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Diffeomorphisms that are Symplectomorphisms

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DIFFEOMORPHISMS THAT ARE SYMPLECTOMORPHISMS

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Abstract. Let $(X, \omega_X)$ and $(Y, \omega_Y)$ be compact symplectic manifolds of dimension $2n > 2$. Let us fix a number $k$ with $0 < k < n$ and assume that a diffeomorphism $\Phi : X \to Y$ transforms all $2k$-dimensional symplectic submanifolds of $X$ onto symplectic submanifolds of $Y$. Then $\Phi$ is a conformal symplectomorphism, i.e., there is a constant $c \neq 0$ such that $\Phi^* \omega_Y = c \omega_X$.

1. Introduction.

Let $(X, \omega_0)$ be a standard symplectic affine space over $\mathbb{R}$ of dimension $2n$, i.e., $X \cong \mathbb{R}^{2n}$ and $\omega_0 = \sum_i dx_i \wedge dy_i$ is the standard non-degenerate skew-symmetric form on $X$. Linear symplectomorphisms of $(X, \omega_0)$ are characterized (cf. [3]) as linear automorphisms of $X$ preserving some minimal, complete data defined by $\omega_0$ on systems of linear subspaces. In this way the linear symplectic group $\text{Sp}(X)$ may be characterized geometrically together with its natural conformal and anti-symplectic extensions.

The purpose of this paper is to put the linear considerations of symplectic invariants into a more general context. Let $(X, \omega_X)$ and $(Y, \omega_Y)$ be compact symplectic manifolds of dimension $2n$ (all manifolds in this paper are assumed to be connected). We say that a diffeomorphism $F : X \to Y$ is a conformal symplectomorphism if there is a non-zero constant $c \in \mathbb{R}$ such that $F^* \omega_Y = c \omega_X$. Recall that a submanifold $Z \subset X$ is a symplectic submanifold of $X$ if it is closed and the pair $(Z, \omega_X|_{TZ})$ is itself a symplectic manifold. Our main result is:

Theorem. Let $(X, \omega_X)$ and $(Y, \omega_Y)$ be compact symplectic manifolds of dimension $2n > 2$. Fix a number $0 < s < n$. Assume that $\Phi : X \to Y$ is a diffeomorphism which transforms all $2s$-dimensional symplectic (closed) submanifolds of $X$ onto symplectic (closed) submanifolds of $Y$. Then $\Phi$ is a conformal symplectomorphism.

In other words, for any fixed $s$ as above, the conformal symplectic structure on $X$ is uniquely determined by the family of all $2s$-dimensional (closed) symplectic submanifolds of $X$.
2. Generators of the group $Sp(2n)$

Here we recall some basic facts about the linear symplectic group. Let $(X, \omega)$ be a symplectic vector space. There exists a basis of $X$, called a symplectic basis, $u_1, \ldots, u_n, v_1, \ldots, v_n$, such that
\[\omega(u_i, u_j) = \omega(v_i, v_j) = 0, \quad \omega(u_i, v_j) = \delta_{ij}.\]

Let $(X, \omega_X)$ and $(Y, \omega_Y)$ be symplectic vector spaces. We say that a linear isomorphism $F : X \to Y$ is a symplectomorphism (or is symplectic on $X$) if $F^* \omega_Y = \omega_X$, i.e., $\omega_X(x, y) = \omega_Y(F(x), F(y))$ for every $x, y \in X$. The group of automorphisms of $(X, \omega)$ is called the symplectic group and is denoted by $Sp(X, \omega)$. Via a symplectic basis, $X$ can be identified with the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ and $Sp(X, \omega)$ can be identified with the group of $2n \times 2n$ real matrices $A$ which satisfy $A^T J_0 A = J_0$, where $J_0$ is the $2n \times 2n$ matrix of $\omega_0$ (in the standard basis), i.e.,
\[
J_0 = \begin{pmatrix}
0 & \ldots & 0 & -1 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & \ldots & -1 \\
1 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 1 & 0 & \ldots & 0
\end{pmatrix}.
\]

