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Equivalence of measures corresponding to the Hilbert space valued Lévy processes

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Equivalence of measures corresponding to the Hilbert space valued Lévy processes*

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Abstract

The goal is to give a Hilbert space version of the well known theorem on equivalence of measures corresponding to the real valued Lévy processes (see e.g. Gihman, Skorohod (1966), Sato (1999)).

Key words: Hilbert-space valued Lévy processes, equivalence of measures, Feldman-Hajek theorem

AMS Subject Classification: 60G51

1 Introduction

The theorem on equivalence of measures corresponding to the real valued Lévy processes can be found e.g. in Gihman, Skorohod (1966) or in a recent monograph Sato (1999). Besides its theoretical value it finds applications for example in detection theory (see e.g. Kailath, Poor (1998)) or mathematical finance (see e.g. Cont, Tankov (2004)). Our goal is to give a counterpart of this theorem in a Hilbert space setup. To the authors' knowlegde the proof of the result is not documented in the literature.

The proof basically follows Sato (1999) (see the proofs of Theorems 33.1 and 33.2). To show the necessary conditions one uses Lévy–Itô decomposition and deals with the gaussian and the jump part separately. As opposed to the Sato (1999) we used Feldman–Hajek theorem and the law of large numbers, similarly as in Gihman, Skorohod (1966). The latter allowed us to avoid the explicit use of the Hellinger integral (which in turn appears in the proof of Feldman–Hajek theorem). The proof of the sufficent conditions benefits from the theory of integration with respect to Poisson random measures.

2 Preliminaries

We assume that H is a separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$. For any $x \in H$ we write $|x| = \sqrt{\langle x, x\rangle}$. Consider $D = D([0,T], H) = \{f\colon [0,T] \to$

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$H\colon f$ is cádág$\}$ together with a measure μ defined on σ–field $\mathcal{F}_D$ of Borel cylinder sets. We define an H–valued process $X = \{X(t)\colon t \geq 0\}$ with cádlág sample paths as a cannonical process on D, i.e. $X(t)(\omega) = \omega(t)$ for all $\omega \in D$.

The H–valued cádlág process X is called a Lévy process if

(i) given $0 < t_0 < \ldots < t_n < \infty$, the H–valued random vairables

$$X(t_1) - X(t_0), \ldots, X(t_n) - X(t_{n-1})$$

are independent;

(ii) given $0 < s < t < \infty$, $X_t - X_s$ is equal in distribution to X_{t-s};

(iii) X is stochastically continuous, i.e.

$$\lim_{s \to t} \mu(|X(t) - X(s)| > \varepsilon) = 0, \qquad \forall \varepsilon > 0;$$

(iv) $\mu(X_0 = 0) = 1$.

The above conditions readly depend on the measure μ, so to emphasize this we will also write that $\{X, \mu\}$ is a Lévy process. From the definition one can deduce that the characteristic function of $X(t)$ is given by the formula

$$\mathbb{E}e^{i\langle x, X_t\rangle} = e^{-t\psi(x)},$$

for some function ψ. The Lévy-Khinchine formula gives the formula for ψ, that is

$$\psi(x) = -i\langle\gamma, x\rangle + \frac{1}{2}\langle Ax, x\rangle - \int_H (e^{i\langle x,y\rangle} - 1 - i\langle x, y\rangle \mathbb{1}_{|y|\leq 1}(y))\nu(dy), \quad \forall x \in H,$$

where $\gamma \in H$, A is nonnegative and trace class operator[1] and ν is measure on H satisfying $\nu(\{0\}) = 0$ and $\int_H (1 \wedge |y|^2)\nu(dy) < \infty$. The tuple (A, ν, γ) is called the generating triplet of X. Moreover, Lévy–Itô decomosition yelds that one can decompose the sample paths of X into the continuous part, X^c, and pure jump part, X^d, in the following way:

$$X(t) = X^c(t) + X^d(t),$$

where

$$X^c(t) = X(t) - \int_0^t \int_{|x|<1} x(\pi(ds, dx) - dt\nu(dx)) - \int_0^t \int_{|x|\geq 1} x\pi(ds, dx),$$

$$X^d(t) = X(t) - X^c(t).$$

Additionally X^c is a Brownian motion (with drift γ and covariance operator A) independet of X^d.

