**INSTITUTE OF MATHEMATICS**
**of the**
**Polish Academy of Sciences**

ul. Śniadeckich 8, P.O.B. 21, 00-956 Warszawa 10, Poland http://www.impan.pl

IM PAN Preprint 692 (2008)

Enrico Priola and Jerzy Zabczyk

# Structual properties of semilinear SPDEs driven by cylindrical stable processes

*Published as manuscript*

*Received 24 October 2008*

# Structural properties of semilinear SPDEs driven by cylindrical stable processes

October 20, 2008

Enrico Priola [1]

*Dipartimento di Matematica, Università di Torino,*
*via Carlo Alberto 10, 10123, Torino, Italy.*
*e-mail enrico.priola@unito.it*

Jerzy Zabczyk [2]

*Instytut Matematyczny, Polskiej Akademii Nauk,*
*ul. Sniadeckich 8, 00-950, Warszawa, Poland.*
*e-mail zabczyk@impan.gov.pl*

**Mathematics Subject Classification (2000):** 47D07, 60H15, 60J75, 35R60.

**Abstract:** We consider a class of semilinear stochastic evolution equations driven by an additive cylindrical stable noise. We investigate structural properties of the solutions like Markov, irreducibility, stochastic continuity, Feller and strong Feller properties, and study integrability of trajectories. The obtained results can be applied to semilinear stochastic heat equations with Dirichlet boundary conditions and bounded and Lipschitz nonlinearities.

## 1 Introduction

The paper is concerned with structural properties of solutions to nonlinear stochastic equations

$$dX_t = AX_tdt + F(X_t)dt + dZ_t, \quad t \ge 0, \quad X_0 = x \in H, \tag{1.1}$$

in a real separable Hilbert space $H$ driven by an infinite dimensional stable process $Z = (Z_t)$. In particular, we study Markov, irreducibility, stochastic continuity, Feller and strong Feller properties for the solutions, and investigate integrability of trajectories. The main results are gradient estimates for the associated transition semigroup

[1] Supported by the M.I.U.R. research projects Prin 2004 and 2006 "Kolmogorov equations" and by the Polish Ministry of Science and Education project 1PO 3A 034 29 "Stochastic evolution equations with Lévy noise".

[2] Supported by the Polish Ministry of Science and Education project 1PO 3A 034 29 "Stochastic evolution equations with Lévy noise".

(see Theorem 4.17 when $F = 0$ and Theorem 5.6 in the general case), from which we deduce the strong Feller property, and a theorem on time regularity of trajectories (see Theorem 4.6). To cover interesting cases, we consider processes $Z$ which take values in a Hilbert space $U$ usually greater than $H$. Moreover $A : \mathrm{dom}(A) \subset H \to H$ is a linear possibly unbounded operator which generates a $C_0$-semigroup $(e^{tA})$ on $H$ and $F : H \to H$ denotes a Lipschitz continuous and bounded function.

In the case when $Z$ is a Wiener process the theory of equations (1.1) is well understood. The situation changes completely in the stable noise case and new phenomena appear. For instance, even in the linear case $F = 0$, it is not known when solutions of (1.1) have càdlàg trajectories. That lack of càdlàg regularity is possible was noted in ([15, Proposition 9.4.4]) in a similar situation. Another difficulty is related to the fact that general necessary and sufficient conditions for absolute continuity of stable measures on Hilbert spaces exist only in the subclass of Gaussian measures.

Structural properties of solutions in the case when $Z$ is a cylindrical Wiener process were an object of a large number of papers (see [5], [6] and the references therein). Some results are also available when $Z$ has a non-trivial Gaussian component (see [19], [16] and the references therein). The situation is different if the Lévy process $Z$ has no a Gaussian part. Even the existence of regular densities for the transition probability functions has been analyzed rather recently and only in finite dimensions, i.e., for ordinary differential stochastic differential equations (see e.g. [17] and the references therein).

According to a well known result due to Doob, see [6, Theorem 4.2.1], our Theorems 5.4 and 5.6 about irreducibility and strong Feller property show that the process $X = (X_t^x)$ in (1.1) has at most one invariant measure. We also deal with a closely related question of existence of regular densities for the transition probability functions. The lack of translation invariant measures in infinite dimensional spaces makes these problems more difficult. We restrict our considerations to SPDEs with additive noise as even in this case some new phenomena, related to the cylindrical Lévy noise, appear. We hope that the results presented here will form a proper starting point to treat general equations with multiplicative Lévy perturbations. Let us also mention that the recent reference [15] is mostly concerned with existence questions for SPDEs driven by Lévy noises rather than with structural properties of the solutions.

In this paper we consider a *cylindrical $\alpha$-stable process* $Z = (Z_t)$, $\alpha \in (0, 2)$, defined by the orthogonal expansion

$$Z_t = \sum_{n \geq 1} \beta_n Z_t^n e_n, \quad t \geq 0, \tag{1.2}$$

where $(e_n)$ is an orthonormal basis of $H$ and $Z_t^n$ are independent, real valued, normalized, symmetric $\alpha$-stable processes defined on a fixed stochastic basis. Moreover, $(\beta_n)$ is a given, possibly unbounded, sequence of *positive* numbers.

The results of the paper apply to stochastic heat equations with Dirichlet boundary conditions

$$\begin{cases} dX(t,\xi) = \big(\triangle X(t,\xi) + f(X(t,\xi))\big)\, dt + dZ(t,\xi), \quad t > 0, \\ X(0,\xi) = x(\xi), \quad \xi \in D, \\ X(t,\xi) = 0, \ \ t > 0, \quad \xi \in \partial D, \end{cases} \tag{1.3}$$

in a given bounded domain $D \subset \mathbb{R}^d$ having Lipschitz-continuous boundary $\partial D$. Here $x(\xi) \in H = L^2(D)$, $f : \mathbb{R} \to \mathbb{R}$ is bounded and Lispchitz continuous and the noise $Z$ is a cylindrical $\alpha$-stable process of the form (1.2), where $(e_n)$ is a basis of eigenfunctions for the Laplace operator $\triangle$ (with Dirichlet boundary conditions).

After short Preliminaries, concerned with notations and basic definitions, in Section 3, we deal with real and Hilbert space valued $\alpha$-stable random variables. We derive some useful lemmas about $\alpha$-stable densities needed in the sequel. The most important result here is a necessary and sufficient condition for the absolute continuity of shifts of infinite products of symmetric $\alpha$-stable, one dimensional distributions (see Theorem 3.4). It is an improvement of an old result by Zinn (see [24]) with a direct proof.

Section 4 is concerned with linear equations

$$dX_t = AX_t dt + dZ_t, \quad t \geq 0, \quad X_0 = x \in H. \tag{1.4}$$

We assume that vectors $(e_n)$ from the representation (1.2) are eigenvectors of $A$. The solutions, called Ornstein-Uhlenbeck processes, have received a lot of attention recently (see, for instance, [4], [3], [11], [7], [19] and [15]).Transition semigroups determined by solutions $X = (X_t^x)$ to (1.4) are also studied under the name of generalized Mehler semigroups.

In Proposition 4.4 we give if and only if conditions under which $X$, the solution of (1.4), takes values in $H$, and establish its basic properties like measurability and markovianity. Then we deal with the time regularity of trajectories. The main result here is Theorem 4.6, which establishes stochastic continuity of the solution and integrability of its trajectories. Better regularity, like right or left continuity of trajectories is established here in very special cases and is an open question for general equations. Note that in [11] it is proved that trajectories of $(X_t^x)$ are càdlàg only in some enlarged Hilbert space $U$ containing $H$. This lack of time regularity introduces additional difficulties into the theory (see also Proposition 9.4.4 in [15]). We establish also irreducibility of the solution. Theorem 4.15 gives conditions under which all transition laws of $X$ are equivalent and establishes a formula for the densities. Moreover (see Theorem 4.17) under the assumptions of Theorem 4.15, the transition semigroup corresponding to $X$ is not only strong Feller but transforms bounded measurable functions onto Fréchet differentiable functions with continuous derivative. Important gradient estimates are established as well.

Theorems on Ornstein-Uhlenbeck processes are based on results about stable measures established in Section 3.

Section 5 is devoted to nonlinear equations (1.1). The proofs of the Markov property and irreducibility require special attention due to the lack of càdlàg regularity of the trajectories. They are given in Theorems 5.3 and 5.4. Then estimates of Section 4 are used to establish the strong Feller property of the solution to the nonlinear equation (see Theorem 5.6). The main tool here is the so called mild version of the Kolmogorov equation and Galerkin's approximation. It is proper to add that the classical approach to get strong Feller using the Bismut-Elworthy-Li formula is not available in the non-Gaussian case. A related formula, but requiring a non trivial Gaussian component in the Lévy noise, was established in finite dimensions in [18].

## 2 Preliminaries

$H$ will denote a real separable Hilbert with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. By $\mathcal{L}(H)$ we denote the space of all bounded linear operators from $H$ into $H$. We will fix an orthonormal basis $(e_n)$ in $H$. Through the basis $(e_n)$ we will often identify $H$ with $l^2$.

More generally, for a given sequence $\rho = (\rho_n)$ of real numbers, we set

$$l^2_\rho = \{(x_n) \in \mathbb{R}^\infty \, : \, \sum_{n \geq 1} x_n^2 \rho_n^2 < \infty\}. \tag{2.1}$$

The space $C_b(H)$ (resp. $B_b(H)$) stands for the Banach space of all real, continuous (resp. Borel) and bounded functions $f : H \to \mathbb{R}$, endowed with the supremum norm: $\|f\|_0 = \sup_{x \in H} |f(x)|$.

The space $\mathcal{C}_b^k(H)$, $k \geq 1$, is the set of all $k$-times differentiable functions $f$, whose Fréchet derivatives $D^i f$, $1 \leq i \leq k$, are continuous and bounded on $H$, up to the order $k$. Moreover we set $C_b^\infty(H) = \cap_{k \geq 1} C_b^k(H)$.

Let us recall that a Lévy process $(Z_t)$ with values in $H$ is an $H$-valued process defined on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, having stationary independent increments, càdlàg trajectories, and such that $Z_0 = 0$, $\mathbb{P}$-a.s..

One has that

$$\mathbb{E}[e^{i\langle Z_t, s\rangle}] = \exp(-t\psi(s)), \; s \in H, \tag{2.2}$$

where $\psi : H \to \mathbb{C}$ is a Sazonov continuous, negative definite function such that $\psi(0) = 0$ (see [14] for more details). We call $\psi$ the *exponent* of $(Z_t)$. Given $\psi$ with the previous properties, there exists a unique in law $H$-valued Lévy process $(Z_t)$, such that (2.2) holds.

The exponent $\psi$ can be expressed by the following infinite dimensional Lévy-Khintchine formula,

$$\psi(s) = \frac{1}{2}\langle Qs, s\rangle - i\langle a, s\rangle - \int_H \Big(e^{i\langle s,y\rangle} - 1 - \frac{i\langle s, y\rangle}{1 + |y|^2}\Big)\nu(dy), \;\; s \in H, \tag{2.3}$$

where $Q$ is a symmetric non-negative trace class operator on $H$, $a \in H$ and $\nu$ is the Lévy measure or the jump intensity measure associated to $(Z_t)$ (see [21] and [15]).

According to Proposition 3.3 (see Remark 4.1) our cylindrical $\alpha$-stable process $Z$ appearing in (1.2) is a Lévy process taking values in a Hilbert space $U = l^2_\rho$, see (2.1), with a properly chosen weight $\rho$.

Let $(P_t)$ be a transition Markov semigroup acting on $B_b(H)$,

$$P_t f(x) = \int_H f(y) p_t(x, dy), \;\; f \in B_b(H), \;\; x \in H.$$

The semigroup $(P_t)$ is called *Feller,* if $P_t f \in C_b(H)$, for any $t \geq 0$, $f \in C_b(H)$. It is called *strong Feller,* if

$$P_t f \in C_b(H), \quad \text{for any } \; t > 0, \; f \in B_b(H). \tag{2.4}$$

If all transition probability functions $p_t(x, \cdot)$, $t > 0$, $x \in H$, are equivalent, the semigroup $(P_t)$ is called *regular.*

# 3 Stable measures on Hilbert spaces

A random variable $\xi$ with values in $H$ is called *$\alpha$-stable* ($\alpha \in ]0,2]$) if, for any $n$, there exists a vector $a_n \in H$, such that, for any independent copies $\xi_1, \ldots, \xi_n$ of $\xi$, the random variable

$$n^{-1/\alpha}(\xi_1 + \ldots + \xi_n) - a_n$$

has the same distribution of $\xi$. A Borel probability measure $\mu$ on $H$ is said to be $\alpha$-stable if it is the distribution of a stable random variable with values in $H$.