Let $c \in \mathbb{R}$ and $i < j$. We can define following "elementary" symplectomorphisms:

1) $L_i(c)(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1, \ldots, x_{i-1}, y_i + cy_i, y_{i+1}, \ldots, y_n),$

2) $L_{ij}(c)(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1, \ldots, x_i, y_i + cy_i, x_{i+1}, \ldots, y_{j-1}, y_{j-1}, y_j + cy_j, y_{j+1}, \ldots, y_n),$

3) $R_i(c)(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1, \ldots, x_{i-1}, x_i + cy_i, x_{i+1}, \ldots, x_n, y_1, \ldots, y_n),$

4) $R_{ij}(c)(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1, \ldots, x_{i-1}, x_i + cy_i, x_{i+1}, \ldots, x_{j-1}, x_j + cy_j, x_{j+1}, \ldots, x_n, y_1, \ldots, y_n).$

We have the following basic result:

**Theorem 2.1.** Let $X = (\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic vector space. Then the group $Sp(X)$ is generated by the following family of elementary symplectomorphisms:

\[\left\{ L_i(c), L_{ij}(c), R_i(c), R_{ij}(c) : 0 < i < j \leq n \text{ and } c \in \mathbb{R} \right\}.\]

**Proof.** We reason by induction. For $n = 1$ we have $Sp(\mathbb{R}^2) = SL(2)$ and the result is well known from linear algebra. Assume $n > 1$.

Let $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear symplectomorphism. Denote coordinates by $x_1, y_1, \ldots, x_n, y_n$ (where $\omega_0 = \sum_i dx_i \wedge dy_i$). We have
\[
S(x_1, y_1, \ldots, x_n, y_n) = (\sum_i a_{1,i}x_i + \sum_j b_{1,j}y_j, \ldots, \sum_i a_{2n,i}x_i + \sum_j b_{2n,j}y_j).
\]

Observe how the rows of the matrix of $S$ are transformed under composition $S \circ L$ with an elementary symplectomorphism $L$ (for simplicity we consider only the first row and we take the coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$). After composition

\[
S(x_1, y_1, \ldots, x_n, y_n) = (\sum_i a_{1,i}x_i + \sum_j b_{1,j}y_j, \ldots, \sum_i a_{2n,i}x_i + \sum_j b_{2n,j}y_j).
\]
with $L_i(c)$ we have:

1) $(a_{11}, ..., a_{1n}, b_{11}, ..., b_{1n}) \to (a_{11}, ..., a_{1i} + cb_{1j}, ..., a_{1n}, b_{11}, ..., b_{1n})$,

with $L_{ij}(c)$ we have:

2) $(a_{11}, ..., a_{1n}, b_{11}, ..., b_{1n}) \to (a_{11}, ..., a_{1i} + cb_{1j}, ..., a_{1n}, b_{11}, ..., b_{1n})$,

with $R_i(c)$ we have:

3) $(a_{11}, ..., a_{1n}, b_{11}, ..., b_{1n}) \to (a_{11}, ..., a_{1n}, b_{11}, ..., b_{1i} + ca_{1j}, ..., b_{1n})$,

with $R_{ij}(c)$ we have:

4) $(a_{11}, ..., a_{1n}, b_{11}, ..., b_{1n}) \to (a_{11}, ..., a_{1n}, b_{11}, ..., b_{1i} + ca_{1j}, ..., b_{1n})$.

Transformations 1) - 4) will be called elementary operations. Now we show that using only elementary operations we can transform the first row of $S$ to $(1, 0, ..., 0)$ and the second to $(0, ..., 0, 1, 0, ..., 0)$ (here the unit corresponds to $b_{1n}$).

