y_n = \sum_{k=1}^n b_{n,k} y_k,\ i = 1, ..., n$, such that $A^{-1}(\{X_1 = 0, ..., X_{n-1} = 0, Y_n = 0\}) = \emptyset$ and $B^{-1}(\{X_1 = 0, ..., X_{n-1} = 0, Y_n = 0\}) = \emptyset$ and additionally for every $0 \le r \le n - 1$ we have dim $A^{-1}(\{X_1 = 0, ..., X_r = 0, Y_n = 0\}) \le 2k - r - 1$ and dim $B^{-1}(\{X_1 = 0, ..., X_r = 0, Y_n = 0\}) \le 2k - r - 1$. Further we can construct $Y_{n-1} = \sum_{k=1}^n b_{n-1,k} y_k,\ i = 1, ..., n$, such that $A^{-1}(\{X_1 = 0, ..., X_{n-2} = 0, Y_{n-1} = 0, Y_n = 0,\}) = \emptyset$ and $B^{-1}(\{X_1 = 0, ..., X_{n-2} = 0, Y_{n-1} = 0, Y_n = 0\}) = \emptyset$ and additionally for every $0 \le r \le n - 2$ we have dim $A^{-1}(\{X_1 = 0, ..., X_r = 0, Y_{n-1} = 0, Y_n = 0\}) \le 2k - r - 2$ and dim $B^{-1}(\{X_1 = 0, ..., X_r = 0, Y_{n-1} = 0, Y_n = 0\}) \le 2k - r - 2$. Continuing in this manner we find a system of coordinates $(X_1, ..., X_n, Y_1, ..., Y_n)$ we are looking for: for all $0 \le r \le n$ we have

$$\Lambda \cap \{X_1 = 0, ..., X_r = 0, Y_{r+1} = 0, ..., Y_n = 0\} = \emptyset,$$

which implies that the mapping

$$q_r : X \ni (X_1, ..., X_n, Y_1, ..., Y_n) \to (X_1, ..., X_r, Y_{r+1}, ..., Y_n) \in \mathbb{R}^n,\ r = 0, ..., n$$

is an immersion. Since X is compact the mapping q_r is an embedding. □

Now we are in a position to prove the first main result of this section:

Theorem 3.5. *Let X be a compact smooth (not necessarily connected) submanifold of $\mathbb{R}^n$ of dimension (not necessarily pure) k. Let $f : X \to \mathbb{R}^n$ be an embedding. If $n \ge 2k + 2$, then there exists a tame diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$F|_X = f.$$

Proof. Apply Lemma 3.3 to X and $f(X)$. Then in virtue of Lemma 3.1 we can assume that there exists tame diffeomorphisms $A, B : \mathbb{R}^n \to \mathbb{R}^n$ such that $A(X) \subset \mathbb{R}^{2k+1} \times \{0\}$ and $B(f(X)) \subset \{0\} \times \mathbb{R}^{2k+1}$ (if necessary we compose A and B with suitable affine transformations with determinants equal to 1). Consider $f' = B \circ f \circ A^{-1}$, of course we can assume that $f = f'$. In particular we can assume that $X \subset \mathbb{R}^{2k+1} \times \{0\}$ and $f(X) \subset \{0\} \times \mathbb{R}^{2k+1}$and that $n = 2k + 2$. Thus $f = (0, f_1, ..., f_{n-1})$.

Applying Lemma 3.4 to the set $X' = graph(f) \subset \mathbb{R}^{2k+1} \times \mathbb{R}^{2k+1}$ we see that (after suitable change of coordinates) all mappings

$$q'_r : X \ni (x_1, ..., x_{2k+1}, 0) \to (x_1, ..., x_r, f_{r+1}, ..., f_{n-1}) \in \mathbb{R}^{2k+1}$$

are embeddings (as a composition of a diffeomorphism $X \to graph(f)$ with q_r-notation as in Lemma 3.4). Since $X \subset \mathbb{R}^{2k+1} \times \{0\}$ there exists a smooth function P_{n-1} such that

$$f_{n-1} = P_{n-1}(x_1, ..., x_{n-1}).$$

Consider a tame diffeomorphism

$$A_{n-1} : \mathbb{R}^n \ni (x_1, ..., x_n) \to (x_1, ..., x_{n-1}, x_n + P_{n-1}(x_1, ..., x_{n-1})) \in \mathbb{R}^n.$$

Thus for $x \in X$ we have

$$A_{n-1}(x) = (x_1, ..., x_{n-1}, f_{n-1}).$$

Now by Lemma 3.4 we have (see remarks above) that the mapping $(x_1, ..., x_{n-2}, f_{n-1})$ restricted to X is also an embedding. Hence there exists a smooth function Q_{n-1} such that we have on X:

$$x_{n-1} = Q_{n-1}(x_1, ..., x_{n-2}, f_{n-1}).$$

Consider a tame diffeomorphism

$$B_{n-1} : \mathbb{R}^n \ni (x_1, ..., x_n) \to (x_1, ..., x_{n-2}, x_{n-1} - Q_{n-1}(x_1, ..., x_{n-2}, x_n), x_n) \in \mathbb{R}^n.$$

Again for $x \in X$ we have

$$B_{n-1} \circ A_{n-1} = (x_1, ..., x_{n-2}, 0, f_{n-1}).$$

In a similar way we can construct tame diffeomorphisms A_{n-2} and B_{n-2} such that

$$A_{n-2}(x_1, ..., x_{n-2}, 0, f_{n-1}) = (x_1, ..., x_{n-2}, f_{n-2}, f_{n-1})$$

and

$$B_{n-2}(x_1, ..., x_{n-2}, f_{n-2}, f_{n-1}) = (x_1, ..., x_{n-3}, 0, f_{n-2}, f_{n-1}).$$

Continuing in this manner, we get a sequence of tame diffeomorphisms $A_{n-1}, A_{n-2}, ..., A_1$ and $B_{n-1}, B_{n-2}, ..., B_1$ such that for $x \in X$ we have

$$B_1 \circ A_1 \circ ... \circ B_{n-1} \circ A_{n-1}(x) = (0, f_1(x), ..., f_{n-1}(x)) = f(x).$$