Finally we define the Poisson random measure corresponding to X as

$$\pi([0, t], \Gamma) = \#\{s \leq t\colon \Delta X(s) \in \Gamma\}, \quad \Gamma \in \mathcal{B}(H).$$

The measure $\widehat{\pi}(ds, dx) = \pi(ds, dx) - ds\nu(dx)$ is called the compensated Poisson random measure.

[1] A linear, bounded operator $\Lambda\colon H \to H$ is called a trace class if there exist $\{a_k\}, \{b_k\} \subset H$, $\sum_k |a_k||b_k| < \infty$ such that $\Lambda u = \sum_k b_k\langle u, a_k\rangle$, see Da Prato, Zabczyk (1992).

3 The main theorem

In the following we consider two measures μ_1 and μ_2 defined on the space $(D, \mathcal{F}_D)$. If $\{X, \mu_i\}$ is a Levy process with generating triplet (A_i, ν_i, γ_i), $i = 1, 2$, then we define the corresponding compensated Poisson random measure

$$\widehat{\pi}_i(ds, dx) = \pi(ds, dx) - ds\nu_i(dx).$$

Theorem 1. Let $\{X, \mu_1\}$ and $\{X, \mu_2\}$ be two H–valued Levy processes with generating triplets (A_1, ν_1, γ_1) and (A_2, ν_2, γ_2) respectively. Then $\mu_1 \sim \mu_2$ if and only if the following conditions hold

(i) $A_1 = A_2 =: A$;

(ii) $\nu_1 \sim \nu_2$ and the function ρ defined by $\frac{d\nu_2}{d\nu_1}(x) = e^{\rho(x)}$ satisfies

$$\int_H (e^{\rho(x)/2} - 1)^2 \nu_1(dx) < \infty;$$

(iii) The integral $\int_{|x|<1} x(d\nu_2 - d\nu_1)(x)$ is well defined and $\gamma_2 - \gamma_1 - \int_{|x|<1} x(d\nu_2 - d\nu_1)(x) \in A^{1/2}(H)$.

Let X_1^c denote the continous part of X with respect to μ_1, that is

$$X_1^c(t) = X(t) - \int_0^t \int_{|x|<1} x\widehat{\pi}_1(ds, dx) + \int_0^t \int_{|x|\geq 1} x\pi(ds, dx).$$

Then, provided that $(i) - (iii)$ hold, we get

$$\frac{d\mu_2}{d\mu_1}\bigg|_{\mathcal{F}_t}(X(\cdot)) = \exp\Bigg\{ \langle b, X_1^c(t)\rangle - \frac{t}{2}|A^{1/2}b|^2 - t\langle \gamma_1, b\rangle$$

$$+ \int_0^t \int_H \rho(x)[\pi(ds, dx) - \mathbb{1}_{(-1,1)}(\rho(x))ds\nu_1(dx)]$$

$$- \int_0^t \int_H [e^{\rho(x)} - 1 - \rho(x)\mathbb{1}_{(-1,1)}(\rho(x))]ds\nu_1(dx) \Bigg\},$$

for any $b \in H$ such that $\gamma_2 - \gamma_1 - \int_{|x|<1} x(d\nu_2 - d\nu_1) = A^{1/2}b$.

Remark 1. The essential difference between arbitrary separable Hilbert space and $H = \mathbb{R}^d$ is that one has to consider the image of $A^{1/2}$ and not A (see condition (iii)). Of course when $H = \mathbb{R}^d$ then $A(H) = A^{1/2}(H)$.