Indeed, consider the first row. Of course it has a non-zero element, say $b_{11}$. Using $L_1(c)$ we can assume that also $a_{11} \neq 0$. Now using $L_{is}(c)$ and $R_{js}(d)$ for sufficiently general $c$ and $d$ we can assume that all elements of the first row are non-zero. Again applying $R_i(c)$ for $i > 1$ we can now transform the first row to $(a_{11}, ..., a_{1n}, 1, 0, ..., 0)$. Using $L_{1j}(c)$ we can transform this row to $(1, 0, ..., 0, 1, 0, ..., 0)$ and finally using $R_1(-1)$ we obtain $(1, 0, ..., 0)$. Now consider the second row (after these transformations): $(a_{21}, ..., a_{2n}, b_{21}, ..., b_{2n})$. We can apply our method to the subrow $(a_{22}, ..., a_{2n}, b_{22}, ..., b_{2n})$ (if it is non-zero) and obtain finally the row $(a_{21}, 1, 0, ..., 0, b_{21}, 0, ..., 0)$ (or $(a_{21}, 0, ..., 0, b_{21}, 0, ..., 0)$). Since the value of $\omega_0$ on these two rows is 1 we conclude that $b_{21} = 1$. Now (in the first case) we can use $L_{12}(-1)$ to obtain a row of the form $(a_{21}, 0, ..., 0, 1, 0, ..., 0)$. Finally applying $L_1(-a_{12})$ we get $(0, ..., 0, 1, 0, ..., 0)$.

Thus under all these compositions the matrix of $S$ in the coordinates $x_1, y_1, ..., x_n, y_n$ has the form

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
* & * & a_{33} & \cdots & b_{3n} \\
* & * & a_{43} & \cdots & b_{4n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & a_{n3} & \cdots & b_{n1}
\end{bmatrix}
\]

Let $r_i$ denote the $i^{th}$ row of the matrix of $S$. For $j > 2$ we have $\omega_0(r_1, r_j) = 0$ and $\omega_0(r_2, r_j) = 0$.

We can easily conclude that all the * in the matrix of $S$ are 0. Since

\[
\begin{bmatrix}
a_{33} & \cdots & b_{3n} \\
a_{43} & \cdots & b_{4n} \\
\vdots & \vdots & \vdots \\
a_{n3} & \cdots & b_{n1}
\end{bmatrix}
\]

is a symplectic matrix we can apply the induction hypothesis. \qed
We conclude this section by recalling (and extending) some result from [3].

**Definition 2.2.** Let $\mathcal{A}_{l,2r} \subset G(l, 2n)$ denote the set of all $l$-dimensional linear subspaces of $X$ on which the form $\omega$ has rank $\leq 2r$.

Of course $\mathcal{A}_{l,2r} \subset \mathcal{A}_{l,2r+2}$ if $2r + 2 \leq l$. We have the following (see [3], Theorem 6.2):

**Proposition 2.3.** Let $(X, \omega)$ be a symplectic vector space of dimension $2n$ and let $F : X \to X$ be a linear automorphism. Let $0 < 2r < 2n$. Assume $F$ transforms $\mathcal{A}_{2r,2r-2}$ into $\mathcal{A}_{2r,2r-2}$. Then there is a non-zero constant $c$ such that $F^*\omega = c\omega$.

From Proposition 2.3 we can deduce the following interesting fact:

**Proposition 2.4.** Let $(X, \omega_X)$ and $(Y, \omega_Y)$ be symplectic vector spaces of dimension $2n$ and let $F : X \to Y$ be a linear isomorphism. Fix a number $s : 0 < s < n$ and assume that $F$ transforms all $2s$-dimensional symplectic subspaces of $X$ onto symplectic subspaces of $Y$. Then there is a non-zero constant $c$ such that $F^*\omega_Y = c\omega_X$.

**Proof.** Via a symplectic basis we can assume that $(X, \omega_X) \cong (\mathbb{R}^{2n}, \omega_0) \cong (Y, \omega_Y)$. By assumption the mapping $F^*$ induced by $F$ transforms the set $A = \mathcal{A}_{2s,2s} \setminus \mathcal{A}_{2s,2s-2}$ into the same set $A$. Of course $F^* : A \to A$ is an injection. Since $A$ is a smooth algebraic variety and $F^*$ is regular, the Borel Theorem (see [1]) implies that $F^*$ is a bijection. This means that $F$ transforms $\mathcal{A}_{2s,2s-2}$ into the same set, and we conclude the proof by applying Proposition 2.3. □

We end this section by:

**Proposition 2.5.** Let $X$ be a vector space of dimension $2n$ and let $\omega_1, \omega_2$ be two symplectic forms on $X$. If $\text{Sp}(X, \omega_1) \subset \text{Sp}(X, \omega_2)$, then there exists a non-zero constant $c$ such that $\omega_2 = c\omega_1$.