□

Corollary 3.6. *With the preceding notation there is a smooth family of tame diffeomorphisms $F_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, such that $F_0 = identity$ and $F_1|_X = f$.*

Proof. Indeed, every triangle diffeomorphism $G = (x_1, ..., x_{n-1}, x_n + P_n(x_1, ..., x_{n-1}))$ is diffeotopic to the identity by $t \to G_t = (x_1, ..., x_{n-1}, x_n + tP_n(x_1, ..., x_{n-1}))$ and the same is true for linear mapping with determinant equal to 1. □

Corollary 3.7. *Let X be a compact smooth manifold of dimension k. In $n \geq 2k+2$, then X has a (tamely) unique embedding into $\mathbb{R}^n$.*

Now note that we can repeat this proof for real analytic submanifolds of $\mathbb{R}^n$ nearly word by word, with one exception - we need a fact that if $f: X \to \mathbb{R}$ is a real analytic function on a real analytic manifold X, then we can extend f to a real analytic function $F: \mathbb{R}^n \to \mathbb{R}$. This follows from a result of Cartan (see [4], p. 89). In this way we have the following interesting:

Theorem 3.8. *Let $X \subset \mathbb{R}^n$ be a compact (not necessarily connected) real analytic submanifold of dimension (not necessarily pure) k. Let $f: X \to \mathbb{R}^n$ be a real-analytic embedding. If $n \geq 2k+2$, then f can be extended to a tame real analytic isomorphism $F: \mathbb{R}^n \to \mathbb{R}^n$.*

Corollary 3.9. *With the preceding notation there is an analytic family of tame analytic isomorphisms $F_t: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, such that $F_0 =$ identity and $F_1|_X = f$.*

Corollary 3.10. *Let X be a compact real analytic manifold of dimension k. In $n \geq 2k+2$, then X has a (tamely) unique analytic embedding into $\mathbb{R}^n$.*

Example 3.11. As shows the example of a non-trivial knot $f: \mathbf{S}^1 \to \mathbb{R}^3$ (note that we can take f as a real analytic mapping!) the assumption $n \geq 2k+2$ in Theorem 3.5 and Theorem 3.8 is essential.

4. Real pseudo-algebraic categories

In this section we apply our results to other categories of manifolds. First we assume that our field is a real field. By $\mathbf{S_0}$ we mean the category of all pairs $(X, \mathbb{R}^{n(X)})$, where $X \subset \mathbb{R}^{n(X)}$ is a smooth closed submanifold of $\mathbb{R}^{n(X)}$ and morphisms are smooth mappings. Let $\mathbf{S}$ be a sub-category of $\mathbf{S_0}$. Every object of $\mathbf{S}$ is the pair $(X, \mathbb{R}^{n(X)})$, in further we will identify such object simply with X. In particular we will identify $(\mathbb{R}^n, \mathbb{R}^n)$ with $\mathbb{R}^n$. We start with:

Definition 4.1. *Let $\mathbf{S}$ be as above and let $X, Y \in \mathbf{S}$. We say that a mapping $f: X \to Y$ is an $\mathbf{S}-$embedding, iff*

1) $(f(X), \mathbb{R}^{n(Y)}) \in \mathbf{S}$

2) f as well as $f^{-1}: f(X) \to X \subset \mathbb{R}^{n(X)}$ are $\mathbf{S}-$morphisms.

Remark 4.2. In particular an $\mathbf{S}-$embedding is always a smooth embbeding, see Definition 2.1.

Definition 4.3. *We say that $\mathbf{S}$ is a fine category iff:*

1) for every $n \in \mathbb{N}: \ \mathbb{R}^n \in \mathbf{S}$,

2) if $f: X \to \mathbb{R}$ is in $\mathbf{S}$, then f can be extended to a mapping $F: \mathbb{R}^{n(X)} \to \mathbb{R}$, which is also in $\mathbf{S}$,

3) linear mappings are in $\mathbf{S}$, moreover, if $X \in \mathbf{S}$, then the restriction of mappings from $\mathbf{S}$ to X are in $\mathbf{S}$,

4) if $X \in \mathbf{S}$, then the set $\mathbf{CS}(X) = \{f: X \to \mathbb{R};\ f \in \mathbf{S}\}$ is an $\mathbb{R}-$algebra,

5) if $X \in \mathbf{S}$ and $\pi: X \to \mathbb{R}^n$ is a projection, which is a smooth embedding, then f is an $\mathbf{S}-$embedding,

6) if $X \in \mathbf{S}$ and $f: X \to \mathbb{R}^n$ and $g: X \to \mathbb{R}^m$ are $\mathbf{S}-$morphisms, then $(f,g): X \to \mathbb{R}^n \times \mathbb{R}^m$ is also an $\mathbf{S}-$morphism.

In the sequel the following definition will be crucial (see e.g., [3]):

Definition 4.4. *We say that a submanifold $X \subset \mathbb{R}^n$ is a Nash manifold if X is a real analytic manifold and a closed semi-algebraic subset of $\mathbb{R}^n$. Moreover, if X, Y are Nash manifolds and $f : X \to Y$ is a mapping, then f is a Nash mapping if f is a real analytic and semi-algebraic mapping.*

Example 4.5. The examples of fine categories are: the category $\mathbf{S_0}$ itself, the category **RA** of smooth real analytic submanifolds with real analytic mappings as morphisms (the category **RA** satisfies (2) by [4]) and the category **NA** of Nash submanifolds with Nash mappings as morphisms (the category **NA** satisfies (2) by [3], Corollary 8.9.13).

