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ON THE CANCELLATION PROBLEM

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ABSTRACT. Let k be an algebraically closed field. For every $n \geq 8$ we give examples of Zariski open, dense, affine subsets of the affine space $A^n(k)$ which do not have the cancellation property.

1. INTRODUCTION.

Let k be an algebraically closed field and let X be an affine variety over k. We say that X has the cancellation property (CP) if for every affine variety Y, if $X \times k \cong Y \times k$, then $X \cong Y$.

There exist smooth affine varieties without CP (see [2], [3], [5], [6]); all the examples given so far are based on the so called Danielewski construction. It is an interesting and in general still open problem whether the affine space $A^n(k)$ has CP. Here, using a new approach, we show that for every $n \geq 8$ there are Zariski open, dense affine subsets of $A^n(k)$ without CP.

Our idea is as follows. Let X be a smooth affine variety. Denote a trivial algebraic vector bundle of rank r on X by $\mathbf{E}_r$. Assume that X admits an algebraic vector bundle $\mathbf{F}$ which is stably trivial (of type 1), i.e., $\mathbf{F} \oplus \mathbf{E}_1 = \mathbf{E}_{n+1}$, but not trivial. Let F denote the total space of $\mathbf{F}$. We have

$$F \times k \cong X \times k^{n+1} \cong (X \times k^n) \times k,$$

but we show that $F \not\cong X \times k^n$ if X is not $k-$uniruled.

Accordingly, to find an affine variety without CP it is enough to find a smooth affine non-$k-$uniruled variety with a stably trivial, but not trivial algebraic vector bundle on it. In particular we obtain in this way the following example. Let $n \geq 3$ be an integer. Consider the polynomial $h(x, y) = \sum_{i=1}^n x_i y_i \in k[x_1, \ldots, x_n; y_1, \ldots, y_n]$. Now let

$$X_{2n} = \{(x, y) \in k^{2n} : \ h(x, y) \neq 0, \ 1+x_i(h(x, y)-1) \neq 0, \ 1+y_i(h(x, y)-1) \neq 0, \ i = 1, \ldots, n\}.$$

Then the cylinder $Y_{3n-1} = X_{2n} \times k^{n-1}$ (which is a Zariski open, dense subset of $A^{3n-1}(k)$) does not have CP. We also show that for every $m \geq 8$, we can find a Zariski open, dense affine subset U_m of $A^m(k)$ which fails CP.

2. PRELIMINARIES

Let X be an affine variety (which is assumed to be irreducible) over k and let $R = k[X]$ be the ring of polynomial functions on X. Let us recall some basic facts about algebraic

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vector bundles over X, which we identify with finitely generated projective $R-$modules. We say that an algebraic vector bundle $\mathbf{E}$ is *stably trivial* (of type t) if

$$\mathbf{E} \oplus \mathbf{E}_t = \mathbf{E}_s$$

for some trivial vector bundles $\mathbf{E}_t$ and $\mathbf{E}_s$. Recall that a sequence $(f_1, \ldots, f_r) \in R^r$ is called a *unimodular row* if $(f_1, \ldots, f_r) = R$ (as an ideal). This is equivalent to the fact that $f_1, \ldots, f_r$ have not common zeros on X. A unimodular row $f = (f_1, \ldots, f_r) \in R^r$ determines uniquely a vector bundle $\mathbf{F}(f) = R^r/Rf$. Of course $\mathbf{F} \oplus \mathbf{E}_1 = \mathbf{E}_r$. It is easy to see that the vector bundle $\mathbf{F}(f)$ is trivial if and only if the unimodular row $f = (f_1, \ldots, f_r)$ can be extended to an $n \times n$ matrix with determinant 1. In other words, $\mathbf{F}(f)$ is trivial if and only if there exists a matrix $[f_{ij}] \in R^{n^2}$ such that

1) $\det\, [f_{ij}] = 1$,
2) $f_i = f_{1i}$ for $i = 1, \ldots, n$.

3. Open subsets of $A^n(k)$ without the cancellation property

Let us recall the definition of a $k-$uniruled variety which was introduced in our paper [7]. First recall that an *affine parametric line* in X is the image of the affine line $A^1(k)$ under a non constant morphism $\phi : A^1(k) \to X$. Now we have:

Definition 3.1. An affine variety X is said to be $k-$*uniruled* if it is of dimension ≥ 1 and there exists a Zariski open, non-empty subset U of X such that for every point $x \in U$ there is a parametric affine line in X passing through x.

We have the following important examples of non-$k-$uniruled varieties:

Proposition 3.2. *Let $h \in k[x_1, \ldots, x_n]$ be a non-constant polynomial. The variety*

$$X(h) = \{x \in k^n : \ h(x) \neq 0, \ 1 + x_i(h(x) - 1) \neq 0 \text{ for } i = 1, \ldots, n \ \}$$

is not $k-$uniruled.