We will mainly consider stable measures which are product of infinitely many one dimensional $\alpha$-stable distributions.

## 3.1 Stable one dimensional densities

Let us consider a one dimensional, normalized, symmetric $\alpha$-stable distribution $\mu_\alpha$, $\alpha \in ]0,2]$, having characteristic functions:

$$\hat{\mu}_\alpha(s) = e^{-|s|^\alpha}, \quad s \in \mathbb{R}. \tag{3.1}$$

The *density of* $\mu_\alpha$, with respect to the Lebesgue measure, will be denoted by $p_\alpha$. This is known in closed form only if $\alpha = 1$ or 2.
We need to know the *precise asymptotic behaviour* of the density $p_\alpha$, $\alpha \in (0,2)$.

We have that for any $\alpha \in (0,2)$, there exists $C_\alpha > 0$ such that

$$p_\alpha(x) \sim \frac{C_\alpha}{x^{\alpha+1}}, \quad \text{as } x \to \infty, \tag{3.2}$$

see [23] and [21, page 88]. According to [10, pages 582-583], one can derive (3.2) using representations of $p_\alpha$ by convergent power series.

The following result concerning the derivative of $p_\alpha$ is straightforward.

**Lemma 3.1.** *Let $p_\alpha$ be the density of the standard one dimensional $\alpha$-stable measure in* (3.1), *$\alpha \in (0,2)$. Then, for any $\alpha \in (0,2)$, $p_\alpha \in C^\infty(\mathbb{R}) \cap C_0(\mathbb{R})$ and moreover*

$$x^2 p'_\alpha(x) \in L^\infty(\mathbb{R}). \tag{3.3}$$

*Proof.* Let $p = p_\alpha$. It is well known that $p \in C^\infty(\mathbb{R}) \cap C_0(\mathbb{R})$ (see, for instance, [21, Chapter 1]). To get the second assertion, we integrate by parts,

$$x^2 p'(x) = i \int_{\mathbb{R}} x^2 e^{ixy} y e^{-|y|^\alpha} dy$$

$$x \int_{\mathbb{R}} \frac{d}{dy}\big(e^{ixy}\big) y e^{-|y|^\alpha} dy = -x \int_{\mathbb{R}} e^{ixy} e^{-|y|^\alpha} \Big(1 + \alpha |y|^\alpha\Big) dy$$

$$= -i\alpha \int_{\mathbb{R}} e^{ixy} e^{-|y|^\alpha} \frac{y}{|y|^{2-\alpha}} dy \; - \; i\alpha^2 \int_{\mathbb{R}} e^{ixy} e^{-|y|^\alpha} \Big(\frac{y}{|y|^{2-2\alpha}} + \frac{y}{|y|^{2-\alpha}}\Big) dy.$$

From this formula the assertion is clear. □

We need the next technical lemma.

**Lemma 3.2.** *Let us consider the function*

$$g(x) = 1 - \int_{\mathbb{R}} p_\alpha^{1/2}(z)\, p_\alpha^{1/2}\big(z - x\big)\, dz, \qquad x \in (-1, 1).$$

*We have*

$$g(x) \sim c_\alpha x^2 \quad \textit{as} \quad x \to 0, \quad \textit{where} \quad c_\alpha = \frac{1}{8} \int_{\mathbb{R}} \frac{p_\alpha'(z)^2}{p_\alpha(z)}\, dz. \tag{3.4}$$

*Proof.* Let $p = p_\alpha$. In order to prove (3.4) we will apply Hopital's rule. To this purpose we prove that $g$ is twice differentiable, with $g'(0) = 0$ and $g''(0) \neq 0$. We have, for $|x| < 1$,

$$g'(x) = \frac{1}{2} \int_{\mathbb{R}} p^{1/2}(z)\, \frac{1}{p^{1/2}\big(z - x\big)}\, p'(z - x)\, dz.$$

The differentiation is justified by (3.2) and (3.3). Indeed, for any $M > 1$, there exists $c > 0$, such that, for any $x \in (-1, 1)$, $|z| > M > 1$,

$$p^{1/2}(z) \frac{1}{p^{1/2}\big(z - x\big)}\, |p'(z - x)| \leq \frac{c}{|z|^{1/2 + \alpha/2}} \frac{(|z| + 1)^{1/2 + \alpha/2}}{||z|^2 - 1|}.$$

The previous estimate also allows to get that $g'(0) = -\frac{1}{2} \int_{\mathbb{R}} p'(z)\, dz = 0$. We show now that there exists the second derivative of $g$. To this purpose, we write

$$g'(x) = \frac{1}{2} \int_{\mathbb{R}} p^{1/2}(z + x)\, \frac{1}{p^{1/2}\big(z\big)}\, p'(z)\, dz.$$

We have, for any $x \in (-1, 1)$,

$$g''(x) = \frac{1}{4} \int_{\mathbb{R}} \frac{p'(z + x)}{p^{1/2}(z + x)}\, \frac{1}{p^{1/2}\big(z\big)}\, p'(z)\, dz.$$

The differentiation can be done, since, for any $M > 1$, there exists $c' > 0$, such that, for any $x \in (-1, 1)$, $|z| > M > 1$,

$$\frac{|p'(z + x)|}{p^{1/2}(z + x)}\, \frac{|p'(z)|}{p^{1/2}\big(z\big)} \leq \frac{c'\, (|z| + 1)^{1 + \alpha}}{||z|^2 - 1|^2}.$$

We have also that

$$g''(0) = \frac{1}{4} \int_{\mathbb{R}} \frac{p'(z)^2}{p(z)}\, dz.$$

and so (3.4) is proved. □

## 3.2 Supports of stable measures

Let us consider independent real random variables $\xi_n$, having all the same law $\mu_\alpha$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Take nonnegative numbers $q_n$ and consider the random variable

$$\xi = (q_1 \xi_1, \ldots, q_n \xi_n, \ldots) \tag{3.5}$$

with values in $\mathbb{R}^\infty$. We start with a preliminary result, which is a special case of [13, Corollary 2.4.2]. We provide a proof for the sake of completeness.

**Proposition 3.3.** *For any $\alpha \in ]0,2]$, the random variable $\xi$ in* (3.5) *takes values in $l^2$, $\mathbb{P}$-a.s., if and only if*

$$\sum_{n\geq 1} q_n^{\alpha} < \infty. \tag{3.6}$$

*If, in addition to* (3.6)*, $q_n > 0$, $k = 1, 2, \ldots$, then the support of the law of $\xi$ is $l^2$.*

*Proof.* We will use the following theorem (see, for instance [12], page 70-71): let $U_n$ be a sequence of independent and symmetric real random variables; then the following statements are equivalent: $\sum_{n\geq 1} U_n$ converge in distribution; $\sum_{n\geq 1} U_n$ converges $\mathbb{P}$-a.s.; $\sum_{n\geq 1} U_n^2$ converges $\mathbb{P}$-a.s..
We have, for any $N \in \mathbb{N}$, $h \in \mathbb{R}$,

$$\mathbb{E}[e^{i\sum_{n=1}^{N} q_n \xi_n h}] = \prod_{n=1}^{N} \mathbb{E}[e^{i q_n \xi_n h}] = e^{-t\sum_{n=1}^{N} q_n^{\alpha} |h|^{\alpha}}.$$

Then it is clear that $\sum_{k=1}^{N} q_k \xi_k$ converges in distribution if and only if (3.6) holds. Moreover if (3.6) holds, then we have convergence in distribution to the random variable $\xi_1 \big(\sum_{k=1}^{\infty} q_k^{\alpha}\big)^{1/\alpha}$. It follows that the series $\sum_{k\geq 1} q_k \xi_k$ converges, $\mathbb{P}$-a.s., and also that

$$\sum_{k\geq 1} q_k^2\, \xi_k^2 < \infty, \qquad \mathbb{P} - a.s., \tag{3.7}$$

and this proves the first part.

To prove the second assertion, we fix an arbitrary ball $B \subset H$, $B = B(y, r)$ with center in $y = (y_k) \in H$ and radius $r > 0$. Using independence, we find

$$\begin{aligned} &\mathbb{P}\Big(\sum_{k\geq 1} (q_k \xi_k - y_k)^2 < r^2\Big) \\ &\geq \mathbb{P}\Big(\sum_{k=1}^{N} (q_k \xi_k - y_k)^2 < \epsilon,\ \sum_{k>N} (q_k \xi_k - y_k)^2 < r^2 - \epsilon\Big) \\ &\geq \mathbb{P}\Big(\sum_{k=1}^{N} (q_k \xi_k - y_k)^2 < \epsilon\Big)\, \mathbb{P}\Big(\sum_{k>N} (q_k \xi_k - y_k)^2 < r^2 - \epsilon\Big). \end{aligned}$$

Now we use that each one dimensional measure $\mu_\alpha$ has a positive density on $\mathbb{R}$. This implies that, for any $N \in \mathbb{N}$, $\epsilon > 0$,

$$\mathbb{P}\Big(\sum_{k=1}^{N} (q_k \xi_k - y_k)^2 < \epsilon\Big) > 0.$$

Since $\mathbb{P}\big(\sum_{k>N} (q_k \xi_k - y_k)^2 < r^2 - \epsilon\big) \to 1$, as $N \to \infty$, the assertion follows. $\square$

## 3.3 Equivalence of shifts of stable measures

In general conditions for equivalence of stable measures in Hilbert spaces are not known. Here we give necessary and sufficient conditions in order that shifts of infinite products, of one dimensional $\alpha$-stable distributions, are equivalent. The equivalence result seems to be new. Moreover, according to Zinn [24] (see Remark 3.5) this result is sharp.

**Theorem 3.4.** *Let us consider the $l^2$-random variable $\xi$ in (3.5) under the condition $q_k > 0$, $k \geq 1$, and $\sum_{k\geq 1} q_k^\alpha < \infty$. Take arbitrary $u, v \in l^2$ such that*

$$\sum_{k\geq 1} \frac{|u_k - v_k|^2}{q_k^2} < \infty. \tag{3.8}$$

*Then the law of the random variable $\xi + u$ and the one of $\xi + v$ are equivalent.*

*In addition, if $\mu$ and $\nu$ denote the laws of $\xi + u$ and $\xi + v$ respectively, the density $\frac{d\mu}{d\nu}$ of $\mu$ with respect to $\nu$ is given by*

$$\frac{d\mu}{d\nu} = \lim_{N\to\infty} \prod_{k=1}^{N} \frac{p_\alpha\left(\frac{z_k - u_k}{q_k}\right)}{p_\alpha\left(\frac{z_k - v_k}{q_k}\right)} \quad in\ \ L^1(\nu), \ \ \alpha \in ]0, 2].$$

The proof requires Lemma 3.2.

*Proof of Theorem 3.4.* Let $p_\alpha = p$ with $\alpha \in (0, 2)$. The measures $\mu$ and $\nu$ can be seen as Borel product measures in $\mathbb{R}^\infty$, i.e.,

$$\mu = \prod_{k\geq 1} \mu^k, \qquad \nu = \prod_{k\geq 1} \nu^k, \quad \text{where } \mu^k,\ \nu^k \text{ have densities, respectively,}$$
$$\frac{1}{q_k} p\Big(\frac{z_k - u_k}{q_k}\Big) \ \text{ and } \ \frac{1}{q_k} p\Big(\frac{z_k - v_k}{q_k}\Big).$$

Now we will use the Hellinger integral. According to [5, Proposition 2.19], $\mu$ and $\nu$ are equivalent if and only if

$$H(\mu,\nu) = \prod_{k\geq 1} \int_{\mathbb{R}} \Big(\frac{d\mu^k}{d\nu^k}\Big)^{1/2} \nu^k(dz_k) \ = \prod_{k\geq 1} \int_{\mathbb{R}} \Big(\frac{d\mu^k}{dz_k}\Big)^{1/2} \Big(\frac{d\nu^k}{dz_k}\Big)^{1/2} (dz_k) \ > 0.$$

Define

$$\begin{aligned} a_k &= \int_{\mathbb{R}} \left(\frac{d\mu^k}{dz_k}(z_k)\right)^{1/2} \left(\frac{d\nu^k}{dz_k}(z_k)\right)^{1/2} dz_k \\ &= \int_{\mathbb{R}} \Big[p^{1/2}\Big(z_k - \frac{u_k}{q_k}\Big) p^{1/2}\Big(z_k - \frac{v_k}{q_k}\Big)\Big]\, dz_k. \end{aligned}$$

Note that

$$\prod_{k\geq 1} a_k = \prod_{k\geq 1} (1 - (1 - a_k)) = e^{\sum_{k\geq 1} \ln(1-(1-a_k))}$$

Note that, if $0 < 1 - a < 1/2$, then

$$\ln(1 - (1 - a)) > -2\log 2\,(1 - a).$$

Consequently, if there exists $k_0$ such that, for all $k \geq k_0$,

$$1 - a_k \leq 1/2 \quad \text{ then } \prod_{k\geq k_0} a_k \geq e^{-2\log 2 \sum_{k\geq k_0} (1-a_k)}.$$

We have, for any $k \geq k_0$

$$1 - a_k = 1 - \int_{\mathbb{R}} p^{1/2}(z_k - \frac{u_k}{q_k}) p^{1/2}(z_k - \frac{v_k}{q_k})\, dz_k 1 - \int_{\mathbb{R}} p^{1/2}(z)\, p^{1/2}\big(z - (\frac{v_k}{q_k} - \frac{u_k}{q_k})\big)\, dz.$$

We write

$$1 - a_k = g\big(\frac{v_k}{q_k} - \frac{u_k}{q_k}\big),$$

where the function $g$ is considered in Lemma 3.2.