A simple but important consequence of Definition 4.3 is:

Proposition 4.6. *Let $\mathbf{S}$ be a fine category and let $X \in \mathbf{S}$. If $f : X \to \mathbb{R}^n$ is an $\mathbf{S}-$embedding, then the induced mapping $f^* : \mathbf{CS}(\mathbb{R}^n) \to \mathbf{CS}(X)$ is an epimorphism.*

Proof. Indeed, let $Y = f(X)$. By definition we have that $Y \in \mathbf{S}$ and the mapping
$$a : \mathbf{CS}(Y) \ni \alpha \to \alpha \circ f \in \mathbf{CS}(X)$$
is an isomorphism. Now let $i : Y \to \mathbb{R}^n$ be the inclusion. Since every $\mathbf{S}-$function $\sigma : Y \to \mathbb{R}$ can be extended to a global $\mathbf{S}-$function $\Sigma : \mathbb{R}^n \to \mathbb{R}$ we have that the mapping $i^* : \mathbf{CS}(\mathbb{R}^n) \to \mathbf{CS}(Y)$ is an epimorphism. But $f^* = a \circ i^*$. □

Now we generalize results from Section 3:

Lemma 4.7. *Let $\mathbf{S}$ be a fine category and let $(X, \mathbb{R}^n) \in \mathbf{S}$ be a submanifold of dimension k. Assume that the projection $\pi : X \ni (x_1, ..., x_n) \to (x_1, ..., x_l, 0, ..., 0) \in \mathbb{R}^l \times \{0\}$ is an embedding. Then, there exists a tame $\mathbf{S}-$diffeomorphism $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Pi|_X = \pi$.*

Proof. Let $X' := p(X)$, it is a closed submanidfold of $\mathbb{R}^n$. Consider the mapping $\pi : X \to X' \subset \mathbb{R}^n$. It is an embedding, so the mapping $\pi^* : \mathbf{CS}(\mathbb{R}^n) \to \mathbf{CS}(X)$ is an epimorphism. In particular for every $k > l$ there exists a function $p_k \in \mathbf{CS}(\mathbb{R}^n)$ such that $x_k = p_k(x_1, ..., x_l)$ (on X). Consider the mapping
$$\Pi(x_1, ..., x_n) = (x_1, ..., x_l, x_{l+1} - p_{l+1}(x_1, ..., x_l), ..., x_n - p_n(x_1, ..., x_l)).$$
The mapping Π is a tame $\mathbf{S}-$diffeomorphism of $\mathbb{R}^n$ and
$$\Pi|_X = \pi.$$
□

In the sequel we need the following:

Definition 4.8. *For a hyperplane $H \subset \mathbb{P}^n$ let us consider the Zariski open affine set $U_H = \mathbb{P}^n \setminus H$. We will say that U_H is a standard open affine subset of $\mathbb{P}^n$. Now let X be a semi-algebraic set and let $f : X \to \mathbb{P}^n$ be a mapping. We say that the mapping f is projectively semi-algebraic if for every standard affine set $U_H \subset \mathbb{P}^n$ the set $f^{-1}(U_H)$ is semi-algebraic and the mapping*
$$f|_{f^{-1}(U_H)} : f^{-1}(U_H) \to U_H \cong \mathbb{R}^n$$
is a semi-algebraic mapping.

Next Lemma is a semi-algebraic variant of Lemma 3.3:

Lemma 4.9. *Let X be a semi-algebraic submanifold of $\mathbb{R}^n$ of dimension k. If $n > 2k+1$, then there exists a system of coordinates $(x_1, ..., x_{2k+1}, x_{2k+2}, ..., x_n)$ such that the projection $\pi : X \ni (x_1, ..., x_{2k+1}, x_{2k+2}, ..., x_n) \to (x_1, ..., x_{2k+1}, 0, ..., 0) \in \mathbb{R}^{2k+1} \times \{0\}$ is an embedding.*

Proof. We follow closely the proof of Lemma 3.3. Let us denote by π_∞ the hyperplane at infinity of $\mathbb{R}^n$. Thus $\pi_\infty \cong \mathbb{P}^{n-1}$ is a real projective space of dimension $n-1 > 2k$. For a non-zero vector $v \in \mathbb{R}^n$ let $[v]$ denote an appropriate point in $\mathbb{P}^{n-1}$.

Let $\Delta = \{(x,y) \in X \times X : x = y\}$ and let TX denote a tangent bundle of X. Set $TX' = TX \setminus X \times \{0\}$. Consider two mappings

$$A : X \times X \setminus \Delta \ni (x,y) \to [x-y] \in \pi_\infty$$

and

$$B : TX' \ni (x,v) \to [v] \in \pi_\infty.$$

Since A, B are semi-algebraic mappings and manifolds $X \times X \setminus \Delta$ and TX' are of dimension $2k$, we have that the set $\Lambda := (A(X \times X \setminus \Delta) \cup B(TX))$ is a projective semi-algebraic set (i.e., it is semi-algebraic in every standard affine open subset of $\mathbb{P}^{n-1}$) of dimension at most $2k$. This means that also the set $\Sigma =$ *closure of* Λ is (projectively) semi-algebraic of dimension at most $2k$ (for details see e.g., [2]). Consequently we have $\pi_\infty \setminus \Sigma \neq \emptyset$.

Let $P \in \pi_\infty \setminus \Sigma$ and let $H \subset \mathbb{R}^{n-1}$ be a hyperplane, which does not contain the point P (at infinity). Since $P \notin \Lambda$ we have that the projection $S : X \ni x \to Px \cap H \in H \cong \mathbb{R}^{n-1}$ is an immersion. Moreover, since $P \notin \Sigma$ we get that the S is also proper, hence it is an embedding.