Proof. It can be proven (see Sato (1999), page 220) that assumption

$$\int_H (e^{\rho(x)/2} - 1)^2 \nu_1(dx) < \infty$$

is equivalent to the following three conditions

$$\int_{|\rho(x)|<1} \rho^2(x)\nu_1(dx) < \infty, \tag{S1}$$

$$\int_{\rho(x)\geq 1} e^{\rho(x)}\nu_1(dx) < \infty, \tag{S2}$$

$$\int_{\rho(x)\leq -1} \nu_1(dx) < \infty. \tag{S3}$$

Necessity. **Step 1**. Suppose $\nu_1 = \nu_2 \equiv 0$. First we prove (i). For any $h \in H$ the process $Y_h(t) = \langle X_t, h\rangle$ is a real–valued brownian motion with drift and $Y_h(t)$ has a normal distribution $N(t\langle\gamma_i, h\rangle, t\langle A_i h, h\rangle)$ under μ_i, $i = 1, 2$. Absolute continuity of μ_1 and μ_2 implies that $\langle A_1 h, h\rangle = \langle A_2 h, h\rangle$ (see e.g Sato (1999), page 229). Since h and A_1, A_2 are covariance operators we have $A_1 = A_2$. Furthermore, the condition (iii) follows directly from the Feldman-Hayek theorem (see Appendix, Theorem 2). Indeed, the measures $\mu_i(X(1) \in \cdot)$, $i = 1, 2$, are equivalent and gaussian $N(\gamma_i, A_i)$.

Step 2. Let π be a Poisson random measure corresponding to X. Then

$$\mu_1(\pi([0,t],\Gamma) = 0) = e^{-t\nu_1(\Gamma)},$$

$$\mu_2(\pi([0,t],\Gamma) = 0) = e^{-t\nu_2(\Gamma)}.$$

By absolute continuity of μ_1 and μ_2 we get that $\nu_1(\Gamma) = 0$ implies $\nu_2(\Gamma) = 0$ and vice versa, so $\nu_1 \sim \nu_2$. As a result there exists function ρ defined by $\frac{d\nu_2}{d\nu_1}(x) = e^{\rho(x)}$. Similarly, $\nu_1(\Gamma) < \infty$ if and only if $\nu_2(\Gamma) < \infty$.

To prove the integrability condition (ii) observe that

$$\int_H (e^{\rho(x)/2} - 1)^2\nu_1(dx) \leq \int_{|\rho(x)|<\varepsilon} (e^{\rho(x)/2} - 1)^2\nu_1(dx) + \int_{|\rho(x)|\geq\varepsilon} (e^{\rho(x)} + 1)\nu_1(dx)$$

$$= \int_{|\rho(x)|<\varepsilon} (e^{\rho(x)/2} - 1)^2\nu_1(dx) + \nu_2(|\rho(x)| \geq \varepsilon) + \nu_1(|\rho(x)| \geq \varepsilon),$$

for $\varepsilon > 0$. It is therefore sufficient to show that the terms on the right hand side are finite.

Suppose that $\nu_2(: \rho(x) \geq \varepsilon) = \infty$. Let $B_n \subset [0,T] \times \{\rho(x) \geq \varepsilon\}$ be disjoint sets such that $\bigcup_{k=1}^{\infty} B_k = [0,T] \times \{\rho(x) \geq \varepsilon\}$ and

$$\iint_{B_n} ds\nu_2(dx) = 1.$$

The family $\{B_n\}$ can be constructed, as the measure $dt \otimes v(dx)$ does not have any atoms. Because π is a Poisson random measure $\pi(B_1), \pi(B_2), \ldots$ are i.i.d. Poisson random variables with intensity 1 under μ_2. By Remark 2, $n^{-1}\sum_{k=1}^{n} \pi(B_k) \to 1$ w.r.t. μ_2 as $n \to \infty$. From absolute continuity it follows that $n^{-1}\sum_{k=1}^{n} \pi(B_k) \to 1$ w.r.t. μ_1 as $n \to \infty$. Since $\pi(B_1), \pi(B_2), \ldots$ are independent Poisson random variables with intensity measure $dt \otimes d\nu_1 = e^{-\rho} dt \otimes d\nu_2$ we get $\mathbb{E}_{\mu_1}\pi(B_k) = \iint_{B_k} dt\nu_1(dx) \leq e^{-\varepsilon}$. Therefore from Remark 2 it follows that

$$n^{-1}\left(\sum_{k=1}^{n} \pi(B_k) - \sum_{k=1}^{n} \mathbb{E}_{\mu_1}\pi(B_k)\right) \to^{\mu_1} 0.$$

This is a contradiction, because the left hand side is greater then $n^{-1}\sum_{k=1}^{n} \pi(B_k) - e^{-\varepsilon}$, which tends to $1 - e^{-\varepsilon}$ w.r.t. μ_1 as $n \to \infty$. Thus $\nu_i(\rho(x) \geq \varepsilon) < \infty$ for $i = 1, 2$. Interchanging ν_2 with ν_1, μ_2 with μ_1 and ρ with $-\rho$ we get the same result with $-\rho$ instead of ρ. As a consequence $\nu_i(|\rho(x)| \geq \varepsilon) < \infty$, for $i = 1, 2$.