Using (3.4) and (3.8), there exists $k_0$ such that, for any $k \geq k_0$,

$$1 - a_k \leq \frac{c_\alpha}{2} \big|\frac{v_k}{q_k} - \frac{u_k}{q_k}\big|^2 \leq 1/2. \tag{3.9}$$

It follows that

$$\prod_{k \geq k_0} a_k \geq e^{-c_\alpha \log 2 \, (\sum_{k \geq k_0} \frac{|u_k - v_k|^2}{q_k^2})} > 0$$

and so $\prod_{k \geq 1} a_k > 0$. The second assertion follows from the first one, applying [5, Proposition 2.19]. □

**Remark 3.5.** The result agrees with [24, Corollary 8.1], which shows that the law of $\xi + u$, $u \in H$, is *absolutely continuous* with respect to the one of $\xi$ if and only if

$$\sum_{k \geq 1} \frac{u_k^2}{q_k^2} < \infty.$$

We point out that in [24], there are no conditions to assure the equivalence of $\alpha$-stable measures.

# 4 The linear stochastic PDE

Let $(e_n)$ be the fixed reference orthonormal basis in $H$. We consider the linear equation

$$dX_t = AX_t dt + dZ_t, \quad x \in H, \tag{4.1}$$

where $Z$ is a cylindrical $\alpha$-stable process, $\alpha \in (0, 2)$, given by (1.2),

$$Z_t = \sum_{n \geq 1} \beta_n Z_t^n e_n, \quad t \geq 0.$$

Here $(\beta_n)$ is a given sequence of positive numbers and $(Z_t^n)$ are independent one dimensional $\alpha$-stable processes $(Z_t^n)$ defined on the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, satisfying the usual assumptions. We have, for any $n \in \mathbb{N}$, $t \geq 0$,

$$\mathbb{E}[e^{iZ_t^n h}] = e^{-t|h|^\alpha}, \quad h \in \mathbb{R}.$$

**Remark 4.1.** Identifying, through the basis $(e_n)$, the Hilbert space $H$ with $l^2$ and using Proposition 3.3, one gets that our cylindrical Lévy process $Z$ is a Lévy process with values in the space $l^2_\rho$, see (2.1), where $(\rho_n)$ is a sequence of positive numbers such that $\sum_{n \geq 1} \beta_n^\alpha \rho_n^\alpha < \infty$.

We make the following assumptions.

**Hypothesis 4.2.** *(i) $A : D(A) \subset H \to H$ is a self-adjoint operator such that the fixed basis $(e_n)$ of $H$ verifies: $(e_n) \subset D(A)$, $Ae_n = -\gamma_n e_n$ with $\gamma_n > 0$, for any $n \geq 1$, and $\gamma_n \to +\infty$.*

*(ii)* $\displaystyle\sum_{n\geq 1} \frac{\beta_n^\alpha}{\gamma_n} < \infty$ *(recall that $\beta_n > 0$, for any $n \geq 1$).*

Clearly, under (i),

$$D(A) = \{x = (x_n) \in H \, : \, \sum_{n\geq 1} x_n^2 \gamma_n^2 < +\infty\}.$$

In addition $A$ generates a compact $C_0$-semigroup $(e^{tA})$ on $H$ such that

$$e^{tA} e_k = e^{-\gamma_k t} e_k, \quad k \in \mathbb{N}, \quad t \geq 0.$$

**Example 4.3.** Consider the following linear stochastic heat equation on $D = [0, \pi]^d$ with Dirichlet boundary conditions (see also (1.3))

$$\begin{cases} dX(t,\xi) = \triangle X(t,\xi)\, dt + dZ(t,\xi), \quad t > 0, \\ X(0,\xi) = x(\xi), \quad \xi \in D, \\ X(t,\xi) = 0, \quad t > 0, \quad \xi \in \partial D, \end{cases} \tag{4.2}$$

where $Z$ is a cylindrical $\alpha$-stable process with respect to the eigenfunctions

$$e_j(\xi_1, \ldots, \xi_d) = (\sqrt{2/\pi})^d \sin(n_1 \xi_1) \cdots \sin(n_d \xi_d), \quad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d,$$

$j = (n_1, \ldots, n_d) \in \mathbb{N}^d$, of the Laplacian $\Delta$ (with Dirichlet boundary conditions). The corresponding eigenvalues are $\gamma_j = -(n_1^2 + \ldots + n_d^2)$. Define the operator $A = -\triangle$ with $D(A) = H^2(D) \cap H_0^1(D)$. It is well known that $A$ verifies condition (i) in Hypothesis 4.2. Moreover (see [22, Section 4.4.3]) we have

$$D(A^{\alpha/2}) = \begin{cases} H^\alpha(D) \cap H_0^1(D) & \text{if } 1 < \alpha \leq 2 \\ H_0^\alpha(D) & \text{if } 1/2 < \alpha \leq 1 \\ H^\alpha(D) & \text{if } 0 < \alpha \leq 1/2. \end{cases}$$

If we identify $H$ with $l^2$ then $D(A^{\alpha/2})$ can be identified with the weighted space $l^2_\rho$ (see (2.1)) where $\rho = (\rho_j)$ and $\rho_j = \gamma_j^{\alpha/2}$. The corresponding dual spaces can be identified with $l^2_{1/\rho}$ or with Sobolev spaces of distributions $H^{-\alpha}(D)$. □

According to Hypothesis 4.2, we may consider our equation as an infinite sequence of independent one dimensional stochastic equations, i.e.,

$$dX_t^n = -\gamma_n X_t^n dt + \beta_n dZ_t^n, \quad X_0^n = x_n, \ \ n \in \mathbb{N}, \tag{4.3}$$

with $x = (x_n) \in l^2 = H$. The solution is a stochastic process $X = (X_t^x)$ which takes values in $\mathbb{R}^\infty$ with components

$$X_t^n = e^{-\gamma_n t} x_n + \int_0^t e^{-\gamma_n (t-s)} \beta_n dZ_s^n, \quad n \in \mathbb{N}, \ \ t \geq 0 \tag{4.4}$$

(the previous stochastic integral can be defined as a limit in probability of Riemann sums). It turns out that the process $X$ takes values in $H$ as the next result shows.

**Proposition 4.4.** *Assume (i) in Hypothesis 4.2. Then the process $X$ given in (4.4) takes values in $H$ if and only if condition (ii) holds. Under (ii) it can be written as*

$$X_t^x = \sum_{n\geq 1} X_t^n e_n = e^{tA}x + Z_A(t), \quad \textit{where} \tag{4.5}$$

$$Z_A(t) = \int_0^t e^{(t-s)A} dZ_s = \sum_{n\geq 1} \Big( \int_0^t e^{-\gamma_n(t-s)} \beta_n dZ_s^n \Big) e_n.$$

*For any $x \in H$, the process $(X_t^x)$ is $\mathcal{F}_t$-adapted. Moreover $X$ is Markovian.*

*Proof.* Let us consider the stochastic convolution

$$Y_t^n = Z_A^n(t) = \int_0^t e^{-\gamma_n(t-s)} \beta_n dZ_s^n, \quad n \in \mathbb{N}, \ \ t \geq 0. \tag{4.6}$$

A direct calculation shows that, for any $h \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}[e^{ihY_t^n}] = \exp\Big[ - \beta_n^\alpha \, |h|^\alpha \int_0^t e^{-\alpha\,\gamma_n\, s} ds \Big] = \exp\Big[ - |h|^\alpha \, c_n^\alpha(t) \Big], \\ \text{where } \ c_n(t) = \beta_n \, \Big( \frac{1 - e^{-\alpha\,\gamma_n t}}{\alpha\,\gamma_n} \Big)^{1/\alpha}. \end{aligned} \tag{4.7}$$

It follows that

$$\mathbb{E}[e^{ih\,Y_t^n}] = \mathbb{E}[e^{ih\,c_n(t)\,L_n}], \quad h \in \mathbb{R}, \tag{4.8}$$

where $L_n$ denotes independent $\alpha$-stable random variables having the same law $\mu_\alpha$ (see (3.1)). Now the first assertion follows directly from Proposition 3.3.

The property that $(X_t^x)$ is $\mathcal{F}_t$-adapted is equivalent to the fact that each real process $\langle X_t^x, e_k \rangle$ is $\mathcal{F}_t$-adapted, for any $k \geq 1$, and this clearly holds.

The Markov property follows easily from the identity

$$Z_A(t+h) - e^{hA} Z_A(t) = \int_t^{t+h} e^{(t+h-s)A} dZ_s, \quad t, h \geq 0.$$

$\square$

**Example 4.5.** *(Continuation of Example 4.3)* By considering sequences $(\beta_j)$ of the form $(\beta_j) = (\gamma_j^\delta)$ one can easily indicate Sobolev spaces of distributions in which the cylindrical Lévy process $Z$ might evolve and, at the same time, the Ornstein-Uhlenbeck process $X$ has trajectories in $L^2(D)$. Assume, for instance, that $Z$ is a standard cylindrical $\alpha$-stable process, that is $\beta_j = 1$, for any $j \in \mathbb{N}$. Then $Z \in H^{-\frac{2}{\alpha}}$ if and only if $\sum_j (\gamma_j)^{-\frac{1}{\alpha}} < +\infty$, thus if and only if $d < \frac{2}{\alpha}$.

## 4.1 Time regularity of trajectories

If the cylindrical Lévy process $Z$ in (4.1) takes values in the Hilbert space $H$ then, by the Kotelenez regularity result (see [15, Theorem 9.20]) trajectories of the process $X$ which solves (4.1) are càdlàg with values in $H$. However $Z_t \in H$, for any $t > 0$, if and only if

$$\sum_{k\geq 1} \beta_k^\alpha < \infty, \tag{4.9}$$

and this is a very restrictive assumption. We conjecture that the càdlàg property holds under much weaker conditions but, at the moment, we are able to establish a weaker time regularity of the solutions.

**Theorem 4.6.** *Assume Hypothesis 4.2. Then the Ornstein-Uhlenbeck process* $X = (X_t^x)$ *satisfies:*

*(i) for any* $x \in H$, $X$ *is stochastically continuous;*

*(ii) for any* $x \in H$, $T > 0$, $X$ *has trajectories in* $L^p(0,T;H)$, *for any* $0 < p < \alpha$, $\mathbb{P}$*-a.s..*

*Proof.* Let $0 < p < \alpha$. We set $Y_t = Z_A(t)$, $t \geq 0$, and first show that

$$\mathbb{E}|Y_t|^p \leq \tilde{c}_p \Big( \sum_{n\geq 1} |\beta_n|^\alpha \frac{(1 - e^{-\alpha\gamma_n t})}{\alpha\gamma_n} \Big)^{p/\alpha}, \tag{4.10}$$

where the constant $\tilde{c}_p$ depends only on $p$. Recall that $(X_t^x)$ and $(Y_t)$ are defined on the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Consider a new probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ where a Rademacher sequence $(r_n)$ is defined (i.e., $r_n : \Omega' \to \{1, -1\}$ are independent and identically distributed with $\mathbb{P}'(r_n = 1) = \mathbb{P}'(r_n = -1) = 1/2$).