Proof. Let $\phi : k \to X(h)$ be a regular mapping. Thus $\phi = (\phi_1(t), \ldots, \phi_n(t))$, where ϕ_i are polynomials. Moreover $h \circ \phi \neq 0$ for every t, which implies that $h \circ \phi$ is a non-zero constant. Similarly $1 + \phi_i(t)(h \circ \phi(t) - 1)$ is a constant. Consequently, either $h \circ \phi(t) = 1$ or all ϕ_i are constant. This means that outside the hypersurface $\{x \in k^n : h(x) = 1\}$ there are no affine parametric curves in $X(h)$. □

In the sequel we need the following nice elementary lemma, which was proved in [4] (for the sake of completeness we include a proof):

Lemma 3.3. *Let $f : U \times k^r \to X$ be a dominant morphism of affine varieties. If there is $u \in U$ such that* $\dim f(\{u\} \times k^r) > 0$ *then X is k-uniruled.*

Proof. We can assume that $X \subset k^m$. Hence $f = (f_1, \ldots, f_m)$. Let $Z = \{u \in U : \dim f(\{u\} \times k^r) = 0$. The set Z is closed in U. Indeed $Z = \bigcap_{s,t \in k^r} \{u \in U : f_i(u, t) = f_i(u, s) \text{ for } i = 1, \ldots, m\}$. By the assumption we have $Z \neq U$. Since the mapping f is dominant and the variety $(U \setminus Z) \times k^r$ is dense in $U \times k^r$, we have that also the mapping $f : (U \setminus Z) \times k^r \to X$ is dominant. Let l_u denote a line in k^r passing through 0, such that the mapping f restricted to $\{u\} \times l$ is not constant. We have $f((U \setminus Z) \times k^r) = \bigcup_{u \in U \setminus Z,\ l_u \subset k^r} f(\{u\} \times l_u)$. Since the set $f((U \setminus Z) \times k^r)$ contains a Zariski open subset of X, the proof is complete. □

The following observation is important for this paper:

Theorem 3.4. *Let X be a non-$k-$uniruled smooth affine variety. Let $\mathbf{F}$ be an algebraic vector bundle on X of rank r. If the total space of $\mathbf{F}$ is isomorphic to $X \times k^r$, then $\mathbf{F}$ is a trivial vector bundle.*

Proof. Let F denote the total space of $\mathbf{F}$. In what follows, we will identify X with the zero section $X \times \{0\} \subset F$. Note that

$$\mathbf{F} = TF|_X/TX.$$

Assume that there exists an isomorphism $\Phi : F \to X \times k^r$. Let $\pi : X \times k^r \to X$ be the projection and take $f = \pi \circ \Phi$. Since the vector bundle $\mathbf{F}$ is locally trivial in the Zariski topology, Lemma 3.3 shows that $\Phi(\mathbf{F}_x) = f(x) \times k^r$ for every $x \in X$. Consequently, the mapping $\sigma := f|X : X \to X$ is an isomorphism. Let $\Sigma = \sigma^{-1} \times identity : X \times k^r \ni (x, t) \mapsto (\sigma^{-1}(x), t) \in X \times k^r$. If we replace Φ by $\Sigma \circ \Phi$ then $\sigma = identity$.

In particular $\Phi : X \in x \mapsto (x, t(x)) \in X \times k^r$. Let us denote coordinates in the product $X \times k^r$ by $(x, t_1, \ldots, t_r)$. By the above, we have $t_i = t_i(x) \ mod \ I(\Phi(X))$. Consider the isomorphism $G : X \times k^r \ni (x, t) \mapsto (x, t - t(x)) \in X \times k^r$. Again we can replace Φ by $G \circ \Phi$ to obtain $\Phi|X : X \times \{0\} \ni (x, 0) \mapsto (x, 0) \in X \times k^r$. Hence we can assume that Φ transforms the zero section into the zero section, and moreover it induces the identity on the zero section. Hence $d\Phi(TX) = TX$ and the mapping

$$d\Phi : TF|_X/TX \cong \mathbf{F} \to T(X \times k^r)|_X/TX \cong \mathbf{E_r}$$

is an isomorphism. Consequently, the bundle $\mathbf{F}$ is trivial. □

Now we give examples of Zariski open, affine subsets of $A^n(k)$ which do not have the cancellation property. We start with the following nice classical example of Raynaud (see [8] for $k = \mathbb{C}$ and [9] for an arbitrary field k):

Example 3.5. *Let $n \geq 3$ and*

$$R = \frac{k[x_1, \ldots, x_n, y_1, \ldots, y_n]}{(\sum_{i=1}^n x_i y_i - 1)}.$$

Then the stably free submodule of R^n given by the unimodular row $(x_1, \ldots, x_n)$ is not free.

Now we can prove our main result.