Suppose now that

$$\int_{0<\rho(x)<\varepsilon} (e^{\rho(x)/2}-1)^2 \nu_1(dx) = \infty.$$

Let $B_n \subset [0,T] \times \{0<\rho(x)<\varepsilon\}$ be disjoint sets such that $\bigcup_{k=1}^{\infty} B_k = [0,T] \times \{0<\rho(x)<\varepsilon\}$ and

$$\iint_{B_n} (e^{\rho(x)/2}-1)^2 ds\nu_1(dx) = 1. \tag{1}$$

From (1) we know that the following random variables are well defined

$$Z_k = \iint_{B_k} (e^{\rho(x)/2}-1)\widehat{\pi}_1(dt,dx). \tag{2}$$

By Theorem 3 we get

$$\mathbb{E}_{\mu_1} Z_k = 0, \qquad D^2_{\mu_1} Z_k = \iint_{B_k} (e^{\rho(x)/2}-1)^2 dt\nu_1(dx) = 1.$$

As the sets $\{B_k\}$ are disjoint the random variables $Z_1, Z_2, \ldots$ are independent. Moreover, since

$$\mathbb{E}_{\mu_2} Z_k = \iint_{B_k} (e^{\rho(x)/2}-1) dt(\nu_2-\nu_1)(dx) = \iint_{B_k} (e^{\rho(x)/2}-1)^2 (e^{\rho(x)/2}+1) dt\nu_1(dx)$$

it follows that $1 \le \mathbb{E}_{\mu_2} Z_k \le 1 + e^{\varepsilon/2}$. Additionally

$$D^2_{\mu_2} Z_k = \iint_{B_n} (e^{\rho(x)/2}-1)^2 dt\nu_2(dx) = \iint_{B_k} (e^{\rho(x)/2}-1)^2 e^{\rho(x)} dt\nu_1(dx) \le e^{\varepsilon}.$$

From Remark 2 we get $n^{-1}\sum_{k=1}^n Z_k \to 0$ w.r.t. μ_1 as $n \to \infty$. By absolute continuity of measures μ_1, μ_2 we get $n^{-1}\sum_{k=1}^n Z_k \to 0$ w.r.t. μ_2 as $n \to \infty$. Moreover, it follows from Remark 2 that

$$n^{-1}\left(\sum_{k=1}^n Z_k - \sum_{k=1}^n \mathbb{E}_{\mu_2} Z_k\right) \to^{\mu_2} 0.$$

But this is impossible since the left hand side is smaller then $n^{-1}\sum_{k=1}^n Z_k - 1$ which tends to -1 w.r.t. μ_2 as $n \to \infty$. Interchanging ν_1 with ν_2, μ_1 with μ_2 and ρ with $-\rho$ we get the same result with $-\rho$ instead of ρ. Thus

$$\int_{|\rho(x)|<\varepsilon} (e^{\rho(x)/2}-1)^2 \nu_1(dx) < \infty.$$

Step 3. By Step 2 the condition (ii) is satisfied. To show that $\int_{|x|<1} x(\nu_2 - \nu_1)(dx) = \int_{|x|<1} x(e^{\rho(x)}-1)\nu_1(dx)$ is well defined we follow Sato (1999):

$$\int_{|x|<1} |x||e^{\rho(x)}-1|\nu_1(dx) \le \int_{|x|<1,|\rho(x)|<1} |x||e^{\rho(x)}-1|\nu_1(dx) + \int_{|\rho(x)|\ge 1} |e^{\rho(x)}-1|\nu_1(dx)$$