Now we can apply the mathematical induction. □

Lemma 4.10. *Let* **S** *be a fine category. Let* $(X, \mathbb{R}^{2n}) \in$ **S**, *where* $n \geq 2k+1$. *Consider mappings*

$$\pi_1 : X \ni (x_1, ..., x_n, y_1, ..., y_n) \to (y_1, ..., y_n) \in \mathbb{R}^n$$

and

$$\pi_2 : X \ni (x_1, ..., x_n, y_1, ..., y_n) \to (x_1, ..., x_n) \in \mathbb{R}^n.$$

If π_1, π_2 *are (closed) embeddings and submanifolds* $\pi_1(X) = X_1$, $\pi_2(X) = X_2$ *are semi-algebraic, then there are affine coordinates* $(X_1, ..., X_n)$ *in* $\mathbb{R}^n \times \{0\}$ *and affine coordinates* $(Y_1, ..., Y_n)$ *in* $\{0\} \times \mathbb{R}^n$ *such that all projection*

$$q_r : X \ni (X_1, ..., X_n, Y_1, ..., Y_n) \to (X_1, ..., X_r, Y_{r+1}, ..., Y_n) \in \mathbb{R}^n,\ i = 0, ..., n$$

are **S**$-$*embeddings.*

Proof. Exactly in the same way as in the proof of Lemma 3.4 we can show that if $(X_1, ..., X_n)$ in $\mathbb{R}^n \times \{0\}$ and $(Y_1, ..., Y_n)$ in $\{0\} \times \mathbb{R}^n$ are sufficiently generic affine coordinates, then all projection

$$q_i : X \ni (X_1, ..., X_n, Y_1, ..., Y_n) \to (X_1, ..., X_{n-i}, Y_{n-i+1}, ..., Y_n) \in \mathbb{R}^n,\ i = 0, ..., n$$

are immersions. The key point now is to prove that they are also proper. Let X_1' be a Zariski projective closure of X_1 in (the first copy of) $\mathbb{P}^n$ and take $W_1 = X_1' \setminus X_1$. In analogous way we define W_2. Of course W_1, W_2 are algebraic sets of dimension $k-1$.

We can choose coordinates $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$ in so generic way, that additionally

$$\dim W_1 \cap \{X_1 = 0, ..., X_t = 0\} \leq k - 1 - t, \text{ for } t = 1, ..., k,$$

and

$$\dim W_2 \cap \{Y_n = 0, ..., Y_{n-t} = 0\} \leq k - t - 2, \text{ for } t = 0, 1, ..., k-1.$$

This gives that mappings

$$T : X_1 \ni (X_1, ..., X_n) \to (X_1, ..., X_k) \in \mathbb{R}^k$$

and

$$R : X_2 \ni (Y_1, ..., Y_n) \to (Y_{n-k+1}, ..., Y_n) \in \mathbb{R}^k$$

are proper. Consequently we obtain that the projection

$$P_1 = T \circ \pi_1 : X \ni (X_1, ..., X_n, Y_1, ..., Y_n) \to (X_1, ..., X_k) \in \mathbb{R}^k$$

is proper. Similarly the projection

$$P_2 = R \circ \pi_1 : X \ni (X_1, ..., X_n, Y_1, ..., Y_n) \to (Y_{n-k+1}, ..., Y_n) \in \mathbb{R}^k$$

is proper. Set

$$T_r : \mathbb{R}^n \ni (X_1, ..., X_r, Y_{r+1}, ..., Y_n) \to (X_1, ..., X_r) \in \mathbb{R}^r$$

and

$$R_r : \mathbb{R}^n \ni (X_1, ..., X_r, Y_{r+1}, ..., Y_n) \to (Y_{r+1}, ..., Y_n) \in \mathbb{R}^{n-r}.$$

It is easy to see that for every r either the map $T_r \circ q_r$ is proper (if $r \geq k$) or the map $R_r \circ q_r$ is proper (if $r \leq n - k + 1$). In both cases this implies that the mapping q_r is proper. This finishes the proof. □

Definition 4.11. *Let* $\mathbf{S}$ *be a fine category and let* $\mathbf{S}' \subset \mathbf{S}$ *be a subcategory. We say that* $\mathbf{S}'$ *is pseudo-algebraic sub-category in* $\mathbf{S}$ *(or shortly a pseudo-algebraic category) iff*

1) if $X \in \mathbf{S}'$, *then* X *is a Nash manifold,*

2) if X, Y *in* $\mathbf{S}'$ *then* $Mor_S(X, Y) = Mor_{S'}(X, Y)$,

where $Mor_S(X, Y) = \{f \in S : f : X \to Y\}$.

Now we can repeat word by word the proof of the Theorem 3.5 to obtain:

Theorem 4.12. *Let* $\mathbf{S}'$ *be a pseudo-algebraic category and let* $(X, \mathbb{R}^n), (Y, \mathbb{R}^n) \in \mathbf{S}'$ *be smooth (not necessarily connected) manifolds of dimension (not necessarily pure)* k. *Let* $f : X \to Y$ *be an* $\mathbf{S}'-$*diffeomorphism. If* $n \geq 2k + 2$, *then* f *can be extended a tame* $\mathbf{S}'-$*diffeomorphism* $F : \mathbb{R}^n \to \mathbb{R}^n$. *Moreover, there is a smooth family of tame* $\mathbf{S}'-$*diffeomorphisms* $F_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, *such that* $F_0 =$ *identity and* $F_1|_X = f$.

It is easy to check that the following categories are pseudo-algebraic: the category of Nash submanifolds of $\mathbb{R}^n$ with Nash mappings as morphisms, (here $\mathbf{S}' = \mathbf{NA}$), the category of Nash submanifolds of $\mathbb{R}^n$ with real analytic mappings as morphisms, (here $\mathbf{S}' \subset \mathbf{RA}$), the category of Nash submanifolds of $\mathbb{R}^n$ with smooth mappings as morphisms (here $\mathbf{S}' \subset \mathbf{S_0}$). In particular we have following:

Theorem 4.13. *Let* $X, Y \subset \mathbb{R}^n$ *be Nash (not necessarily connected) manifolds of dimension (not necessarily pure)* k. *Let* $f : X \to Y$ *be a diffeomorphism. If* $n \geq 2k + 2$, *then* f *can be extended to a tame diffeomorphism* $F : \mathbb{R}^n \to \mathbb{R}^n$. *Moreover, there is a smooth family of tame diffeomorphisms* $F_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, *such that* $F_0 =$ *identity and* $F_1|_X = f$.