**Proof.** If $n = 1$, then theorem is obvious. Assume that $n > 1$. Let $A_1$ ($A_2$) be a set of all $\omega_1$ ($\omega_2$) symplectic $2$ dimensional subspaces of $X$. These sets are open and dense in the Grassmannian $G(2, 2n)$. Hence $A_1 \cap A_2 \neq \emptyset$. Take $H \in A_1 \cap A_2$. We have $A_1 = \text{Sp}(X, \omega_1)H \subset \text{Sp}(X, \omega_2)H = A_2$. Now apply Proposition 2.4 to $X = (X, \omega_1)$, $Y = (X, \omega_2)$ and $F = \text{identity}$. □

3. **Technical Results**

Let $X = (\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic vector space. In $X$ we consider the norm $\| (a_1, \ldots, a_{2n}) \| = \max_{i=1}^{2n} |a_i|$. Take a smooth function $H : X \times \mathbb{R} \ni (z, t) \to \mathbb{R}$ and consider a system of differential equations

$$\phi'(t, x) = J_0(\nabla_x H)(\phi(t), t), \quad \phi(0, x) = x.$$  
Assume that this system has a solution $\phi(t, x)$ for every $x$ and every $t$ (this is satisfied, e.g., if supports of all functions $H_t$, $t \in \mathbb{R}$ are contained in a compact set). Then we can define the diffeomorphism...
(3.1) \( \Phi(x) = \phi(1, x) \)

It is not difficult to check that \( \Phi \) is a symplectomorphism.

**Definition 3.1.** Let \( \Phi : X \to X \) be a symplectomorphism. We say that \( \Phi \) is a *hamiltonian symplectomorphism* if it is given by the formula (3.1) for some smooth function \( H \). We also say that \( H \) is a Hamiltonian of \( \Phi \).

**Lemma 3.2.** All elementary linear symplectomorphisms are hamiltonian symplectomorphisms.

**Proof.** Indeed, we have:

1) \( L_i(c) \) is given by the Hamiltonian \( H(x, y) = (c/2)x_i^2 \),
2) \( L_{ij}(c) \) is given by the Hamiltonian \( H(x, y) = cx_ix_j \),
3) \( R_i(c) \) is given by the Hamiltonian \( H(x, y) = -(c/2)y_i^2 \),
4) \( R_{ij}(c) \) is given by the Hamiltonian \( H(x, y) = -cy_iy_j \). \( \square \)

Now we show how to compute a Hamiltonian of a linear symplectomorphism:

**Theorem 3.3.** Let \( L : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) be a linear symplectomorphism. Then \( L \) has a polynomial Hamiltonian

\[
H_L(z, t) = \sum_{i,j=1}^{2n} a_{i,j}(t)z_iz_j,
\]

where \( a_{i,j}(t) \in \mathbb{R}[t] \) are polynomials of one variable \( t \). Moreover, we can compute \( H_L \) effectively.

**Proof.** Let \( L = L_m \circ \cdots \circ L_1 \) where the \( L_i \) are elementary symplectomorphisms. We proceed by induction with respect to \( m \). If \( m = 1 \) then we can use Lemma 3.2. In this case the flow \( L_1(t) \) depends linearly on \( t \).