The following Khintchine inequality holds, for arbitrary real numbers $c_1, \ldots, c_n$, for any $p > 0$,

$$\Big( \sum_{n\geq 1} c_n^2 \Big)^{1/2} \leq c_p \Big( \mathbb{E}' \Big| \sum_{n\geq 1} r_n c_n \Big|^p \Big)^{1/p},$$

where the constant $c_p$ depends only on $p$ (for $p = 1$, we have $c_1 = \sqrt{2}$) and $\mathbb{E}'$ indicates the expectation with respect to $\mathbb{P}'$.

We fix $\omega \in \Omega$, $t \geq 0$, and write

$$\Big( \sum_{n\geq 1} |Y_t^n(\omega)|^2 \Big)^{1/2} \leq c_p \Big( \mathbb{E}' \Big| \sum_{n\geq 1} r_n Y_t^n(\omega) \Big|^p \Big)^{1/p}.$$

Integrating with respect to $\omega$ and using the Fubini theorem on the product space $\Omega \times \Omega'$, we find

$$\begin{aligned} \mathbb{E}|Y_t|^p &\leq c_p^p \, \mathbb{E} \Big[ \mathbb{E}' \Big| \sum_{n\geq 1} r_n Y_t^n \Big|^p \Big] = c_p^p \, \mathbb{E}' \Big[ \mathbb{E} \Big| \sum_{n\geq 1} r_n Y_t^n \Big|^p \Big] \\ &= c_p^p \, \mathbb{E}' \Big[ \mathbb{E} \Big| \sum_{n\geq 1} r_n \int_0^t e^{-\gamma_n (t-s)} \beta_n dZ_s^n \Big|^p \Big]. \end{aligned} \tag{4.11}$$

Since, for any $t \geq 0$, $\lambda \in \mathbb{R}$ (using also that $|r_n| = 1$, $n \geq 1$),

$$\mathbb{E}[e^{i\lambda \sum_{n\geq 1} r_n Y_t^n}] = e^{-|\lambda|^\alpha \sum_{n\geq 1} |\beta_n|^\alpha \int_0^t e^{-\alpha \gamma_n (t-s)} ds},$$

we get easily assertion (4.10).

(i) Let us prove the stochastic continuity. We will show that, for any $\epsilon > 0$,

$$\lim_{h\to 0^+} \sup_{t\geq 0} \mathbb{P}(|Y_{t+h} - Y_t| > \epsilon) = 0. \tag{4.12}$$

This will imply the stochastic continuity.

Note that, for any $t \geq 0$, $h \geq 0$,

$$Y_{t+h} - Y_t = \int_t^{t+h} e^{(t+h-s)A} dZ_s + e^{hA} \int_0^t e^{(t-s)A} dZ_s - \int_0^t e^{(t-s)A} dZ_s$$

$$= e^{hA} Y_t - Y_t + \int_t^{t+h} e^{(t+h-s)A} dZ_s.$$

Let us choose $p \in (0, \alpha)$. We have

$$\mathbb{P}(|Y_{t+h} - Y_t| > \epsilon) \leq \mathbb{P}(\big|e^{hA} Y_t - Y_t\big| > \frac{\epsilon}{2}) + \mathbb{P}(\Big|\int_t^{t+h} e^{(t+h-s)A} dZ_s\Big| > \frac{\epsilon}{2})$$

$$\leq 2^p \frac{\mathbb{E}|e^{hA} Y_t - Y_t|^p}{\epsilon^p} + 2^p \frac{\mathbb{E}\big|\int_0^h e^{sA} dZ_s\big|^p}{\epsilon^p} = I_1(t,h) + I_2(h).$$

But (see (4.10))

$$\mathbb{E}|Y_t|^p \leq c_p \Big( \sum_{n \geq 1} |\beta_n|^\alpha \frac{(1 - e^{-\alpha \gamma_n t})}{\alpha \gamma_n} \Big)^{p/\alpha}$$

and so

$$[I_2(h)]^{\alpha/p} \to 0, \quad \text{as } \ h \to 0^+.$$

Concerning $I_1$, we find, using again the Khintchine inequality,

$$|e^{hA} Y_t - Y_t| = \Big( \sum_{n \geq 1} |(e^{-\gamma_n h} - 1) Y_t^n|^2 \Big)^{1/2} \leq c_p \Big( \mathbb{E}' \Big| \sum_{n \geq 1} r_n (e^{-\gamma_n h} - 1) Y_t^n \Big|^p \Big)^{1/p}$$

and, reasoning as in (4.11) with $\beta_n$ replaced by $(1 - e^{-\gamma_n h})\beta_n$,

$$\mathbb{E}|e^{hA} Y_t - Y_t|^p \leq c_p^p \, \mathbb{E}' \mathbb{E} \Big| \sum_{n \geq 1} r_n (e^{-\gamma_n h} - 1) Y_t^n \Big|^p \leq C_p \Big( \sum_{n \geq 1} |(1 - e^{-\gamma_n h}) \beta_n|^\alpha \frac{(1 - e^{-\alpha \gamma_n t})}{\alpha \gamma_n} \Big)^{p/\alpha}$$

$$\leq \frac{C_p}{\alpha^{p/\alpha}} \Big( \sum_{n \geq 1} \frac{|(1 - e^{-\gamma_n h}) \beta_n|^\alpha}{\gamma_n} \Big)^{p/\alpha}, \qquad t \geq 0.$$

Since

$$\lim_{h \to 0^+} \Big( \sum_{n \geq 1} \frac{|(1 - e^{-\gamma_n h}) \beta_n|^\alpha}{\gamma_n} \Big)^{p/\alpha} = 0,$$

we get

$$\lim_{h \to 0^+} \sup_{t \geq 0} I_1(t,h) = 0$$

and so assertion (4.12) is proved.

(ii) We need to show that, for any $x \in H$, for any $p \in (0, \alpha)$,

$$\int_0^T |X_t^x|^p dt < \infty, \quad \mathbb{P} - a.s..$$

To this purpose, it is enough to show that

$$\mathbb{E} \int_0^T \Big( \sum_{n \geq 1} |Y_t^n|^2 \Big)^{p/2} dt < \infty, \tag{4.13}$$

where $Y_t = Z_A(t)$, $t \ge 0$. Using (4.10) we get

$$\int_0^T \mathbb{E}|Y_t|^p dt \le \tilde{c}_p \int_0^T \Big( \sum_{n\ge 1} |\beta_n|^\alpha \frac{(1-e^{-\alpha\gamma_n t})}{\alpha\gamma_n} \Big)^{p/\alpha} dt \le C_{p,\alpha}\, T \Big( \sum_{n\ge 1} \frac{|\beta_n|^\alpha}{\gamma_n} \Big)^{p/\alpha} < +\infty.$$

The proof is complete. □

**Remark 4.7.** In the limiting Gaussian case of $\alpha = 2$, the previous proof allows to get the well known result that trajectories of $X$ are in $L^2(0, T; H)$, for any $T > 0$.

Recall that the $\sigma$-algebra $\mathcal{P}$ of predictable sets is the smallest $\sigma$-algebra on $[0, \infty[\times\Omega$ containing the sets $\{0\} \times A_0$, $]s, t] \times A_s$, for any $0 < s < t$, $A_0 \in \mathcal{F}_0$, $A_s \in \mathcal{F}_s$. A stochastic process with values in $H$ is said to be *predictable* if it is measurable as an application from $([0, \infty[\times\Omega, \mathcal{P})$ with values in $(H, \mathcal{B}(H))$, where $\mathcal{B}(H)$ is the Borel $\sigma$-algebra of $H$.

Using that $X = (X_t^x)$, $x \in H$, is stochastically continuous and $\mathcal{F}_t$-adapted (see Theorem 4.6 and Proposition 4.12) we can apply [5, Proposition 3.6] and obtain

**Corollary 4.8.** *For any $x \in H$, the process $(X_t^x)$ has a predictable version.*

For $p \in (0, 1)$, $L^p(0, T; H)$ is a linear complete and separable metric space with respect to the distance

$$d_p(f, g) = \int_0^T |f(t) - g(t)|^p dt, \qquad f, g \in L^p(0, T; H).$$

From Theorem 4.6 it is straightforward to obtain

**Corollary 4.9.** *Assume Hypothesis 4.2. Then, for any $T > 0$, $x \in H$, $\mathbb{P}$-a.s., the Ornstein-Uhlenbeck process $X = (X_t^x)_{t\in[0,T]}$ is a random variable with values in $L^p(0, T; H)$, for any $0 < p < \alpha$.*

## 4.2 Support

We start with a preliminary one dimensional result.

**Proposition 4.10.** *Let $L = (L_t)$ be a one dimensional $\alpha$-stable process, $\alpha \in (0, 2)$, $\gamma \in \mathbb{R}$, and set*

$$K(t) = \int_0^t e^{\gamma(t-s)} dL_s, \qquad t \ge 0.$$

*Then, for any $p > 0$, $T > 0$, the random variable $(K, K_T)$ has full support in $L^p(0, T) \times \mathbb{R}$.*

The proposition is a direct corollary of the following general lemma. Recall that for an arbitrary Borel measure $\gamma$ on $\mathbb{R}$, we have the unique measure decomposition

$$\gamma = \gamma_{ac} + \gamma_s \tag{4.14}$$

where $\gamma_{ac}$ has a density and $\gamma_s$ is singular with respect to the Lebesgue measure.

**Lemma 4.11.** *Let $L = (L_t)$ be a real valued Lévy process with intensity measure $\nu$ (see (2.3)). Suppose that there exists $R > 0$ such that $\nu$ restricted to $(-R, R)$ has an absolutely continuous part with a strictly positive density (see (4.14)). Then, for any $p > 0$, $T > 0$, the random variable $(L, L_T)$ has full support in $L^p(0, T) \times \mathbb{R}$.*

*Proof.* We write $\nu = \nu_0 + \nu_1$, where $\nu_0, \nu_1$ are positive measures and $\nu_1$ is a finite measure with strictly positive density $g$ on $(-R, R)$. We can assume, by using the Ito-Lévy-Khinchine decomposition (see [2]), that $\mathbb{P}$-a.s.,

$$L = L^1 + L^0, \quad \text{i.e.,} \;\; L_t = L_t^1 + L_t^0, \;\; t \geq 0$$

where $L^1$ and $L^0$ are independent Lévy processes and $L^1$ is a compound Lévy process with the intensity measure $\nu_1$.

Since the law of $(L, L_T)$ is the convolution of the laws of $(L^0, L_T^0)$ and $(L^1, L_T^1)$, our assertion will follow from the fact that $(L^1, L_T^1)$ has full support in $L^p(0,T) \times \mathbb{R}$.

Taking into account that pice-wise constant functions taking value 0 at $t = 0$ are dense in $L^p(0,T)$, for any $p > 0$, we only have to prove that for a fixed pice-wise constant function $\phi : [0,T] \to \mathbb{R}$, with $\phi(0) = 0$, for a fixed $a \in \mathbb{R}$ and $\epsilon > 0$,

$$\mathbb{P}\Big( \int_0^T |L_t^1 + \phi(t)|^p dt + |L_T^1 - a| < \epsilon \Big) > 0. \tag{4.15}$$

We may assume that $\phi(T) = a$ and that $\phi$ takes real values $0, x_1, \ldots, x_{k-1}$, $x_k = a$, respectively on intervals $[0, t_1[, \ldots, [t_k, T[$, with $0 < t_1 < \ldots t_k < T$. Define

$$S = \sup\{|x_i|, \quad i = 1, \ldots, k\}.$$

Let $0 < \tau_1 < \ldots < \tau_k$ be the first $k$ consecutive moments of jumps for the process $L^1$ and denote by $Y_1, \ldots, Y_k$ the random variables $L_{\tau_1}^1, \ldots, L_{\tau_k}^1$; set $Y_0 = 0$ and $\tau_0 = 0$.

Note that $\tau_j - \tau_{j-1}$, $j = 1, \ldots, k$, and $Y_j - Y_{j-1}$, $j = 1, \ldots, k$, are independent random variables. Moreover, $\tau_j - \tau_{j-1}$ have the same exponential distribution and $Y_j - Y_{j-1}$ have the positive density $g$ on $(-R, R)$.