This gives the following nice application to study complex algebraic varieties:

Corollary 4.14. *Let* $X, Y \subset \mathbb{C}^n$ *be smooth (not necessarily connected) complex algebraic manifolds of complex dimension (not necessarily pure)* k. *Let* $f : X \to Y$ *be a diffeomorphism. If* $n \geq 2k + 1$, *then* f *can be extended to a tame diffeomorphism* $F : \mathbb{C}^n \to \mathbb{C}^n$. *In particular, if two smooth algebraic complex curves* $X, Y \subset \mathbb{C}^3$ *are diffeomorphic, then they are (topologically) embedded into* $\mathbb{C}^3$ *in the same way.*

Proof. Indeed, we can treat X, Y as $2k$ dimensional real algebraic smooth submanifold of $\mathbb{C}^n \cong \mathbb{R}^{2n}$. By the assumption $2n \geq 2(2k + 1) = 2(2k) + 2$. □

We have also the analytic variant of Theorem 4.13:

Theorem 4.15. *Let $X, Y \subset \mathbb{R}^n$ be Nash (not necessarily connected) submanifolds of dimension (not necessarily pure) k. Let $f : X \to Y$ be a Nash isomorphism (real analytic isomorphism). If $n \geq 2k + 2$, then f can be extended to a tame Nash isomorphism (real analytic isomorphism) $F : \mathbb{R}^n \to \mathbb{R}^n$. Moreover, there is an analytic family of tame Nash isomorphisms (real analytic isomorphisms) $F_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, such that $F_0 =$ identity and $F_1|_X = f$.*

Example 4.16. Since some non-trivial knots $f : \mathbb{R} \to \mathbb{R}^3$ can be realized as a polynomial embedding (see e.g. [11]), we see that the assumption $n \geq 2k + 2$ in Theorem 4.13 and Theorem 4.15 is optimal.

5. Complex pseudo-algebraic categories

Now assume that our field is a complex field. By $\mathbf{S_0}$ we mean the category of all pairs $(X, \mathbb{C}^{n(X)})$, where $X \subset \mathbb{C}^{n(X)}$ is a smooth closed submanifold of $\mathbb{C}^{n(X)}$ and morphisms are smooth mappings. Every object of $\mathbf{S_0}$ is the pair $(X, \mathbb{C}^{n(X)})$, in further we will identify such object simply with X. In particular we will identify $(\mathbb{C}^n, \mathbb{C}^n)$ with $\mathbb{C}^n$. Let $\mathbf{S}$ be a subcategory of $\mathbf{S_0}$. We can easily extend Definition 4.3 to:

Definition 5.1. *We say that $\mathbf{S}$ is a fine category iff:*

1) for every $n \in \mathbb{N} :\ \mathbb{C}^n \in \mathbf{S}$,

2) if $f : X \to \mathbb{C}$ is in $\mathbf{S}$, then f can be extended to a mapping $F : \mathbb{C}^{n(X)} \to \mathbb{C}$ which is also in $\mathbf{S}$,

3) $\mathbb{C}-$linear mappings are in $\mathbf{S}$, moreover, if $X \in \mathbf{S}$, then the restriction of mappings from $\mathbf{S}$ to X are in $\mathbf{S}$,

4) if $X \in \mathbf{S}$, then the set $\mathbf{CS}(X) = \{f : X \to \mathbb{C};\ f \in \mathbf{S}\}$ is an $\mathbb{C}-$algebra,

5) if $X \in \mathbf{S}$ and $\pi : X \to \mathbb{C}^n$ is a projection, which is a smooth embedding, then f is an $\mathbf{S}-$embedding,

6) if $X \in \mathbf{S}$ and $f : X \to \mathbb{C}^n$ and $g : X \to \mathbb{C}^m$ are $\mathbf{S}-$morphisms, then $(f, g) : X \to \mathbb{C}^n \times \mathbb{C}^m$ is also an $\mathbf{S}-$morphism.

From now on our models of fine categories will be a category $\mathbf{St}$ of smooth Stein submanifolds $X \subset \mathbb{C}^{n(X)}$, with holomorphic mappings as morphisms and the category $\mathbf{Pl}$ of smooth algebraic submanifolds with polynomial mappings as morphisms. Note that again the fact that the category $\mathbf{St}$ satisfies property (2) is a non-trivial fact, which follows from the Cartan Theorem B, for details see [4]. We can also define a complex pseudo-algebraic category:

Definition 5.2. *Let $\mathbf{S}$ be a (complex) fine category and let $\mathbf{S}' \subset \mathbf{S}$ be a subcategory. We say that $\mathbf{S}'$ is a (complex) pseudo-algebraic sub-category in $\mathbf{S}$ (or shortly a pseudo-algebraic category) iff*

1) if $X \in \mathbf{S}'$, then X is a complex algebraic manifold,

2) if X, Y in $\mathbf{S}'$ then $Mor_S(X, Y) = Mor_{S'}(X, Y)$,

where $Mor_S(X, Y) = \{f \in S : f : X \to Y\}$.

Now we can repeat word by word results of the previous section to obtain:

Theorem 5.3. *Let $\mathbf{S}'$ be a pseudo-algebraic category and let $(X, \mathbb{C}^n), (Y, \mathbb{C}^n) \in \mathbf{S}'$ be smooth (not necessarily connected) complex manifolds of dimension (not necessarily pure) k. Let $f : X \to Y$ be an $\mathbf{S}'-$diffeomorphism. If $n \geq 2k + 2$, then f can be extended* *a tame* $\mathbf{S}'-$*diffeomorphism* $F : \mathbb{C}^n \to \mathbb{C}^n$. *Moreover, there is a smooth family of tame* $\mathbf{S}'-$*diffeomorphisms* $F_t : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$, *such that* $F_0 =$ *identity and* $F_1|_X = f$.

It is easy to check that the the category of smooth complex algebraic submanifolds with holomorphic mappings as morphisms, is a pseudo-algebraic category (here $\mathbf{S}' \subset \mathbf{S} = \mathbf{St}$). Similarly the category of smooth complex algebraic submanifolds with polynomial mappings as morphisms, is a pseudo-algebraic category (here $\mathbf{S}' = \mathbf{S} = \mathbf{Pl}$). In particular we have:

Theorem 5.4. *Let* $X, Y \subset \mathbb{C}^n$ *be smooth (not necessarily connected) complex algebraic submanifolds of dimension (not necessarily pure)* k. *Let* $f : X \to Y$ *be a biholomorphism (polynomial isomorphism). If* $n \geq 2k + 2$, *then* f *can be extended to a tame biholomorphism (polynomial automorphism)* $F : \mathbb{C}^n \to \mathbb{C}^n$. *Moreover, there is a smooth family of tame biholomorphisms (polynomial isomorphisms)* $F_t : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$, *such that* $F_0 =$ *identity and* $F_1|_X = f$.