Now consider \( L' = L_{m-1} \circ \cdots \circ L_1 \). By the induction hypothesis \( L'(t) = L_{m-1}(t) \circ \cdots \circ L_1(t) \) is given by the Hamiltonian \( H' \) of the form 3.2. Let \( H'' \) be the Hamiltonian of \( L_m \) (as in Lemma 3.2). Now the flow \( L(t) = L_m(t) \circ L'(t) \) is given by the Hamiltonian

\[
H(z, t) = H''(z) + H'(L_m(t)^{-1}(z), t).
\]

Of course it has also the form 3.2. Since we can decompose \( L \) into the product \( L = L_m \circ \cdots \circ L_1 \) effectively (see the proof of Theorem 2.1), we can also compute \( H \) in effective way. \( \square \)

**Proposition 3.4.** Let \( L : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) be a hamiltonian symplectomorphism given by the flow \( x \to \phi(t, x) \); \( t \in \mathbb{R} \). Assume that \( \phi(t, 0) = 0 \) for \( t \in [0, 1] \). For every \( \eta > 0 \) there is an \( \epsilon > 0 \) and a hamiltonian symplectomorphism \( \Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) such that

1) \( \Phi(x) = L(x) \) for all \( x \) with \( \|x\| \leq \epsilon \),
2) \( \Phi(x) = x \) for all \( x \) with \( \|x\| \geq \eta \).
Proof. We know that $L(x) = \phi(1, x)$, where $\phi(t, x)$ is the solution of some differential equation

$$
\phi'(t) = J_0(\nabla_z H)(\phi(t), t); \quad \phi(0) = x.
$$

Since $\phi(t, 0) = 0$ for every $t \in [0, 1]$, we can find $\epsilon > 0$ so small, that all trajectories $\{\phi(t, x), 0 \leq t \leq 1\}$, which start from the ball $B(0, \epsilon)$ are contained in the ball $B(0, \eta/2)$. Let $\sigma : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function such that

$$
\sigma(z) = \begin{cases} 
1 & \text{if } ||z|| \leq \eta/2, \\
0 & \text{if } ||z|| \geq \eta.
\end{cases}
$$

Take $S = \sigma H$. The Hamiltonian symplectomorphism $\Phi$ given by the differential equation

$$
\phi'(t) = J_0(\nabla_z S)(\phi(t), t), \quad \phi(0) = x,
$$

is well defined on the whole of $\mathbb{R}^{2n}$ and

$$
\Phi(x) = \begin{cases} 
L(x) & \text{if } ||x|| \leq \epsilon, \\
x & \text{if } ||x|| \geq \eta.
\end{cases}
$$

□

Now Theorem 3.3 easily yields the following important:

**Corollary 3.5.** Let $L : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear symplectomorphism. For every $\eta > 0$ there is an $\epsilon > 0$ and a Hamiltonian symplectomorphism $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that

1) $\Phi(x) = L(x)$ for all $x$ with $||x|| \leq \epsilon$,

2) $\Phi(x) = x$ for all $x$ with $||x|| \geq \eta$.

Before we formulate our next result we need the following:

**Lemma 3.6.** Let $X = (\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic vector space. Fix $\eta > 0$ and let $a, b \in B(0, \eta)$. Then there exists a symplectomorphism $\Phi : X \to X$ such that

$$
\Phi(a) = b \text{ and } \Phi(x) = x \text{ for } ||x|| \geq 2\eta.
$$

**Proof.** Let $c = (c_1, \ldots, c_{2n}) = b - a$. Define a sequence of points as follows:

1) $a_0 = a$,

2) $a_i = a_{i-1} + (0, \ldots, 0, c_i, 0, \ldots, 0)$.

Of course $a_i \in B(0, \eta)$ and $a_{2n} = b$. Now consider the translation

$$
T_i : \mathbb{R}^{2n} \ni (x, y) \mapsto (x, y) + (0, \ldots, 0, c_i, 0, \ldots, 0) \in \mathbb{R}^{2n}.
$$

We have $T_i(a_{i-1}) = a_i$ for $i = 1, \ldots, 2n$.