Theorem 3.6. *Let $n \geq 3$ be an integer and let $h = \sum_{i=1}^n x_i y_i \in k[x_1, \ldots, x_n; y_1, \ldots, y_n]$. Define*

$$X_{2n}(h) = \{(x, y) \in k^{2n} : \ h(x, y) \neq 0, \ 1 + x_i(h(x, y) - 1) \neq 0, \\ 1 + y_i(h(x, y) - 1) \neq 0, \ i = 1, \ldots, n \ \}.$$

Then the cylinder $Y_{3n-1} = X_{2n} \times k^{n-1}$ (which is a Zariski open, dense affine subset of k^{3n-1}) fails CP.

Proof. Let $Z = \{(x, y) \in k^{2n} : h(x, y) = 1\}$. By the Raynaud example we know that a row $(x_1, \ldots, x_n)$ which is unimodular on Z cannot be extended on Z to an $n \times n$ matrix with determinant 1 (cf. section 2). Let us note that this row is also unimodular on the variety $X_{2n}(h)$. Indeed, we have $\sum_{i=1}^n x_i y_i = h \neq 0$ on $X_{2n}(h)$. I claim that this row cannot be extended on $X_{2n}(h)$ to an $n \times n$ matrix with determinant 1. To see this, first note that $Z \subset X_{2n}(h)$. Now, the restriction to Z of such a matrix would give a similar matrix on Z, a contradiction. From this we conclude that a unimodular row $(x_1, \ldots, x_n)$ determines on $X_{2n}(h)$ a non-trivial algebraic vector bundle $\mathbf{F}$. In particular we have $\mathbf{F} \oplus \mathbf{E}_1 = \mathbf{E}_n$ and $\mathbf{F}$ is a non-trivial vector bundle. Let us denote by F the total space of $\mathbf{F}$. Then

$$F \times k \cong X_{2n}(h) \times k^n \cong (X_{2n}(h) \times k^{n-1}) \times k.$$

Since the variety $X_{2n}(h)$ is not $k-$uniruled (see Proposition 3.2), Theorem 3.4 shows that the variety F is not isomorphic to the cylinder $X_{2n}(h) \times k^{n-1}$. □

Remark 3.7. If we have one open subvariety $X_{2n}(h) \subset k^{3n-1}$ without CP, we can easily construct infinitely many pairwise non-isomorphic open subvarieties of this type. Indeed, choose sufficiently general polynomials $a_i \in k[x,y]$, $i = 1, 2, \ldots$ Let $Y_k = \{(x,y) \in X_{2n}(h) : 1 \neq a_i(1-h),\ i = 1, \ldots, k\}$. In this way we obtain a strictly descending sequence of open subvarieties $Y_0 \supset Y_1 \supset Y_2 \supset \ldots$, which do not have CP. They are pairwise non-isomorphic by the Ax Theorem (see [1]).

By a slight modification of the proof of Theorem 3.6 we get:

Theorem 3.8. *For every $n \geq 8$, we can find a Zariski open, dense affine subset U_n of $A^n(k)$ which fails CP.*

Proof. Let $n = 8 + s$. Consider the ring $R = k[x_1, x_2, x_3, y_1, y_2, y_3, z_1, \ldots, z_s]$. Let $h = \sum_{i=1}^{3} x_i y_i \in R$. Define

$$Y_{n-2}(h) = \{(x,y,z) \in k^3 \times k^3 \times k^s :\ h(x,y) \neq 0,\ 1+x_i(h(x,y)-1) \neq 0,\ 1+y_i(h(x,y)-1) \neq 0,$$

$$i = 1, 2, 3;\ z_j \neq 0 \text{ for } j = 1, \ldots, s\ \}.$$

As in the proof of Proposition 3.2 we see that the variety $Y_{n-2}(h)$ is not $k-$uniruled.

Let $Z' = \{(x,y,z) \in k^3 \times k^3 \times k^s : h(x,y) = 1;\ z_j = 1 \text{ for every } j = 1, \ldots, s\ \}$. By the Raynaud example we know that a row (x_1, x_2, x_3) which is unimodular on Z' cannot be extended on Z' to a 3×3 matrix with determinant 1. This row is also unimodular on the variety $Y_{n-2}(h)$ and it also cannot be extended on $Y_{n-2}(h)$ to a 3×3 matrix with a determinant 1. Indeed, the restriction to Z' of such a matrix would give a similar matrix on Z', a contradiction. From this, as before, we conclude that a unimodular row (x_1, x_2, x_3) determines on $Y_{n-2}(h)$ a non-trivial algebraic vector bundle $\mathbf{F}$. Now arguing as in the proof of Theorem 3.6 we see that the variety $U_n = Y_{n-2}(h) \times k^2$ does not have CP. □

Corollary 3.9. *Let $n \geq 8$. Then there exists a non-zero polynomial $g \in k[x_1, \ldots, x_n]$ and a finitely generated $k-$algebra F such that*

$$k[x_1, \ldots, x_n]_g \otimes_k k[T] = F \otimes_k k[T]$$

but $k[x_1, \ldots, x_n]_g \ncong F$.

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