Example 5.5. Let

$$X = \{(x_1, ..., x_n) \in \mathbb{C}^n : x_1 \cdot ... \cdot x_n = 1\}$$

and

$$Y = \{(x_1, ..., x_n) \in \mathbb{C}^n : x_1 \cdot (x_2 \cdot ... \cdot x_n)^s = 1\},$$

where $s \geq n \geq 3$. Let us consider the biholomorphic mapping

$$f : X_1 \ni (x_1, ..., x_n) \to (x_1^s, x_2, ..., x_{n-2}, e^{-x_1}x_{n-1}, e^{x_1}x_n) \in X_2.$$

We show that f can not be extend to a global biholomorphism $F : \mathbb{C}^n \to \mathbb{C}^n$. Indeed, assume that $F = (F_1, ..., F_n)$ is a such extension. Then

$$X = \{(x_1, ..., x_n) \in \mathbb{C}^n : F_1 \cdot (F_2 ... \cdot F_n)^s = 1\}$$

and the equation

$$F_1 \cdot (F_2 ... \cdot F_n)^s = 1$$

is reduced. This means that the first non-constant homogeneous term in an expansion of $F_1 \cdot (F_2 ... \cdot F_n)^s$ into a power series is $x_1 \cdot ... \cdot x_n$ (up to a non-zero constant factor). Since every F_i has a non-trivial linear part, this is impossible.$\square$

Example 5.6. (see [9]) Let $n \geq 4$ be an even number and consider the variety $S_{2n-1} = \{(x, y) \in \mathbb{C}^{2n} : \sum_{i=1}^n x_i y_i = 1\}$. Then the embeddings

$$\iota : S_{2n-1} \times \mathbb{C}^2 \ni ((x, y), (s, t)) \to ((x, y), s, t, 0, ..., 0) \in \mathbb{C}^{2n} \times \mathbb{C}^n,$$

and $\phi : S_{2n-1} \times \mathbb{C}^2 \to \mathbb{C}^{2n} \times \mathbb{C}^n$ given by

$$((x, y), (s, t)) \to ((x, y), y_1 s + x_2 t, y_2 s - x_1 t, y_3 s + x_4 t, y_4 s - x_3 t, ..., y_{n-1} s + x_n t, y_n s - x_{n-1} t),$$

are non-equivalent, i.e., there does not exist a biholomorphism

$$\Phi : \mathbb{C}^{2n} \times \mathbb{C}^n \to \mathbb{C}^{2n} \times \mathbb{C}^n,$$

such that $\Phi \circ \iota = \phi$.

Example 5.7. For every $n \geq 2$ there is a (closed) holomorphic embedding $f : \mathbb{C} \times \{0, ..., 0\} \to \mathbb{C}^n$, which cannot be extended to a biholomorphism $F : \mathbb{C}^n \to \mathbb{C}^n$ (for details see [5]). Of course, the reason is that the smooth Stein curve $Y = f(\mathbb{C})$ is far from being algebraic.

Example 5.8. Let $X, Y \subset \mathbb{C}$ be finite sets of points with $\#X = \#Y \geq 3$. Since every biholomorpism of $\mathbb{C}$ is a $\mathbb{C}-$linear mapping, we have that in general the non-trivial bijection $f : X \to Y$ can not be extended to a global biholomorphism $F : \mathbb{C} \to \mathbb{C}$. This means that at least for $k = 0$ the assumption $n \geq 2k + 2$ of Theorem 5.4 is optimal.

6. Semi-algebraic category

At the end of this paper we consider the category, which is not smooth. Let **SE** be a category of closed semi-algebraic subsets of $\mathbb{R}^n$, i.e., objects of this category are pairs $(X, \mathbb{R}^{n(X)})$ and $X \subset \mathbb{R}^{n(X)}$ is a closed semi-algebraic subsets of $\mathbb{R}^{n(X)}$. The morphisms in **SE** are continuous semi-algebraic mappings. This is in some sense a fine category, since we have:

1) for every $n \in \mathbb{N}: \ \mathbb{R}^n \in \mathbf{SE}$,

2) if $f : X \to \mathbb{R}$ is in **SE**, then f can be extended to a mapping $F : \mathbb{R}^{n(X)} \to \mathbb{R}$, which is also in **SE** (this is a semi-algebraic version of Tietze Extension Theorem, see e.g., [3], Proposition 2.6.9),

3) linear mappings are in **SE**, moreover, if $X \in \mathbf{SE}$, then the restriction of mappings from **SE** to X are in **SE**,

4) if $X \in \mathbf{SE}$, then the set $\mathbf{CSE}(X) = \{f : X \to \mathbb{R};\ f \in \mathbf{SE}\}$ is an $\mathbb{R}-$algebra,

5) if $X, \in \mathbf{SE}$ and $\pi : X \to \mathbb{R}^n$ is a projection which is a topological embedding, then f is an **SE**−embedding,

6) if $X \in \mathbf{SE}$ and $f : X \to \mathbb{R}^n$ and $g : X \to \mathbb{R}^m$ are **SE**−morphisms, then $(f, g) : X \to \mathbb{R}^n \times \mathbb{R}^m$ is also an **SE**−morphism.

By Proposition 4.6 we have:

Lemma 6.1. *If X is a semi-algebraic sets and $f : X \to \mathbb{R}^n$ is an* **SE**−*embedding, then the mapping*
$$f^* : \mathbf{CSE}(\mathbb{R}^n) \ni h \to h \circ f \in \mathbf{CSE}(X)$$
is an epimorphism.

Moreover, using basic properties of semi-algebraic sets it is not difficult to prove a topological counterparts of Lemma 4.9, Lemma 4.7 and Lemma 4.10 (the main idea is the same, we have to use the lemma below).