The translation $T_i$ is a Hamiltonian symplectomorphism given by the Hamiltonian

$$
H_i(x, y) = \begin{cases} 
-c_iz_i & \text{if } i \leq n, \\
c_ix_{i-n} & \text{if } i > n.
\end{cases}
$$
Let \( V_i \) be the symplectic vector field which is determined by the Hamiltonian \( H_i \). Since the ball \( B(0,r) \) is a convex set, all trajectories \( \phi(t) \), \( 0 \leq t \leq 1 \), of the symplectic vector fields \( V_i \), which begin at \( a_i \) lie in the ball \( B(0,\eta) \). Let \( \sigma: \mathbb{R}^{2n} \to \mathbb{R} \) be a smooth function such that

\[
\sigma(x) = \begin{cases} 
1 & \text{if } \|x\| \leq \eta, \\
0 & \text{if } \|x\| \geq 2\eta.
\end{cases}
\]

Now let \( F_i: \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) be the hamiltonian symplectomorphism given by the Hamiltonian \( G_i = \sigma H_i \). Then

\[
G_i(a_{i-1}) = a_i \text{ and } G_i(x) = x \text{ if } \|x\| \geq 2\eta.
\]

Now it is enough to take \( \Phi = G_{2n} \circ G_{2n-1} \circ \cdots \circ G_{1} \).

We apply Proposition 3.5 to the general case:

**Theorem 3.7.** Let \((X,\omega)\) be a symplectic manifold. Let \( a_1, \ldots, a_m \) and \( b_1, \ldots, b_m \) be two families of points of \( X \). For every \( i = 1, \ldots, n \) choose a linear symplectomorphism \( L_i: T_{a_i}X \to T_{b_i}X \). Then there is a symplectomorphism \( \Phi: X \to X \) such that

1) \( \Phi(a_i) = b_i \),

2) \( d_{a_i}\Phi = L_i \).

**Proof.** By the Darboux Theorem every point \( x \in X \) has an open neighborhood \( V_x \) which is symplectically isomorphic to the ball \( B(0,r_x) \) in the standard vector space \((\mathbb{R}^{2n},\omega_0)\). Denote by \( U_x \subset V_x \) the open set which corresponds to the ball \( B(0,r_x/3) \).

Since \( \dim X \geq 2 \) the manifold \( X \setminus \{a_2, \ldots, a_m\} \) is also connected. Hence there exists a smooth path \( \gamma: I \to X \) such that \( \gamma(0) = a_1, \gamma(1) = b_1 \) and \( \{a_2, \ldots, a_m\} \cap \gamma(I) = \emptyset \). Additionally we can assume that the sets \( V_x \) which cover \( \gamma(I) \) are also disjoint from \( \{a_2, \ldots, a_m\} \).

Let \( \epsilon \) be a Lebesgue number for the function \( \gamma: I \to X \) with respect to the cover \( \{U_x\}_{x \in X} \) and choose an integer \( N \) with \( 1/N < \epsilon \). If \( I_k := [k/N,(k+1)/N] \), then \( \gamma(I_k) \) is contained in some \( \{U_x\} \); denote it by \( U_k \), the set \( V_x \) by \( V_k \), and \( r_x \) by \( r_k \). Let \( A_k := \gamma(k/N) \), in particular \( A_0 = a_1, A_N = b_1 \).

Since \( V_k \cong B(0,r_k) \) and \( A_k, A_{k+1} \in B(0,r_k/3) \) we can apply Lemma 3.6 to obtain a symplectomorphism \( \Phi: B(0,r_k) \to B(0,r_k) \) such that

\[
\Phi(A_k) = A_{k+1} \text{ and } \Phi(x) = x \text{ for } \|x\| \geq (2/3)r_k.
\]

We can extend \( \Phi \) to the whole of \( X \) (we glue it with the identity); denote this extension by \( \Phi_k \). Put

\[
\Psi = \Phi_N \circ \Phi_{N-1} \circ \cdots \circ \Phi_0.
\]

Then \( \Psi(a_1) = b_1 \) and \( \Psi(a_i) = a_i \) for \( i > 1 \). Repeating this process, we finally arrive at a symplectomorphism \( \Sigma: X \to X \) such that \( \Sigma(a_i) = b_i \) for \( i = 1, \ldots, m \). In a similar way using Proposition 3.5 we can construct a symplectomorphism \( \Pi: X \to X \) such that

1) \( \Pi(b_i) = b_i \).
2) $d_b \Pi = L_i \circ (d_{a_i} \Sigma)^{-1}$.