For arbitrary $i, j \in \{0, \ldots, k\}$, $\delta > 0$, $M > S - R$ the independent events

$$\{|\tau_i - t_j| \leq \delta\}, \quad \{|Y_i - x_j| \leq M\}$$

have all positive probabilities. Using this fact and the property of independence, we get easily (4.15). $\square$

**Proof of Proposition 4.10.** We consider $\gamma \neq 0$ (the case $\gamma = 0$ follows from Lemma 4.11). Using [20, Theorem 3.1], we know that there exists an $\alpha$-stable process $Z = (Z_t)$ such that

$$\int_0^t e^{-\gamma s} dL_s = Z(h(t)), \quad \text{where} \;\; h(t) = \frac{1 - e^{-\alpha\gamma t}}{\alpha\gamma}, \quad t \geq 0.$$

Consequently, $K(t) = e^{\gamma t} Z(h(t))$. Using Lemma 4.11 and the fact that $h \in C^\infty([0, +\infty[)$ with $h'(t) \neq 0$, $t \geq 0$, we get the assertion. $\square$

**Theorem 4.12.** *Assume Hypothesis 4.2 and fix $T > 0$, $x \in H$ and $p \in (0, \alpha)$. Consider the Ornstein-Uhlenbeck process $X = (X_t^x)_{t \in [0,T]}$, solving (4.1). The support of the random variable $(X, X_T) : \Omega \to L^p(0,T;H) \times H$ is $L^p(0,T;H) \times H$.*

*Proof.* It is enough to prove that, for any $\epsilon > 0$, and for any $(\phi, a) \in L^p(0,T;H) \times H$, one has

$$\mathbb{P}\Big( \int_0^T \Big( \sum_{n \geq 1} |X_t^n - \phi_n(t)|^2 \Big)^{p/2} dt < \epsilon, \; \sum_{n \geq 1} |X_T^n - a_n|^2 < \epsilon \Big) > 0.$$

By using a density argument, we may assume that $(\phi, a)$ is of the form

$$\phi(t) = \sum_{k=1}^{N} \phi_k(t) e_k, \quad a = \sum_{k=1}^{N} a_k e_k,$$

for some $N \in \mathbb{N}$. We write, using that $p/2 < 1$,

$$\mathbb{P}\Big( \int_0^T \Big( \sum_{n=1}^{N} |X_t^n - \phi_n(t)|^2 \Big)^{p/2} dt < \epsilon, \ \sum_{n=1}^{N} |X_T^n - a_n|^2 < \epsilon \Big)$$

$$\geq \mathbb{P}\Big( \int_0^T \sum_{n=1}^{N} |X_t^n - \phi_n(t)|^p dt < \epsilon, \ \sum_{n=1}^{N} |X_T^n - a_n|^2 < \epsilon \Big)$$

$$\geq \mathbb{P}\Big( \int_0^T |X_t^1 - \phi_1(t)|^p dt < \epsilon/N, \ |X_T^1 - a_1|^2 < \epsilon/N \Big) \cdots$$

$$\cdots \mathbb{P}\Big( \int_0^T |X_t^N - \phi_N(t)|^p dt < \epsilon/N, \ |X_T^N - a_N|^2 < \epsilon/N \Big),$$

using independence. By Proposition 4.10 we know that the previous product of probabilities is positive. The proof is complete. □

**Corollary 4.13.** *Under Hypothesis 4.2, for any $x \in H$, the OU process $(X_t^x)$ is irreducible, i.e., for any open ball $B \subset H$, $t > 0$,*

$$\mathbb{P}(X_t^x \in B) > 0.$$

### 4.3 Equivalence of transition probabilities

Here we will assume Hypothesis 4.2 together with

**Hypothesis 4.14.** *For any $t > 0$,*

$$\sup_{n \geq 1} \frac{e^{-\gamma_n t} \, \gamma_n^{1/\alpha}}{\beta_n} = C_t < \infty. \tag{4.16}$$

**Theorem 4.15.** *Assume Hypotheses 4.2 and 4.14. Then the laws $\mu_t^x$ and $\mu_t^y$ of $X_t^x$ and $X_t^y$, respectively, are equivalent, for any $t > 0$, $x, y \in H$. Moreover the density $\frac{d\mu_t^x}{d\mu_t^y}$ of $\mu_t^x$ with respect to $\mu_t^y$ is given by*

$$\frac{d\mu_t^x}{d\mu_t^y} = \lim_{N \to \infty} \prod_{k=1}^{N} \frac{p_\alpha\Big( \frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)} \Big)}{p_\alpha\Big( \frac{z_k - e^{-\gamma_k t} y_k}{c_k(t)} \Big)} \quad \text{in } L^1(\mu_t^y), \ \text{ where } c_k(t) = \beta_k \Big( \frac{1 - e^{-\alpha \gamma_k t}}{\alpha \, \gamma_k} \Big)^{1/\alpha}. \tag{4.17}$$

*where $p_\alpha$ is the density of the one dimensional $\alpha$-stable measure, $\alpha \in (0,2)$, considered in (3.1).*

*If (4.14) does not hold then for some $x \in H$, $\mu_t^x$ is not absolutely continuous with respect to $\mu_t^0$.*

**Remark 4.16.** If we assume Hypothesis 4.2, then Hypothesis 4.14 is sharp in the limiting Gaussian case of $\alpha = 2$. Indeed, under Hypothesis 4.2 and $\alpha = 2$, Hypothesis 4.14 is equivalent to each of the following facts:

(i) the laws of $X_t^x$ and $X_t^y$ are equivalent, for any $t > 0$, $x, y \in H$;

(ii) the Gaussian Ornstein-Uhlenbeck semigroup $(R_t)$ is strong Feller (see [5, Section 9.4.1]).

In addition, under Hypothesis 4.2, the following regularizing property

$$R_t f \in C_b^\infty(H), \quad t > 0, \quad f \in B_b(H),$$

holds if and only if $e^{-\gamma_n t}\sqrt{\frac{\gamma_n}{\beta_n^2}}$ is a bounded sequence.

*Proof of Theorem 4.15.* Fix $x = (x_n)$ and $y = (y_n)$. Let $p = p_\alpha$ and consider formulas (4.4), (4.6) and (4.8). The density of the random variable $Y_t^k$ is clearly $\frac{1}{c_k(t)} p\Big(\frac{z_k}{c_k(t)}\Big)$ so that the density of $X_t^k$ is

$$\frac{1}{c_k(t)} p\Big(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\Big).$$

Note that $\mu_t^x$ and $\mu_t^y$ can be seen as Borel product measures in $\mathbb{R}^\infty$, i.e.,

$$\mu_t^x = \prod_{k\geq 1} \mu_t^{x_k}, \qquad \mu_t^y = \prod_{k\geq 1} \mu_t^{y_k}, \quad \text{where } \mu_t^{x_k},\ \mu_t^{y_k} \text{ have densities, respectively,}$$

$$\frac{1}{c_k(t)} p\Big(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\Big) \text{ and } \frac{1}{c_k(t)} p\Big(\frac{z_k - e^{-\gamma_k t} y_k}{c_k(t)}\Big).$$

To get the assertion we will apply Theorem 3.4. To this purpose, one has to check that

$$\sum_{k\geq 1} \frac{e^{-2\gamma_k t}|x_k - y_k|^2}{c_k(t)^2} < \infty.$$

This follows easily from (4.16).

If (4.16) does not hold, for some $t > 0$, then it is easy to see that there exists $\hat{x} = (\hat{x}_n) \in H$ such that

$$\sum_{k\geq 1} \frac{e^{-2\gamma_k t}\, \hat{x}_k^2}{c_k(t)^2} = \infty.$$

According to Remark 3.5, this condition means that $\mu_t^{\hat{x}}$, the law of $X_t^{\hat{x}}$, is not absolutely continuous with respect to $\mu_t^0$, the law of $X_t^0 = Z_A(t)$. □

## 4.4 Smoothing effect

We now consider the transition Markov semigroup $(R_t)$ associated to $(X_t^x)$, i.e. $R_t : B_b(H) \to B_b(H)$,

$$R_t f(x) = \mathbb{E}[f(X_t^x)], \quad x \in H, \quad f \in B_b(H), \quad t \geq 0.$$

The next result shows not only that $(R_t)$ is strong Feller, but also that it has a smoothing effect, i.e., that gradient estimates hold for it.

**Theorem 4.17.** *Assume Hypotheses 4.2 and 4.14. Then, for any $t > 0$, the transition semigroup $(R_t)$ maps Borel and bounded functions into $C_b^1(H)-$functions. Moreover, for any $k \in H$, $|k| \le 1$, we have*

$$\sup_{x\in H} |\langle DR_t f(x), k\rangle| \le 8c_\alpha\, C_t \|f\|_0, \quad f \in B_b(H), \qquad \text{where} \quad C_t = \sup_{n\ge 1} \frac{e^{-\gamma_n t}\, \gamma_n^{1/\alpha}}{\beta_n}, \quad t > 0 \tag{4.18}$$

*($c_\alpha$ is defined in (3.4)). Finally, for any $t > 0$, $f \in C_b(H)$, $x = (x_n)$, $h = (h_n) \in H$, we have (see (4.17))*

$$\langle DR_t f(x), h\rangle = \int_H f(e^{tA}x + y) \sum_{k\ge 1} \frac{p'_\alpha(\frac{y_k}{c_k(t)})}{p_\alpha(\frac{y_k}{c_k(t)})}\, \frac{e^{-\gamma_k t}}{c_k(t)}\, h_k\, \mu_t^0(dy), \tag{4.19}$$

*where $\mu_t^0$ is the law of $X_t^0 = Z_A(t)$.*

*Proof.* We fix $t > 0$. The proof is divided into some steps. By the first three steps, we will show that, for any $f \in C_b(H)$, $R_t f$ is Gâteaux differentiable at any $x \in H$ and moreover that equality (4.19) holds.

*I Step.* We assume that $f \in C_b(H)$ is cylindrical, i.e., it depends only on a finite numbers of coordinates. Identifying $H$ with $l^2$ through the basis $(e_n)$, we have

$$f(x) = \tilde f(x_1, \ldots, x_j), \quad x \in H, \tag{4.20}$$

for some $j \ge 1$, and $\tilde f : \mathbb{R}^j \to \mathbb{R}$ continuous and bounded. In this first step we also assume that $\tilde f$ has bounded support in $\mathbb{R}^j$.

Fix arbitrary $x, h \in H$. We want to show that there exists $D_h R_t f(x)$, the directional derivative of $R_t f$ at $x$, along the direction $h$. Set $h_N = \sum_{k=1}^N h_k e_k$ so that $h_N \to h$ in $H$.

Since $f$ is cylindrical, for $m \ge \max(j, N)$, we get

$$R_t f(x) = \int_H f(y) \prod_{k\ge 1} \mu_t^{x_k}(dy) = \int_{\mathbb{R}^m} \tilde f(z) \prod_{k=1}^m p_\alpha\Big(\frac{z_k - e^{-\gamma_k t}x_k}{c_k(t)}\Big) \frac{1}{c_k(t)}\, dz_k.$$

Using our assumptions on $\tilde f$, it is not difficult to show that there exists

$$\begin{aligned} D_{h_N} R_t f(x) &= -\int_{\mathbb{R}^m} \tilde f(z) \Big( \sum_{k=1}^N \frac{p'_\alpha\Big(\frac{z_k - e^{-\gamma_k t}x_k}{c_k(t)}\Big)}{p_\alpha\Big(\frac{z_k - e^{-\gamma_k t}x_k}{c_k(t)}\Big)} \frac{e^{-\gamma_k t} h_k}{c_k(t)} \Big) \cdot \\ &\qquad\qquad \cdot \prod_{k=1}^m p_\alpha\Big(\frac{z_k - e^{-\gamma_k t}x_k}{c_k(t)}\Big) \frac{1}{c_k(t)}\, dz_k \\ &= -\int_H f(z) \Big( \sum_{k=1}^N \frac{p'_\alpha\Big(\frac{z_k - e^{-\gamma_k t}x_k}{c_k(t)}\Big)}{p_\alpha\Big(\frac{z_k - e^{-\gamma_k t}x_k}{c_k(t)}\Big)} \frac{e^{-\gamma_k t} h_k}{c_k(t)} \Big) \prod_{k\ge 1} \mu_t^{x_k}(dz_k), \quad N \in \mathbb{N}. \end{aligned}$$

In order to pass to the limit, as $N \to \infty$, we show that

$$g_N(t, x) = \sum_{k=1}^N \frac{p'_\alpha\Big(\frac{z_k - e^{-\gamma_k t}x_k}{c_k(t)}\Big)}{p_\alpha\Big(\frac{z_k - e^{-\gamma_k t}x_k}{c_k(t)}\Big)} \frac{e^{-\gamma_k t} h_k}{c_k(t)} \quad \text{converges in } \ L^2(\mu_t^x). \tag{4.21}$$