$$\leq \left(\int_{|x|<1} |x|^2 \nu_1(dx)\right)^{1/2} \left(\int_{|\rho(x)|<1} |e^{\rho(x)} - 1|^2 \nu_1(dx)\right)^{1/2}$$

$$+ \int_{|\rho(x)|\geq 1} |e^{\rho(x)} - 1| \nu_1(dx) < \infty,$$

where we have used the Hölder's inequality. The first integral in the last expression is finite from the definition of Lévy measure ν_1. The finiteness of the second term follows from the inequality $(S1)$ and the basic fact that for any $a \in \mathbb{R}$, $|a| < 1$, $|e^a - 1| < e|a|$. The finiteness of the last term follows from $(S2)$ and $(S3)$.

By Lévy–Itô decomposition it follows that

$$X_1^c(t) = X(t) - \int_0^t \int_{|x|<1} x\widehat{\pi}_1(ds, dx) - \int_0^t \int_{|x|\geq 1} x\pi(ds, dx)$$

$$- \int_0^t \int_{|x|<1} xds(\nu_2 - \nu_1)(x) + \int_0^t \int_{|x|<1} xds(\nu_2 - \nu_1)(dx)$$

$$= X(t) - \int_0^t \int_{|x|<1} x\widehat{\pi}_2(ds, dx) - \int_0^t \int_{|x|\geq 1} x\pi(ds, dx) - \int_0^t \int_{|x|<1} xds(\nu_2 - \nu_1)(dx).$$

The last equality can be justified by the approximation argument. Consequently

$$X_1^c(t) = X_2^c(t) - \int_0^t \int_{|x|<1} xds(\nu_2 - \nu_1)(dx)$$

meaning that, under the measure μ_2, X_1^c is a Brownian motion with drift $\gamma_2 - \int_{|x|<1} x(\nu_2 - \nu_1)(dx)$ and covariance matrix A_2. Finally, because measures induced on D by (X_1^c, μ_1) and (X_1^c, μ_2) are equivalent, the whole statement follows from the Step 1 and Step 2.

Sufficiency. Define

$$U(t) = \langle b, X_1^c(t)\rangle - \frac{t}{2}|A^{1/2}b|^2 - t\langle \gamma_1, b\rangle \tag{3}$$

$$+ \int_0^t \int_{|\rho|<1} \rho(x)\widehat{\pi}_1(ds, dx) + \int_0^t \int_{|\rho|\geq 1} \rho(x)\pi(ds, dx)$$

$$- \int_0^t \int_H [e^{\rho(x)} - 1 - \rho(x)\mathbb{1}_{(-1,1)}(\rho(x))]ds\nu_1(dx).$$

Step 1. We will show that the integrals in (3) are well defined and that the process $e^{U(t)}$ defines a change of measure. The conditions $(S1) - (S3)$ imply that

$$\int_0^t \int_H |e^{\rho(x)} - 1 - \rho(x)\mathbb{1}_{(-1,1)}(\rho(x))|ds\nu_1(dx) < \infty.$$

Thus by Theorem 3

$$\mathbb{E}_{\mu_1} \exp\left\{\int_0^t \int_{|\rho(x)|\geq 1} \rho(x)\pi(ds, dx)\right\} = \exp\left\{\int_0^t \int_{|\rho(x)|\geq 1} \left(e^{\rho(x)} - 1\right) ds\nu_1(dx)\right\}$$

and

$$\mathbb{E}_{\mu_1} \exp\left\{\int_0^t \int_{|\rho(x)|<1} \rho(x)\widehat{\pi}_1(ds,dx)\right\} = \exp\left\{\int_0^t \int_{|\rho(x)|<1} \left(e^{\rho(x)} - 1 - \rho(x)\right) ds\nu_1(dx)\right\}.$$