Now it is enough to take $\Phi = \Pi \circ \Sigma$. \hfill $\square$

Now we need the following result which is due to S.K. Donaldson (see [2]):

**Theorem 3.8.** Let $(X, \omega_X)$ be a compact symplectic manifold of dimension $2n > 2$. Fix a number $0 < s < n$. There exists a closed $2s$-dimensional symplectic submanifold $Z \subset X$.

Using Theorem 3.7 we can restate this result as follows:

**Proposition 3.9.** Let $(X, \omega)$ be a compact symplectic manifold of dimension $2n > 2$. Let $a_1, \ldots, a_m$ be a family of points of $X$. Take $0 < s < n$. For every $i = 1, \ldots, m$ choose a linear $2s$-dimensional symplectic subspace $H_i \subset T_{a_i}X$. Then there is a closed symplectic $2s$-dimensional submanifold $Y \subset X$ such that

1) $a_i \in Y$,
2) $T_{a_i}Y = H_i$.

**Proof.** Let $Z \subset X$ be as in Theorem 3.8. Take points $b_1, \ldots, b_m \in Z$. Let $S_i = T_{b_i}Z$. There are linear symplectomorphisms $L_i : T_{b_i}X \to T_{a_i}X$ such that $L_i(S_i) = H_i$ for $i = 1, \ldots, m$. By Theorem 3.7 there is a symplectomorphism $\Phi : X \to X$ such that

1) $\Phi(b_i) = a_i$,
2) $d_b \Phi = L_i$.

Now it is enough to take $Y = \Phi(Z)$. \hfill $\square$

4. Main result

Finally we show that a symplectomorphism can be described as a diffeomorphism which preserves symplectic submanifolds.

**Theorem 4.1.** Let $(X, \omega_X)$ and $(Y, \omega_Y)$ be compact symplectic manifolds of dimension $2n > 2$. Fix a number $0 < s < n$. Assume that $\Phi : X \to Y$ is a diffeomorphism which transforms all $2s$-dimensional symplectic submanifolds of $X$ onto symplectic submanifolds of $Y$. Then $\Phi$ is a conformal symplectomorphism, i.e., there exists a non-zero number $c \in \mathbb{R}$ such that

$$\Phi^* \omega_Y = c \omega_X.$$ 

**Proof.** Fix $x \in X$ and let $H \subset T_x X$ be a $2s$-dimensional symplectic subspace of $T_x X$. By Proposition 3.9 (applied for $m = 1$, $a_1 = x$ and $H_1 = H$) there exists a $2s$-dimensional symplectic submanifold $M$ of $X$ such that $x \in M$ and $T_x M = H$.

Let $\Phi(M) = M'$, $x' = \Phi(x)$. By assumption the submanifold $M' \subset Y$ is symplectic. This means that the space $d_{x'} \Phi(H) = T_{x'} M'$ is symplectic. Hence the mapping $d_x \Phi$ transforms all linear $2s$-dimensional symplectic subspaces of $T_x X$ onto subspaces of the same type. By Proposition 2.4
this implies that $d_x \Phi$ is a conformal symplectomorphism, i.e.,
\[(d_x \Phi)^* \omega_Y = \lambda(x) \omega_X,\]
where $\lambda(x) \neq 0$. This means that there is a smooth function $\lambda : X \to \mathbb{R}^* (= \mathbb{R} \setminus \{0\})$ such that
\[\Phi^* \omega_Y = \lambda \omega_X.\]
But since the form $\omega_X$ is closed, so is $\Phi^* \omega_Y$. Since $n > 1$ this implies that the derivative $d\lambda$ vanishes, i.e., the function $\lambda$ is constant. □