Using that, for $j \neq k$,

$$\frac{e^{-\gamma_k t}h_k}{c_k(t)}\frac{e^{-\gamma_j t}h_j}{c_j(t)}\int_{\mathbb{R}^2}\frac{p'_\alpha\Big(\frac{z_k-e^{-\gamma_k t}x_k}{c_k(t)}\Big)}{p_\alpha\Big(\frac{z_k-e^{-\gamma_k t}x_k}{c_k(t)}\Big)}\frac{p'_\alpha\Big(\frac{z_j-e^{-\gamma_j t}x_j}{c_j(t)}\Big)}{p_\alpha\Big(\frac{z_j-e^{-\gamma_j t}x_j}{c_j(t)}\Big)}\cdot$$
$$\cdot p_\alpha\Big(\frac{z_k-e^{-\gamma_k t}x_k}{c_k(t)}\Big)p_\alpha\Big(\frac{z_j-e^{-\gamma_j t}x_j}{c_j(t)}\Big)dz_k dz_j$$
$$=\int_{\mathbb{R}}p'_\alpha(y_k)dy_k\int_{\mathbb{R}}p'_\alpha(y_j)dy_j=0$$

(since $p'_\alpha$ is odd) we get, for $N,p\in\mathbb{N}$,

$$\int_H\Big|\sum_{k=N}^{N+p}\frac{p'_\alpha\Big(\frac{z_k-e^{-\gamma_k t}x_k}{c_k(t)}\Big)}{p_\alpha\Big(\frac{z_k-e^{-\gamma_k t}x_k}{c_k(t)}\Big)}\frac{e^{-\gamma_k t}h_k}{c_k(t)}\Big|^2\mu_t^x(dz)$$
$$\int_{\mathbb{R}^{p+1}}\Big|\sum_{k=N}^{N+p}\frac{p'_\alpha\Big(\frac{z_k-e^{-\gamma_k t}x_k}{c_k(t)}\Big)}{p_\alpha\Big(\frac{z_k-e^{-\gamma_k t}x_k}{c_k(t)}\Big)}\frac{e^{-\gamma_k t}h_k}{c_k(t)}\Big|^2\prod_{k=N}^{N+p}p_\alpha\Big(\frac{z_k-e^{-\gamma_k t}x_k}{c_k(t)}\Big)\frac{1}{c_k(t)}\,dz_k$$
$$=\int_H\sum_{k=N}^{N+p}\frac{(p'_\alpha)^2\Big(\frac{z_k-e^{-\gamma_k t}x_k}{c_k(t)}\Big)}{p^2_\alpha\Big(\frac{z_k-e^{-\gamma_k t}x_k}{c_k(t)}\Big)}\frac{e^{-2\gamma_k t}h_k^2}{c_k^2(t)}\,\mu_t^x(dz)$$
$$=\sum_{k=N}^{N+p}\frac{e^{-2\gamma_k t}h_k^2}{c_k^2(t)}\int_{\mathbb{R}}\frac{p'^2_\alpha(y_k)}{p_\alpha(y_k)}\,dy_k\ \leq\ 8c_\alpha C_t^2\sum_{k=N}^{N+p}h_k^2,$$

where $8c_\alpha=\int_{\mathbb{R}}\frac{p'^2_\alpha(y)}{p_\alpha(y)}\,dy$ (see (3.4)). This proves (4.21).

Note that, for any $N\in\mathbb{N}$,

$$D_{h_N}R_tf(x)=-\int_H f(z+e^{tA}x)\Big(\sum_{k=1}^{N}\frac{p'_\alpha\Big(\frac{z_k}{c_k(t)}\Big)}{p_\alpha\Big(\frac{z_k}{c_k(t)}\Big)}\frac{e^{-\gamma_k t}h_k}{c_k(t)}\Big)\,\mu_t^0(dz).$$

Up to now we have showed that

$$\frac{R_tf(x+sh_N)-R_tf(x)}{s}=\frac{1}{s}\int_0^s D_{h_N}R_tf(x+rh_N)dr,\quad s\in(-1,1).\tag{4.22}$$

Using also (4.21), it is not difficult to show that, for any $r\in(-1,1)$, $N\in\mathbb{N}$,

$$\lim_{N\to\infty}D_{h_N}R_tf(x+rh_N)=-\int_H f(z+e^{tA}(x+rh))\Big(\sum_{k=1}^{\infty}\frac{p'_\alpha\Big(\frac{z_k}{c_k(t)}\Big)}{p_\alpha\Big(\frac{z_k}{c_k(t)}\Big)}\frac{e^{-\gamma_k t}h_k}{c_k(t)}\Big)\,\mu_t^0(dz).\tag{4.23}$$

Moreover, $|D_{h_N}R_tf(x+rh_N)|\leq 8c_\alpha C_t|h|\|f\|_0$, for any $r\in(-1,1)$. Thus we can pass to the limit, as $N\to\infty$, in (4.22) and get

$$\frac{R_tf(x+sh)-R_tf(x)}{s}=\frac{1}{s}\int_0^s u(t,x+rh)\,dr,\quad s\in(-1,1),\tag{4.24}$$

where $u(t, x + rh)$ is the right-hand side of (4.23). This shows that $R_t f$ is Gâteaux differentiable at $x \in H$ along the direction $h$ and moreover that (4.19) holds.

*II Step.* We consider $f \in C_b(H)$ which is only cylindrical (i.e., $f$ is given by (4.20) but the function $\tilde{f}$ is not assumed to have bounded support in $\mathbb{R}^j$).

Define $\tilde{f}_n(y) = \tilde{f}(y)\phi(\frac{|y|}{n})$, for any $y \in \mathbb{R}^j$, where $\phi : [0, +\infty[\to \mathbb{R}_+$ is a continuous function such that, $\phi(s) = 1$, $s \in [0, 1]$, $\phi(s) = 0$, $s \geq 2$.

We have that $\|\tilde{f}_n\|_0 \leq \|\tilde{f}\|_0$, $n \in \mathbb{N}$, and moreover $\tilde{f}_n(y) \to \tilde{f}(y)$, as $n \to \infty$, for any $y \in \mathbb{R}^j$.

Let $f_n : H \to \mathbb{R}$, $f_n(x) = \tilde{f}_n(x_1, \ldots, x_j)$, for any $x \in H$, $n \in \mathbb{N}$.

We find by the previous step, for any $n \in \mathbb{N}$ and $x \in H$,

$$\frac{R_t f_n(x + sh) - R_t f_n(x)}{s} = \frac{1}{s} \int_0^s D_h R_t f_n(x + rh) dr, \quad s \in (-1, 1). \tag{4.25}$$

Passing to the limit, as $n \to \infty$, it is easy to see that (4.24) holds for $f$. This shows the Gâteaux differentiability of $R_t f$ on $H$ and also the equality (4.19).

*III Step.* We consider an arbitrary $f \in C_b(H)$. Let us introduce the cylindrical functions $g_n$,

$$g_n(x) = f\Big(\sum_{k=1}^{n} x_k e_k\Big), \quad n \in \mathbb{N}, \ \ x \in H.$$

It is clear that $\|g_n\|_0 \leq \|f\|_0$, $n \in \mathbb{N}$, and moreover $g_n(x) \to f(x)$, for any $x \in H$. Repeating the argument of the previous step, with $f_n$ replaced by $g_n$, and passing to the limit, we get that the assertion of the previous step holds even for any $f \in C_b(H)$.

*IV Step.* Let $f \in C_b(H)$ and consider the Gâteaux derivative of $R_t f$ in $x \in H$

$$DR_t f(x) = \int_H f(e^{tA}x + y) \sum_{k \geq 1} \frac{p'_\alpha(\frac{y_k}{c_k(t)})}{p_\alpha(\frac{y_k}{c_k(t)})} \frac{e^{-\gamma_k t}}{c_k(t)} e_k \ \mu_t^0(dy).$$

It is not difficult to show that $DR_t f : H \to H$ is continuous. This gives that $R_t f$ *is Fréchet differentiable at any* $x \in H$.

Moreover, we have the required gradient estimate

$$\|DR_t f\|_0 \leq 8 c_\alpha C_t \|f\|_0$$

*V Step.* To complete the proof, take $g \in B_b(H)$. A well known argument (see [6, Chapter 7]) shows that $R_t g$ is Lipschitz continuous on $H$. Then the semigroup law gives that $R_t g \in C_b^1(H)$, for any $t > 0$. The proof is complete. □

**Remark 4.18.** Under the assumptions of Theorem 4.17, one could show the following regularizing property

$$R_t f \in C_b^\infty(H), \ \ t > 0, \ \ f \in B_b(H).$$

This generalizes the well known smoothing property of the Gaussian Ornstein-Uhlenbeck semigroup (see also Remark 4.16).

**Remark 4.19.** Note that Theorem 4.15 can be also deduced from Theorem 4.17 and Corollary 4.13 if one applies the Hasminkii theorem (see [6, Proposition 4.1.1]).

## 5 Nonlinear stochastic PDEs

We pass now to the main object of our paper, namely nonlinear SPDEs of the form

$$dX_t = AX_t dt + F(X_t)dt + dZ_t, \quad X_0 = x \in l^2 = H, \tag{5.1}$$

where $Z = (Z_t)$ is a cylindrical $\alpha$-stable Lévy process. Throughout the section, we will assume Hypothesis 4.2 and also that

$$F : H \to H \text{ is Lipschitz continuous and bounded.} \tag{5.2}$$

### 5.1 Existence, uniqueness and Markov property

We say that a predictable $H$-valued stochastic process $X = (X_t^x)$, depending on $x \in H$, is a *mild solution* to equation (1.1) if, for any $t \geq 0$, $x \in H$, it holds:

$$\begin{aligned} X_t^x = e^{tA}x + \int_0^t e^{(t-s)A}F(X_s^x)ds + Z_A(t), \quad \mathbb{P}-a.s., \quad \text{where} \\ Z_A(t) = \int_0^t e^{(t-s)A}dZ_s, \end{aligned} \tag{5.3}$$

see (4.5). In formula (5.3) we are considering a predictable version of the process $(Z_A(t))$ (see Corollary 4.8).

Note that, since $F$ is bounded, the deterministic integral in (5.3) is a well defined continuous process. Moreover, as far as the regularity of trajectories is concerned, the mild solution will have the same regularity as $(Z_A(t))$. In particular, according to Theorem 4.6, any mild solution $X$ will be stochastically continuous.

To show existence and uniqueness we need the following deterministic result which is not standard in the case $p \in (0, 1)$.

**Proposition 5.1.** *Let $F : H \to H$ be Lipschitz continuous and bounded and $f \in L^p(0, T; H)$, for some $p > 0$. Let $A : D(A) \subset H \to H$ be the generator of a $C_0$-semigroup $(e^{tA})$.*

*(i) For any $x \in H$, the equation*

$$y(t) = e^{tA}x + \int_0^t e^{(t-s)A}F(y(s) + f(s))ds \tag{5.4}$$

*has a unique continuous solution $y : [0, T] \to H$.*

*(ii) There exists a constant $C > 0$ such that for solutions $y$ and $z \in C([0, T]; H)$ corresponding respectively to functions $f$, $g \in L^p(0, T; H)$ and to the same $x \in H$, we have the estimates*

$$(a) \quad \|y - z\|_{C([0,T];H)} \leq C\Big(\int_0^T |f(t) - g(t)|^p dt\Big)^{1/p}, \quad p \geq 1;$$

$$(b) \quad \|y - z\|_{C([0,T];H)} \leq C\int_0^T |f(t) - g(t)|^p dt, \quad p \in (0, 1).$$

*Proof.* Assertion (i) follows easily by a fixed point argument. Let us consider (ii). The proof of (ii) when $p \geq 1$ is an easy application of the Gronwall lemma. Thus we only prove (b).