Lemma 6.2. *Let W be a semi-algebraic subset of $\mathbb{P}(\mathbb{R}^n)$. Let $(x_1, ..., x_k)$ be a system of linear homogenous polynomial on $\mathbb{P}(\mathbb{R}^n)$ such that $V(x_1, ..., x_k) \cap W = \emptyset$. Then for generic $\lambda = (\lambda_1, ..., \lambda_k) \in \mathbb{R}^k$ we have* $\dim\, W \cap V(\sum_{i=1}^k \lambda_i x_i) \leq \dim\, W - 1$.

Proof. Let $W = \bigcup_{i=1}^r W_i$ be the decomposition of W into irreducible (in a semi-algebraic sense) components. Let L_i be a linear subspace of $\mathbb{P}^n(\mathbb{R})$ spanned by W_i, $i = 1, ..., s$. A hyperplane H satisfies $\dim\, W \cap H = \dim\, W$ if and only if it contains some of L_i. If $\mathbb{P}(\lambda)$ parametrizes all hyperplanes of the type $\sum_{i=1}^k \lambda_i x_i = 0$, then those that contain some L_i form a linear subspace Λ_i of $\mathbb{P}(\lambda)$. By our assumption we have $\Lambda_i \neq \mathbb{P}(\lambda)$ for every i (since otherwise $W_i \subset V(x_1, ..., x_k) \cap W$). Hence the union $\bigcup_{i=1}^s \Lambda_i$ is a proper subset of $\mathbb{P}(\lambda)$ and the proof is finished. □

For example we give a proof of:

Lemma 6.3. *Let X be a closed semi-algebraic subset of $\mathbb{R}^n$ of dimension k. If $n > 2k + 1$, then there exists a system of coordinates $(x_1, ..., x_{2k+1}, x_{2k+2}, ..., x_n)$ such that the projection $\pi : X \ni (x_1, ..., x_{2k+1}, x_{2k+2}, ..., x_n) \to (x_1, ..., x_{2k+1}, 0, ..., 0) \in \mathbb{R}^{2k+1} \times \{0\}$ is a topological embedding.*

Proof. Again we follow the proof of Lemma 3.3. Let us denote by π_∞ the hyperplane at infinity of $\mathbb{R}^n$. Thus $\pi_\infty \cong \mathbb{P}^{n-1}$ is a real projective space of dimension $n - 1 > 2k$. For a non-zero vector $v \in \mathbb{R}^n$ let $[v]$ denote an appropriate point in $\mathbb{P}^{n-1}$.

Let $\Delta = \{(x, y) \in X \times X : x = y\}$. Consider a mapping

$$A : X \times X \setminus \Delta \ni (x, y) \to [x - y] \in \pi_\infty.$$

Since A is a semi-algebraic mapping and the semi-algebraic set $X \times X \setminus \Delta$ is of dimension $2k$, we conclude that the set $\Lambda := A(X \times X \setminus \Delta)$ is (projectively) semi-algebraic of dimension at most $2k$. This means that also the set $\Sigma = closure\ of\ \Lambda$ is (projectively) semi-algebraic of dimension at most $2k$ (for details see e.g., [2]). Consequently we have $\pi_\infty \setminus \Sigma \neq \emptyset$.

Let $P \in \pi_\infty \setminus \Sigma$ and let $H \subset \mathbb{R}^{n-1}$ be a hyperplane, which does not contain the point P (at infinity). Since $P \notin \Lambda$ we have that the projection $S : X \ni x \to Px \cap H \in H \cong \mathbb{R}^{n-1}$ is an injection. Moreover, since $P \notin \Sigma$ we get that the S is also proper, hence it is a topological embedding.

Now we can apply the mathematical induction. □

Lemma 6.4. *Let $(X, \mathbb{R}^n) \in$ **SE** be a closed subset of dimension k. Assume that the projection $\pi : X \ni (x_1, ..., x_n) \to (x_1, ..., x_l, 0, ..., 0) \in \mathbb{R}^l \times \{0\}$ is a topological embedding. Then, there exists a tame **SE**−homeomorphism $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Pi|_X = \pi$.*□

Lemma 6.5. *Let $(X, \mathbb{R}^{2n}) \in$ **SE**, where $n \geq 2k + 1$. Consider mappings*

$$\pi_1 : X \ni (x_1, ..., x_n, y_1, ..., y_n) \to (y_1, ..., y_n) \in \mathbb{R}^n$$

and

$$\pi_2 : X \ni (x_1, ..., x_n, y_1, ..., y_n) \to (x_1, ..., x_n) \in \mathbb{R}^n.$$

If π_1, π_2 are (closed) embeddings, then there are affine coordinates $(X_1, ..., X_n)$ in $\mathbb{R}^n \times \{0\}$ and affine coordinates $(Y_1, ..., Y_n)$ in $\{0\} \times \mathbb{R}^n$ such that all projection

$$q_r : X \ni (X_1, ..., X_n, Y_1, ..., Y_n) \to (X_1, ..., X_r, Y_{r+1}, ..., Y_n) \in \mathbb{R}^n,\ r = 0, ..., n$$

are topological embeddings. □

Now we can repeat nearly word by word the proof of the Theorem 3.5 to obtain:

Theorem 6.6. *Let $X, Y \subset \mathbb{R}^n$ be closed semi-algebraic subsets of dimension k. Let $f : X \to Y$ be a semi-algebraic homeomorphism. If $n \geq 2k + 2$, then f can be extended to a tame semi-algebraic homeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$.*

Since every triangle homeomorphism $G = (x_1, ..., x_{n-1}, x_n + P_n(x_1, ..., x_{n-1}))$ is homeotopic to the identity by $t \to G_t = (x_1, ..., x_{n-1}, x_n + tP_n(x_1, ..., x_{n-1}))$ and the same is true for linear mapping with determinant equal to 1, we have:

Corollary 6.7. *Let $X, Y \subset \mathbb{R}^n$ be closed semi-algebraic subsets of dimension k. Let $f : X \to Y$ be a semi-algebraic homeomorphism. If $n \geq 2k + 2$, then X, Y are semi-algebraically homeotopic, i.e., there is a continuous semi-algebraic family $t \to G_t$ of semi-algebraic homeomorphisms $G_t : \mathbb{R}^n \to \mathbb{R}^n$, such that $G_0 = identity$ and $G_1|_X = f$.*

Remark 6.8. Example 4.16 shows that assumption $n \geq 2k+2$ in Theorem 6.6 is essential.