**Corollary 4.2.** Let $X$ be a compact manifold of dimension $2n > 2$. Let $\omega_1$ and $\omega_2$ be two symplectic forms on $X$. Fix a number $0 < k < n$. Assume that the family of all $2k$-dimensional $\omega_1$-symplectic submanifolds of $X$ is contained in the family of all $2k$-dimensional $\omega_2$-symplectic submanifolds of $X$. Then there exists a non-zero number $c \in \mathbb{R}$ such that
\[\omega_1 = c \omega_2.\]

*Proof.* It is enough to apply Theorem 4.1 to $X = (X, \omega_1)$, $Y = (X, \omega_2)$ and $\Phi = \text{identity}$. □

**Corollary 4.3.** Let $(X, \omega)$ be a compact symplectic manifold of dimension $2n > 2$. Fix a number $0 < s < n$. Assume that $\Phi : X \to X$ is a diffeomorphism which transforms all $2s$-dimensional symplectic submanifolds of $X$ onto symplectic submanifolds. Then $\Phi$ is a symplectomorphism or antisymplectomorphism, i.e., $\Phi^* \omega = \pm \omega$. If $\Phi$ preserves an orientation and $n$ is odd, then $\Phi$ is a symplectomorphism. Moreover, if $n$ is even, then $\Phi$ has to preserve the orientation.

*Proof.* Indeed, we have $\Phi^* \omega = c \omega$. We have

\[(4.1) \quad \text{vol}(X) = \int_X \omega^n = \pm \int_X \Phi^* \omega^n = \pm c^n \int_X \omega^n\]

hence $c = \pm 1$. Moreover, if $\Phi$ preserves an orientation and $n$ is odd, then we get that $c = 1$. If $n$ is even then $(-\omega)^n = \omega^n$ and $\Phi$ has to preserve the orientation. □

**Example 4.4.** We show that in the general case $\Phi$ do not need be a symplectomorphism. Let $Y = (S^2, \omega)$ (where $\omega$ is a standard volume form on the sphere) and let $(X_n, \omega_n) = \prod_{i=1}^n Y$ be a standard symplectic product. Further let $\sigma : S^2 \ni (x, y, z) \to (x, y, -z) \in S^2$ be a mirror symmetry. Of course $\sigma^* \omega = -\omega$. More general if $\Sigma = \prod_{i=1}^n \sigma : X_n \to X_n$, then $\Sigma^* \omega_n = -\omega_n$. Hence it is possible that $\Phi$ from Corollary 4.3 is an antisymplectomorphism.

However, in any case either $\Phi$ or $\Phi \circ \Phi$ is a symplectomorphism.

Now let $(X, \omega)$ be a symplectic manifold and let us denote by $\text{Symp}(X, \omega)$ the group of symplectomorphisms of $X$. At the end of this note we show that this group also determine a conformal symplectic structure on $X$: 
**Theorem 4.5.** Let $X$ be a smooth manifold of dimension $2n > 2$ and let $\omega_1$, $\omega_2$ be two symplectic forms on $X$. If $\text{Symp}(X, \omega_1) \subset \text{Symp}(X, \omega_2)$, then there exists a non-zero constant $c$ such that $\omega_2 = c\omega_1$.

**Proof.** Take $x \in X$ and consider symplectic vector spaces $V_1 = (T_xX, \omega_1)$ and $V_2 = (T_xX, \omega_2)$. By Theorem 3.7 we have that for every linear symplectomorphism $S$ of $V_1$, there is a symplectomorphism $\Phi_S \in \text{Symp}(X, \omega_1)$, such that

a) $\Phi_S(x) = x$,

b) $d_x\Phi_S = S$.

Since $\text{Symp}(X, \omega_1) \subset \text{Symp}(X, \omega_2)$ we easily obtain that $\text{Sp}(V_1) \subset \text{Sp}(V_2)$. Consequently by Proposition 2.5 there exist a non-zero number $\lambda(x)$ such that $\omega_2(x) = \lambda(x)\omega_1(x)$. Now we finish the proof as in the proof of Theorem 4.1. □

**References**


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