This proves that the required integrals are well defined. Moreover, by Theorem 3, all term in (3) are independent and we get

$$\mathbb{E}_{\mu_1} e^{U(t)} = 1.$$

Step 2. We will show that for every $u \in \mathbb{R}$ the following equality holds

$$\mathbb{E}_{\mu_1} e^{i\langle z, X(t)\rangle + iuU(t)} = \exp\Big\{it\langle\gamma_1, z\rangle - \frac{itu}{2}|A^{1/2}b|^2 - \frac{t}{2}\langle A(z+ub), (z+ub)\rangle \quad (4)$$

$$-iu\int_0^t \int_H (e^{\rho(x)} - 1 - \rho(x)\mathbb{1}_{[-1,1]}(\rho(x)))ds\nu_1(dx)$$

$$+ \int_0^t \int_H (e^{i\langle x, z\rangle + iu\rho(x)} - 1 - i\langle x, z\rangle \mathbb{1}_{\{|x|<1\}}(x) - iu\rho(x)\mathbb{1}_{(-1,1)}(\rho(x)))ds\nu_1(dx)\Big\}.$$

Indeed, from Lévy–Itô decomposition and the linearity of the integral we get

$$\mathbb{E}_{\mu_1} e^{i\langle z, X(t)\rangle + iuU(t)} = \mathbb{E}_{\mu_1} \exp\Bigg\{i\langle X_1^c(t), z+ub\rangle - \frac{iut}{2}|A^{1/2}b|^2 - iut\langle\gamma_1, b\rangle$$

$$+i\int_0^t \int_{|x|<1} \langle x, z\rangle\widehat{\pi}_1(ds,dx) + i\int_0^t \int_{|x|\geq 1} \langle x, z\rangle\pi(ds,dx)$$

$$+iu\int_0^t \int_{|\rho(x)|<1} \rho(x)\widehat{\pi}_1(ds,dx) + iu\int_0^t \int_{|\rho(x)|\geq 1} \rho(x)\pi(ds,dx)$$

$$-iu\int_0^t \int_H [e^{\rho(x)} - 1 - \rho(x)\mathbb{1}_{(-1,1)}(\rho(x))]ds\nu_1(dx)\Bigg\}$$

$$= \exp\left\{it\langle\gamma_1, z\rangle - \frac{iut}{2}\langle Ab, b\rangle - \frac{t^2}{2}\langle A(z+ub), (z+ub)\rangle\right\}$$

$$\times \exp\left\{-iu\int_0^t \int_H [e^{\rho(x)} - 1 - \rho(x)\mathbb{1}_{(-1,1)}(\rho(x))]ds\nu_1(dx)\right\}$$

$$\times\mathbb{E}_{\mu_1} \exp\Bigg\{i\int_0^t \int_{|x|<1} \langle x, z\rangle\widehat{\pi}_1(ds,dx) + i\int_0^t \int_{|x|\geq 1} \langle x, z\rangle\pi(ds,dx)$$

$$+iu\int_0^t \int_{|\rho(x)|<1} \rho(x)\widehat{\pi}_1(ds,dx) + iu\int_0^t \int_{|\rho(x)|\geq 1} \rho(x)\pi(ds,dx)\Bigg\} = I_1 I_2 I_3.$$

Define the following disjoint sets

$$B_{00} = [0,t]\times\{|x|<1,\ |\rho(x)|<1\}, B_{01} = [0,t]\times\{|x|<1,\ |\rho(x)|\geq 1\},$$

$$B_{10} = [0,t]\times\{|x|\geq 1,\ |\rho(x)|<1\}, B_{11} = [0,t]\times\{|x|\geq 1,\ |\rho(x)|\geq 1\}.$$

Then

$$I_3 = \mathbb{E}_{\mu_1} \exp\left\{ i \iint_{B_{00}} (\langle x, z\rangle + u\rho(x))\widehat{\pi}_1(ds, dx) + i \iint_{B_{11}} (\langle x, z\rangle + u\rho(x))\pi(ds, dx) \right.$$

$$+ i \iint_{B_{01}} \langle x, z\rangle \widehat{\pi}_1(ds, dx) + i \iint_{B_{01}} u\rho(x)\pi(ds, dx)$$

$$\left. + i \iint_{B_{10}} u\rho(x)\widehat{\pi}_1(ds, dx) + i \iint_{B_{10}} \langle x, z\rangle \pi(ds, dx) \right\}$$