We consider a family of equivalent norms $\|\cdot\|_\lambda$ on the Banach space $E = C([0,T];H)$, for $\lambda \geq 0$,

$$\|h\|_\lambda = \sup_{t\in[0,T]} e^{-\lambda t}|h(t)|, \quad h \in E$$

(for $\lambda = 0$ we get the usual sup norm). For a fixed $f \in L^p(0,T;H)$, let us define the operator $K_f : E \to E$,

$$(K_f y)(t) = e^{tA}x + \int_0^t e^{(t-s)A}F(y(s) + f(s))ds, \quad y \in E, \;\; t \in [0,T].$$

Note that there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|e^{tA}\|_{\mathcal{L}(H)} \leq Me^{\omega t}$, $t \geq 0$. We find for $\lambda > \omega$, for any $y, z \in E$,

$$\|K_f y - K_f z\|_\lambda \leq C \sup_{t\in[0,T]} e^{-\lambda t} \int_0^t e^{\omega(t-s)A}|y(s) - z(s)|ds$$

$$\leq C \sup_{t\in[0,T]} \int_0^t e^{-(\lambda-\omega)(t-s)}ds \, \|y - z\|_\lambda \leq \frac{C}{\lambda - \omega}\|y - z\|_\lambda.$$

Let us choose $\lambda_0$ large enough such that $c_0 = \frac{C}{\lambda_0 - \omega} < 1$. We have

$$\|K_f y - K_f z\|_{\lambda_0} \leq c_0\|y - z\|_{\lambda_0}, \quad y, z \in E. \tag{5.5}$$

Let now $f$ and $g \in L^p(0,T;H)$. We get, for any $t \in [0,T]$, $y \in E$,

$$|(K_f y)(t) - (K_g y)(t)| \leq M \int_0^t e^{\omega(t-s)}|F(y(s) + f(s)) - F(y(s) + g(s))|ds.$$

Since $F$ is bounded and Lipschitz continuous, it is also Hölder continuous of order $p$ and we find

$$\|K_f y - K_g y\|_{\lambda_0} \leq cMe^{\omega T} \int_0^T |f(s) - g(s)|^p ds.$$

If we have solutions $y$ and $z$ corresponding to $f$ and $g$, then $y = K_f y$ and $z = K_g z$. It follows

$$\|y - z\|_{\lambda_0} = \|K_f y - K_g z\|_{\lambda_0}$$

$$\leq \|K_f y - K_f z\|_{\lambda_0} + \|K_f z - K_g z\|_{\lambda_0} \leq C_T \int_0^T |f(s) - g(s)|^p ds + c_0\|y - z\|_{\lambda_0}$$

and the assertion follows.

□

**Remark 5.2.** *Clearly the previous result holds when $F$ is only Lipschitz continuous and $f \in L^p(0,T;H)$ with $p \geq 1$.*

**Theorem 5.3.** *Assume Hypothesis 4.2 and that $F : H \to H$ is Lipschitz continuous and bounded. Then there exists a unique mild solution $(X_t^x)$ to the equation (5.3). Moreover $(X_t^x)$ is a Markov process and its transition semigroup is Feller.*

*Proof. Step 1. Existence and uniqueness.* Uniqueness follows by the Gronwall lemma. Let us prove existence. By using Proposition 5.1 we find that for $x \in H$ there exists $\mathbb{P}$-a.s. a continuous functions $Y_t = Y_t^x$ with values in $H$ which solves

$$Y_t = e^{tA}x + \int_0^t e^{(t-s)A} F(Y_s + Z_A(s))ds, \quad t \geq 0.$$

Since $Z_A(t)$ is predictable it follows that $Y = (Y_t^x)$ is predictable as well. Let us define

$$X_t^x = Y_t^x + Z_A(t).$$

Clearly $(X_t^x)$ is the unique mild solution.

*Step 2. Markov property.* The proof of the Markov property is quite involved. Indeed since our solution is not assumed to have càdlàg trajectories, we have to proceed differently from [5, Theorem 7.10].

For any measurable function $\psi : [0,T] \to H$, let $y(t)$ be the unique continuous function with values in $H$ which solves the equation

$$y(t) = \int_0^t e^{(t-s)A} F(y(s) + \psi(s))ds.$$

Set $y(t) = y(t,\psi)$, $t \in [0,T]$, to indicate the dependence on $\psi$. We have, for $t, t+h \in [0,T]$,

$$y(t+h,\psi) = \int_0^{t+h} e^{(t+h-s)A} F(y(s,\psi) + \psi(s))ds$$

$$= e^{hA} \int_0^t e^{(t-s)A} F(y(s,\psi) + \psi(s))ds + \int_t^{t+h} e^{(t+h-s)A} F(y(s,\psi) + \psi(s))ds$$

$$= e^{hA}[y(t,\psi)] + \int_0^h e^{(h-s)A} F(y(t+s,\psi) + \psi(t+s))ds.$$

Define a new function on $[0, T-t]$,

$$v(\cdot,\psi) := y(t+\cdot,\psi) - e^{(\cdot)\,A}[y(t,\psi)]$$

We have

$$v(h,\psi) = \int_0^h e^{(h-s)A} F(v(s,\psi) + e^{sA}[y(t,\psi)] + \psi(t+s))ds, \quad h \in [0, T-t].$$

By uniqueness, $v(h,\psi) = y(h, e^{(\cdot)\,A}[y(t,\psi)] + \psi(t+\cdot))$ and so we get for $t, t+h \in [0,T]$,

$$y\big(h, e^{(\cdot)\,A}[y(t,\psi)] + \psi(t+\cdot)\big) + e^{h\,A}[y(t,\psi)] = y(t+h,\psi). \tag{5.6}$$

Defining $u(t,\psi) = y(t,\psi) + \psi(t)$, $t \in [0,T]$, we find

$$u(t+h,\psi) - \psi(t+h) = u\big(h, e^{(\cdot)A}[u(t,\psi) - \psi(t)] + \psi(t+\cdot)\big)$$

$$-e^{hA}[u(t,\psi) - \psi(t)] - \psi(t+h) + e^{hA}[u(t,\psi) - \psi(t)]$$

and so we get

$$u(t+h,\psi) = u(h, e^{(\cdot)A}[u(t,\psi) - \psi(t)] + \psi(t+\cdot))$$

which is the same formula of the end of [5, page 256]. Now the Markov property follows arguing as in [5, page 257].

*Step 3. Feller property.* Fix $x, y \in H$. From the estimate,

$$|X_t^x - X_t^y| \le \|e^{tA}\| |x - y| + \int_0^t \|e^{(t-s)A}\| \, |F(X_s^x) - F(X_s^y)| ds, \tag{5.7}$$

using the Lipschitz continuity of $F$ and the Gronwall lemma, we find that, for any $T > 0$, $|X_t^x - X_t^y| \le M_T |x - y|$, $t \in [0, T]$, $x, y \in H$, $\mathbb{P}$-a.s.. The Feller property follows easily.

□

## 5.2 Irreducibility

We establish now irreducibility of the solutions to (5.3). In fact we have the following result.

**Theorem 5.4.** *Assume Hypothesis 4.2 and $F : H \to H$ bounded and Lipschitz continuous. Then, for any $x \in H$, the mild solution $X = (X_t^x)$ to the equation (5.3) is irreducible.*

*Proof.* Fix $x \in H$, $T > 0$, and denote by $X = (X_t)$ the solution to (5.3) starting from $x$. Set

$$Y_t = X_t - Z_A(t), \quad t \in [0, T],$$

where

$$\begin{cases} dZ_A(t) = AZ_A(t)dt + dZ_t, \\ Z_A(0) = 0, \quad t \ge 0. \end{cases} \tag{5.8}$$

Note that

$$Y_t = e^{tA}x + \int_0^t e^{(t-s)A} F(Y_s + Z_A(s)) ds.$$

Let $z^u$ and $y^{u,x}$ be the solutions, driven by a control function $u \in L^2(0, T; H)$, of the following control systems, respectively,

$$\begin{cases} \dfrac{dz}{dt} = Az(t) + u(t), \\ z(0) = 0, \quad t \in [0, T], \end{cases} \tag{5.9}$$

$$\begin{cases} \dfrac{dy}{dt} = Ay(t) + F(y(t)) + u(t), \\ y(0) = x \in H, \quad t \in [0, T]. \end{cases} \tag{5.10}$$

Thus

$$z^u(t) = \int_0^t e^{(t-s)A} u(s) ds, \quad t \in [0, T], \tag{5.11}$$

and $y^{u,x}$ is the solution of the following integral equation

$$y(t) = e^{tA}x + \int_0^t e^{(t-s)A} F(y(s)) ds + z^u(t), \quad t \in [0, T].$$

By Theorem 7.4.2 of [6] we know that the system (5.10) is approximately controllable at time $T > 0$ in the sense that, for any $x, a \in H$ and for any $\epsilon > 0$, there exists a control function $u \in L^2(0, T; H)$ such that $|y^{u,x}(T) - a| < \epsilon$.

Let

$$\bar{y}(t) = y^{u,x}(t) - z^u(t), \quad t \in [0, T].$$

Note that

$$\bar{y}(t) = e^{tA}x + \int_0^t e^{(t-s)A} F(\bar{y}(s) + z^u(s))ds.$$

Take $p < \alpha$ with $p \in (0, 1)$. By estimate (b) in Proposition 5.1 we get, $\mathbb{P}$-a.s.,

$$\sup_{t \in [0,T]} |Y_t - \bar{y}(t)| \leq C \int_0^T |Z_A(t) - z^u(t)|^p dt.$$

and so $|Y_T - \bar{y}(T)| \leq C \int_0^T |Z_A(t) - z^u(t)|^p dt$ or, equivalently,

$$|X_T - Z_A(T) - y^{u,x}(T) + z^u(T)| \leq C \int_0^T |Z_A(t) - z^u(t)|^p dt.$$

We write, for any $a \in H$,

$$|X_T - a| \leq |X_T - Z_A(T) - y^{u,x}(T) + z^u(T)| + |Z_A(T) + y^{u,x}(T) - z^u(T) - a|$$

$$\leq C \int_0^T |Z_A(t) - z^u(t)|^p dt + |y^{u,x}(T) - a| + |Z_A(T) - z^u(T)| = I_1 + I_2 + I_3.$$

For a given $\epsilon > 0$, let us fix a control function $u$ such that $I_2 = |y^{u,x}(T) - a| < \epsilon/3$. Using Proposition 4.12, we get with positive probability that $I_1 < \epsilon/3$ and $I_3 < \epsilon/3$. The result follows. □

### 5.3 Strong Feller property

Let $(P_t)$ be the Markov semigroup associated to $X = (X_t^x)$, i.e. $P_t : B_b(H) \to B_b(H)$,

$$P_t f(x) = \mathbb{E}[f(X_t^x)], \quad x \in H, \ f \in B_b(H), \ t \geq 0. \tag{5.12}$$

To show the strong Feller property of $(P_t)$, we will assume Hypotheses 4.2, assumption (5.2) and

**Hypothesis 5.5.** *There exist $\hat{c} > 0$, $\gamma \in (0, 1)$, $T_0 > 0$, such that*

$$\sup_{n \geq 1} \frac{e^{-\gamma_n t} \gamma_n^{1/\alpha}}{\beta_n} = C_t \leq \frac{\hat{c}}{t^\gamma}, \qquad t \in (0, T_0).$$

Note that this hypothesis is stronger than Hypothesis 4.14. Our aim is to prove the following result.

**Theorem 5.6.** *Assume that Hypotheses 4.2 and 5.5 hold. Then, for any $t > 0$, the transition semigroup $(P_t)$ associated to (5.3) maps Borel and bounded functions into Lipschiz continuous functions. Moreover, there exists $C = C(\gamma, c_\alpha, \hat{c}, \|F\|_0, T_0) > 0$, such that, for any $x, y \in H$, we have*

$$|P_t f(x) - P_t f(y)| \leq C \|f\|_0 \frac{1}{\min(t^\gamma, 1)} |x - y|, \qquad t > 0. \tag{5.13}$$

To prove the result we first investigate generalised solutions to the Kolmogorov equation associated to $(P_t)$ (or to $(X_t^x)$) as in [5, Section 9.4.2].