Remark 6.9. In the paper [6] Herman Gluck have obtained similar results for "mild" topological embeddings of compact polyhedrons. In some sense this section is an extension of his results to the non-compact case.

7. Example

If X be a $k-$dimensional Nash submanifold of $\mathbb{R}^n$ and $n > 2k+1$ then X has a unique Nash embedding into $\mathbb{R}^n$. We know that for $k = 1$ and $n = 3$ this result is optimal. It is interesting, whether it is also optimal for large k and n.

We give examples of Nash manifolds $X_{n+1} \subset \mathbb{R}^{2n}$, (where n is any even number different from $2, 4, 8$) which has at least two different Nash embedding into $\mathbb{R}^{2n}$. This means that our results can not be much improved for large n.

Theorem 7.1. *Let* $\mathbf{S}^{n-1} \subset \mathbb{R}^n$ *be a sphere where* n *is an even number different from* $2, 4, 8$. *The embeddings*

$$\iota : \mathbf{S}^{n-1} \times \mathbb{R}^2 \ni (x, (s, t)) \to (x, s, t, 0, ..., 0) \in \mathbb{R}^n \times \mathbb{R}^n,$$

and $\phi : \mathbf{S}^{n-1} \times \mathbb{R}^2 \to \mathbb{R}^n \times \mathbb{R}^n$ *given by*

$$(x, (s, t)) \to (x, x_1 s + x_2 t, x_2 s - x_1 t, x_3 s + x_4 t, x_4 s - x_3 t, ..., x_{n-1} s + x_n t, x_n s - x_{n-1} t),$$

are non-equivalent, i.e., does not exist a diffeomorphism

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n,$$

such that $\Phi \circ \iota = \phi$.

Proof. Let n be an even number and $\mathbf{S}^{n-1} \subset \mathbb{R}^n$ be a sphere. It is well known that for $n \neq 2, 4, 8$ the tangent bundle $\mathbf{A} = T\mathbf{S}^{n-1}$ is not trivial. However, since the normal bundle $\mathbf{N}(\mathbf{S}^{n-1})$ is trivial, we have that $\mathbf{A}$ is stably trivial. In fact, if $\mathbf{E_r}$ denote a trivial bundle of rank r on the sphere, then $\mathbf{A} \oplus \mathbf{E_1} = \mathbf{E_n}$.

More precisely, let $x_1, ..., x_n$ be standard coordinates in $\mathbb{R}^n$. Let $\mathbf{E}' \subset \mathbf{E_n}$ be a subbundle of rank 1 generated by the vector $\mathbf{a} = (x_1, ..., x_n)$ and $\mathbf{E}''$ be a subbundle generated by the vector $\mathbf{b} = (x_2, -x_1, x_4, -x_3, ..., x_n, -x_{n-1})$.

It is easy to see that

$$F : \mathbf{E_n} \ni (v_1, ..., v_n) \to \sum_{i=1}^{n} x_i v_i \in \mathbf{E_1}$$

and

$$G : \mathbf{E_n} \ni (v_1, ..., v_n) \to x_2 v_1 - x_1 v_2 + ... + x_n v_{n-1} - x_{n-1} v_n \in \mathbf{E_1}$$

are morphisms of vector bundles. Moreover, $F(\mathbf{a}) = \mathbf{1}$ and $G(\mathbf{b}) = \mathbf{1}$. This means that $\mathbf{E}'$ and $\mathbf{E}''$ are prime factors in $\mathbf{E_n}$.

Since $ker\ F = \mathbf{A}$, we have $\mathbf{E}' \oplus \mathbf{A} = \mathbf{E_n}$. Moreover, since $F(\mathbf{b}) = \mathbf{0}$ we have $\mathbf{E}'' \subset \mathbf{A}$. In particular, this means that there exist a subbundle $\mathbf{C} \subset \mathbf{E_n}$, such that $\mathbf{A} = \mathbf{E}'' \oplus \mathbf{C}$. By the construction we have $\mathbf{C} \oplus < \mathbf{a}, \mathbf{b} > = \mathbf{E_n}$, where $< \mathbf{a}, \mathbf{b} >$ denote the subbundle generated by vectors $\mathbf{a}$ and $\mathbf{b}$ (please check that it is really a subbundle!).

Now consider the embedding

$$\phi : \mathbf{S}^{n-1} \times \mathbb{R}^2 \ni (x, (s, t)) \to (x, x_1 s + x_2 t, x_2 s - x_1 t, ..., x_{n-1} s + x_n t, x_n s - x_{n-1} t) \in \mathbb{R}^n \times \mathbb{R}^n.$$

By direct computations we see that the normal bundle $\mathbf{N}(\phi(\mathbf{S}^{n-1} \times \mathbb{R}^2))$ restricted to the submanifold $\mathbf{S}^{n-1} \times \{0\}$ is equal to

$$\mathbf{N}(\mathbf{S}^{n-1}) \oplus (\mathbf{E_n} / < \mathbf{a}, \mathbf{b} >) = \mathbf{E_1} \oplus \mathbf{C} = \mathbf{A} = T\mathbf{S}^{n-1}.$$

This means that this normal bundle is not trivial along $\mathbf{S}^{n-1} \times \{0\}$.

However it is easy to see that the normal bundle $\mathbf{N}(\iota(\mathbf{S}^{n-1} \times \mathbb{R}^2))$ restricted to the submanifold $\mathbf{S}^{n-1} \times \{0\}$ is trivial. Since ϕ and ι coincide along $\mathbf{S}^{n-1} \times \{0\}$, this implies that there is not a diffeomorphism

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n,$$

such that $\Phi \circ \iota = \phi$. □

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