Since $\nu_1(|x| \geq 1) < \infty$ and $\nu_1(|\rho| \geq 1) < \infty$ (see $(S2) - (S3)$) we have

$$\int_{B_{10}} |\rho(x)|\nu_1(dx) \leq \nu_1(|x| \geq 1) < \infty,$$

$$\int_{B_{01}} |\langle x, z\rangle|\nu_1(dx) \leq |z| \int_{B_{01}} |x|\nu_1(dx) \leq |z|\nu_1(|\rho| \geq 1) < \infty,$$

and the integrals $\iint_{B_{01}} \langle x, z\rangle \pi(ds, dx)$, $\iint_{B_{10}} \langle x, z\rangle \pi(ds, dx)$, $\iint_{B_{01}} \rho(x)\pi(ds, dx)$, $\iint_{B_{10}} \rho(x)\pi(ds, dx)$ are μ_1–well defined. Thus we can write

$$I_3 = \mathbb{E}_{\mu_1} \exp\left\{ i \iint_{B_{00}} (\langle x, z\rangle + u\rho(x))\widehat{\pi}_1(ds, dx) + i \iint_{B_{11}} (\langle x, z\rangle + u\rho(x))\pi(ds, dx) \right.$$

$$+ i \iint_{B_{01}} (\langle x, z\rangle + u\rho(x))\pi(ds, dx) - i \iint_{B_{01}} \langle x, z\rangle ds\nu_1(dx)$$

$$\left. + i \iint_{B_{10}} (\langle x, z\rangle + u\rho(x))\pi(ds, dx) - i \iint_{B_{10}} u\rho(x) ds\nu_1(dx) \right\}$$

Because the sets $B_{00}, B_{10}, B_{01}, B_{11}$ are disjoint it follows from the Theorem 3 that

$$I_3 = \exp\left\{ \iint_{B_{00}} (e^{i(\langle x, z\rangle + u\rho(x))} - 1 - i(\langle x, z\rangle + u\rho(x))) ds v_1(dx) \right\}$$

$$\times \exp\left\{ \iint_{B_{11}} (e^{i(\langle x, z\rangle + u\rho(x))} - 1) ds v_1(dx) \right\}$$

$$\times \exp\left\{ \iint_{B_{01}} (e^{i(\langle x, z\rangle + u\rho(x))} - 1) ds\nu_1(dx) - i \iint_{B_{01}} \langle x, z\rangle ds v_1(dx) \right\}$$

$$\times \exp\left\{ \iint_{B_{10}} (e^{i(\langle x, z\rangle + u\rho(x))} - 1) ds\nu_1(dx) - i \iint_{B_{10}} u\rho(x) ds v_1(dx) \right\}.$$

Thus the identity (4) holds.

Step 4. Let $f_z(c)$ and $h_z(c)$ denote the left hand side and the right hand side of the equation (4), respectively, with fixed z and c instead of u. Denote $F = \{c \in \mathbb{C} \colon Imc \in [-1, 0]\}$. Then f_z is continuous on F. This follows from the dominated convergence and the estimate

$$|e^{i\langle z, X(t)\rangle + icU(t)}| = e^{-ImcU(t)} \leq (1 + Imc)e^{U(t)} - Imc \leq e^{U(t)} + 1,$$

where the first inequality follows from convexity of the function $t \to e^t$. It is also analytic in $intF$ since it is the limit of analytic functions

$$\mathbb{E}_{\mu_1}(e^{i\langle z,X(t)\rangle+icU(t)}\mathbb{1}_{\{|U(t)|<n\}}).$$

Similar arguments show that h is also continuous on F and analytic on $intF$. As a result they can be analytically extended to $\widehat{F} = \{c \in \mathbb{C}\colon Imc \in [-1,1]\}$. Moreover, by Step 3, $f_z(c) = h_z(c)$ for all c such that $Imc = 0$. Since the set $\{c \in \mathbb{C}\colon Imc = 0\}$ has a accumulation point belonging to $int\widehat{F}$ the functions f_z and h_z coincide on $\widehat{F}$.

Step 5. By Step 4 we get

$$h_z(-i) = \exp\left\{it\langle\gamma_2, z\rangle - \frac{t}{2}\langle A_2 z, z\rangle + t\int_H (e^{i\langle x,z\rangle} - 1 - i\langle x, z\rangle\mathbb{1}_{\{|x|<1\}}(x))v_2(dx)\right\}.$$

Define the measure $\widetilde{\mu}_2(A) = \mathbb{E}_{\mu_1}(e^{U_t}\mathbb{1}_A)$, for $A \in \mathcal{F}_t$. Then

$$\mathbb{E}_{\widetilde{\mu}_2} e^{i\langle z,X(t)\rangle} = h_z(-i). \tag{5}$$

Observe that the process (X, U) is a Levy process under μ_1, as a sum of independent Levy processes (corresponding to the continuous and discontinuous part). Thus by (5) it follows that under $\widetilde{\mu}_2$, the process X is a Levy process with generating triplet (A_2, γ_2, ν_2). Therefore, $\widetilde{\mu}_2$ and μ_2 must coincide and as a result $\mu_2 \sim \mu_1$. □

4 Appendix

For the convinience of the reader we recall some well known facts.

Theorem 2 (Feldman-Hajek, see Da Prato, Zabczyk (1992), Theorem)**.** Let μ, ν be two measures on separable Hilbert space. Then the following statements hold

(i) Suppose that $\mu = N(m_1, Q_1)$, $\nu = N(m_2, Q_2)$. Then μ and ν are either singular or equivalent.

(ii) μ and ν are equivalent if and only if the following conditions hold:

 (a) $Q_1^{1/2}(H) = Q_2^{1/2}(H) =: H_0$

 (b) $m_1 - m_2 \in H_0$;

 (c) $Q_1^{-1/2} Q_2 Q_1^{-1/2} - I$ is a Hilbert-Schmidt operator[2] defined on $\overline{H_0}$.

Let π be the Poisson random measure with intensity measure $ds\nu(dx)$. The following theorem summarizes properties of the integrals with respect to Poisson random measures and can be found in many texts, like Kingman (1993), Applebaum (2004), Peszat, Zabczyk (2007), to name a few.

Theorem 3. Let $f\colon H \to \mathbb{R}_+$ such that $\int_H |f(x)|\nu(dx) < \infty$. Then

[2]We say that a linear, bounded operator $\Lambda\colon H \to H$ is a Hilbert-Schmidt operator if there exists in H an orthonormal and complete basis (e_k) such that $\sum_{k,j=1}^{\infty} |\langle \Lambda e_j, e_k\rangle|^2 < \infty$, see Da Prato, Zabczyk (1992).

(i) $\int_0^t \int_H f(x)\pi(ds,dx)$ is well defined.

(ii) $\mathbb{E}\exp\left\{c\int_0^t \int_H f(x)\pi(ds,dx)\right\} = \exp\left\{\int_0^t \int_H (e^{cf(x)}-1)ds\nu(dx)\right\}$ for all $c \in \mathbb{C}$ such that the right hand side converges.

(iii) $\mathbb{E}|\int_0^t \int_H f(x)\pi(ds,dx)| < \infty$ and $\mathbb{E}\int_0^t \int_H f(x)\widehat{\pi}(ds,dx) = 0$.

(iv) Assume additionally that $\int_H |f(x)|^2\nu(dx) < \infty$. Then

$$\mathbb{E}|\int_0^t \int_H f(x)\widehat{\pi}(ds,dx)|^2 = \int_0^t \int_H |f(x)|^2 ds\nu(x).$$

(v) Let $f_1, \ldots, f_n$ satisfy the same conditions as f and $A_1, \ldots, A_n$ be the disjoint subsets of $[0,T]\times H$. Then $\int_{A_1} f_1(x)\pi(ds,dx), \ldots, \int_{A_n} f_n(x)\pi(ds,dx)$ are independent.

Remark 2 (Law of Large Numbers)**.** Let $X_1, \ldots, X_n$ be a sequence of real random variables on a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that

$$\lim_{n\to\infty} n^{-2}D^2(\sum_{k=1}^n X_k) = 0.$$

Then Chebyshev's inequality implies that

$$n^{-1}\left(\sum_{k=1}^n X_k - \mathbb{E}\sum_{k=1}^n X_k\right) \to_{n\to\infty} 0 \quad \text{w.r.t. } \mathbb{P}.$$

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