Note that the generator $\mathcal{A}_0$ of $(P_t)$ is formally given by

$$\mathcal{A}_0 f(x) = \langle Ax + F(x), Df(x)\rangle + \sum_{k\geq 1} \beta_k^\alpha \int_{\mathbb{R}} (f(x+e_n z) - f(e_n z)) \frac{1}{|z|^{1+\alpha}} dz, \quad (5.14)$$

for regular and cylindrical functions $f$. The associated Kolmogorov equation is

$$\begin{cases} \partial_t u(t,x) = \mathcal{A}_0 u(t,\cdot)(x), \quad t>0,\ x\in H, \\ u(0,x) = f(x), \quad x \in H. \end{cases} \quad (5.15)$$

Let us consider the space

$$\Lambda(0,T) = \{u \in C(]0,T]; C_b^1(H)) \ : \ \sup_{t\in]0,T]} t^\gamma \|u(t,\cdot)\|_1 < \infty\},$$

where $\|u(t,\cdot)\|_1 = \|u(t,\cdot)\|_0 + \|D_x u(t,\cdot)\|_0$ and $\gamma \in (0,1)$ is fixed in Hypothesis 5.5. According to [5] a *mild solution* to the Kolmogorov equation (5.15) (on $[0,T]$ with initial datum $f \in B_b(H)$) is a function $u \in \Lambda(0,T)$ such that

$$u(t,x) = R_t f(x) + \int_0^t R_{t-s}\big(\langle F(\cdot), Du(s,\cdot)\rangle\big)(x)\, ds, \ \ t\in[0,T],\ x\in H, \quad (5.16)$$

where $D = D_x$ and $(R_t)$ is the transition semigroup corresponding to the linear equation (4.3). To stress the dependence on $f$, we will also write

$$u = u(t,x) = u^f(t,x), \quad t\in[0,T], \ \ x \in H.$$

Note that using Theorem 4.17 and Hypothesis 5.5, we get that for any $f \in B_b(H)$,

$$\|DR_t f\|_0 \leq \frac{C_0}{t^\gamma}\|f\|_0, \quad t \in ]0,T_0], \ \ \text{where} \ \ C_0 = 8c_\alpha\, \hat{c}. \quad (5.17)$$

Thanks to (5.17), we can adapt the proof of [5, Theorem 9.24] and obtain that the mapping $S : \Lambda(0,T) \to \Lambda(0,T)$,

$$S(u)(t,x) = R_t f(x) + \int_0^t R_{t-s}\big(\langle F(\cdot), Du(s,\cdot)\rangle\big)(x) ds, \ \ u \in \Lambda(0,T), \quad (5.18)$$

is a contraction for $T$ small enough. Therefore, we obtain

**Proposition 5.7.** *For any $f \in B_b(H)$, $T>0$, there exists a unique mild solution $u = u^f$ to (5.15). Moreover, if we define*

$$\tilde{P}_t f(\cdot) = u^f(t,\cdot), \quad t \geq 0, \ \ f \in B_b(H),$$

*then $(\tilde{P}_t)$ is a semigroup of bounded linear operators on $B_b(H)$.*

In the proof of the next lemma on smoothing properties of the semigroup $(\tilde{P}_t)$, we will use the following Gronwall lemma. Let $a, b, \gamma$ be non-negative constants, with $\gamma < 1$. Let $T > 0$. For any integrable function $v : [0,T] \to \mathbb{R}$,

$$0 \leq v(t) \leq at^{-\gamma} + b\int_0^t (t-s)^{-\gamma} v(s) ds, \ \ t \in [0,T[ \ \ a.e., \quad \text{implies} \quad v(t) \leq aMt^{-\gamma}, \quad (5.19)$$

$t \in [0,T[$, a.e.. (where $M = M(b,\gamma,T)1 + b\, k_\gamma\, T^{1-\gamma}$).

**Lemma 5.8.** *There exists $c = c(\gamma, c_\alpha, \hat{c}, \|F\|_0) > 0$ such that, for any $f \in B_b(H)$, $t \in ]0, T_0]$,*

$$\|D\tilde{P}_t f\|_0 \le \frac{c}{t^\gamma}\|f\|_0.$$

*Proof.* We have

$$Du(t,x) = DR_t f(x) + \int_0^t DR_{t-s}\big(\langle F(\cdot), Du(s,\cdot)\rangle\big)(x)\, ds, \quad x \in H.$$

Hence, by (5.17),

$$\|Du(t,\cdot)\|_0 \le \frac{C_0}{t^\gamma}\|f\|_0 + \int_0^t \frac{C_0}{(t-s)^\gamma}\|F\|_0 \|Du(s,\cdot)\|_0 ds, \quad t \in ]0, T_0].$$

The Gronwall lemma implies that

$$\|Du(t,\cdot)\|_0 \le \frac{C_0 M}{t^\gamma}\|f\|_0, \ \ t \in ]0, T_0], \quad M = M(\gamma, c_\alpha, \hat{c}, \|F\|_0) > 0.$$

□

**Galerkin's approximation.** To show the regularizing effect of $(P_t)$, it would be enough as in [5, Theorem 9.27] to show that $(P_t)$ and $(\tilde{P}_t)$ coincide. However the proof of [5, Theorem 9.27] is not complete and we are unable to fill the gap in our situation.

We therefore resort to Galerkin's approximations and will only identify suitable approximating semigroups of $(P_t)$ and $(\tilde{P}_t)$.

Let $\pi_n : H \to \{e_1, \ldots, e_n\}$ be an orthogonal projection. For any $n \in \mathbb{N}$, define the process

$$Y_t^n = \pi_n(X_t^{\pi_n x}), \quad x \in H, \ t \ge 0. \tag{5.20}$$

Note that $(Y_t^n)$ is the unique mild solution to

$$Y_t = e^{tA_n}x + \int_0^t e^{(t-s)A_n}(\pi_n \circ F \circ \pi_n)(Y_s) ds + Z_{A_n}(t), \tag{5.21}$$

$A_n = \pi_n \circ A$. Let $F_n = \pi_n \circ F \circ \pi_n$. Note that, for any $n \in \mathbb{N}$, it holds:

$$\|F_n\|_0 \le \|F\|_0, \quad Lip(F_n) \le Lip(F), \tag{5.22}$$

where $Lip(F_n)$ denotes the Lipschitz constant of $F_n$.

Consider the mild solution $u_n$ to the Kolmogorov equation corresponding to $Y_t^n$, i.e.,

$$\begin{gathered} u_n(t,x) = u_n^f(t,x) = R_t^n f(x) + \int_0^t R_{t-s}^n\big(\langle F_n(\cdot), Du_n(s,\cdot)\rangle\big)(x) ds, \quad x \in H, \\ \text{where } \ R_t^n f(x) = \mathbb{E}[f(e^{tA_n}x + \pi_n Z_A(t))] = \int_H f(e^{tA_n}x + \pi_n y)\mu_t^0(dy). \end{gathered} \tag{5.23}$$

Define two approximating semigroups on $B_b(H)$:

$$P_t^n f(x) = \mathbb{E}[f(Y_t^n)], \quad \tilde{P}_t^n f(x) = u_n^f(t,x), \ \ f \in B_b(H), \tag{5.24}$$

see (5.21) and (5.23).

**Lemma 5.9.** *For any function $f \in B_b(H)$, $n \in \mathbb{N}$, we have*

$$P_t^n f = \tilde{P}_t^n f, \quad t \geq 0.$$

*Proof.* We fix $n \in \mathbb{N}$. It is enough to prove the assertion for any function $f \in B_b(H)$, which depends only on the first $n$-coordinates. Identifying $(P_t^n)$ and $(\tilde{P}_t^n)$ with the corresponding semigroups acting on $B_b(\mathbb{R}^n)$, we have to prove that

$$P_t^n f = \tilde{P}_t^n f, \quad f \in B_b(\mathbb{R}^n), \ \ t \geq 0. \tag{5.25}$$

To this purpose, first note that it is well known that $(P_t^n)$ is a strongly continuous semigroup of positive contractions on $C_0(\mathbb{R}^n)$ (see [2, Section 6.7]). Here $C_0(\mathbb{R}^n)$ denotes the space of all real continuous functions on $\mathbb{R}^n$ vanishing at infinity.

Let us consider now $(\tilde{P}_t^n)$. We first show that $\tilde{P}_t^n(C_0(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n)$, $t \geq 0$.

Let $f \in C_0(\mathbb{R}^n)$ and $t \in ]0, T_0]$; we will use an inductive argument to prove that $\tilde{P}_t f \in C_0(\mathbb{R}^n)$. By (5.18), we know that

$$\tilde{P}_t^n f = \lim_{m \to \infty} S^m(0) = \lim_{m \to \infty} (S \circ \ldots \circ S)(0) \quad in \ \ \Lambda(0, T).$$

We prove that for any $m \in \mathbb{N}$, $S^m(0)(t, \cdot)$ and $D_x S^m(0)(t, \cdot) \in C_0(\mathbb{R}^n)$. We have (for $m = 1$) $S^1(0)(t, \cdot)(x) = R_t f(x)$, and so

$$\begin{aligned} & D_x S^1(0)(t, \cdot)(x) = D R_t^n f(x) = \int_{\mathbb{R}^n} f(e^{tA_n} x + y)\, U_n(y,t)\, \mu_t^n(dy), \ \ x \in \mathbb{R}^n, \\ & \text{where } \ \mu_t^n \ \text{ has density } \ \prod_{k=1}^n p_\alpha\Big(\frac{y_k}{c_k(t)}\Big)\frac{1}{c_k(t)} \quad \text{and} \\ & U_n(y,t) = \sum_{k=1}^n \frac{p'_\alpha(\frac{y_k}{c_k(t)})}{p_\alpha(\frac{y_k}{c_k(t)})}\, \frac{e^{-\gamma_k t}}{c_k(t)}\, e_k \ \in L^2(\mu_t^n; \mathbb{R}^n). \end{aligned} \tag{5.26}$$

It follows that $S^1(0)(t, \cdot)$ and $D_x S^1(0)(t, \cdot) \in C_0(\mathbb{R}^n)$. Assume that the assertion holds for an arbitrary $m \in \mathbb{N}$. Since

$$\begin{aligned} & S^{m+1}(0)(t, \cdot)(x) = R_t^n f(x) + \int_0^t R_{t-s}^n\big(\langle F^n(\cdot), DS^m(s, \cdot)\rangle\big) ds, \\ & D_x S^{m+1}(0)(t, \cdot)(x) = D R_t^n f(x) + \int_0^t D R_{t-s}^n\big(\langle F^n(\cdot), DS^m(s, \cdot)\rangle\big) ds \\ & = D R_t^n f(x) \\ & + \int_0^t ds \int_{\mathbb{R}^n} \langle F^n(e^{(t-s)A_n} x + y), DS^m(s, e^{(t-s)A_n} x + y)\rangle\, U_n(y, t-s)\, \mu_{t-s}^n(dy), \end{aligned}$$

$x \in \mathbb{R}^n$, we have easily that the assertion holds also for $m + 1$.

Using Lemma 5.8, we get that $(\tilde{P}_t^n)$ is a strongly continuous semigroup of positive bounded linear operators on $C_0(\mathbb{R}^n)$.

We will prove (5.25) when $f \in C_0(\mathbb{R}^n)$. Indeed, by a standard argument (see [9, Chapter 4]) this is enough to get (5.25).

By Ito formula $D_0 = C_0^2(\mathbb{R}^n) = \{f \in C_0(\mathbb{R}^n) \ : \ Df \ and \ D^2 f \in C_0(\mathbb{R}^n)\}$ is invariant for $(P_t^n)$ (compare with [2, Theorem 6.7.4]). Moreover, $D_0 \subset$dom $(\mathcal{A}_n)$, where $\mathcal{A}_n$ is the generator of $(P_t^n)$. By a well known result, $D_0$ is a core for $(P_t^n)$. Note that

$$\mathcal{A}_n f = \langle A_n x + F_n(x), Df(x)\rangle + \sum_{k=1}^n \beta_k^\alpha \int_{\mathbb{R}} (f(x+e_n z) - f(e_n z)) \frac{1}{|z|^{1+\alpha}} dz, \quad f \in D_0.$$

Let us consider $(\tilde{P}_t^n)$. If $f \in D_0$, we can solve (by the contraction principle)

$$u(t,x) = R_t^n f(x) + \int_0^t R_{t-s}^n\big(\langle F_n(\cdot), Du(s,\cdot)\rangle\big)(x) ds, \quad x \in \mathbb{R}^n,$$

in the space $C([0,T_0]; C_0^2(\mathbb{R}^n))$ and get that $D_0$ is also invariant for $(\tilde{P}_t)$.

A straightforward calculation, shows that $D_0 \subset$dom $(\tilde{\mathcal{A}}_n)$, where $\tilde{\mathcal{A}}_n$ is the generator of $(\tilde{P}_t^n)$. Thus $D_0$ is a core also for $(\tilde{P}_t^n)$ and $(\tilde{\mathcal{A}}_n)$ coincides with $(\mathcal{A}_n)$ on $D_0$. It follows that $(P_t^n)$ and $(\tilde{P}_t^n)$ coincide on $C_0(\mathbb{R}^n)$ (see Corollary III.5.15 in [8]) and this finishes the proof. $\square$

**Proof of Theorem 5.6.**

*I Step.* We have for any $f \in C_b(H)$ (see (5.23), Lemmas 5.8 and 5.9)

$$\begin{aligned} &|u_n(t,x) - u_n(t,y)| = |P_t^n f(x) - P_t^n f(y)| \\ &\le |R_t^n f(x) - R_t^n f(y)| \\ &+ \int_0^t |R_{t-s}^n\big(\langle F_n(\cdot), Du_n(s,\cdot)\rangle\big)(x) - R_{t-s}^n\big(\langle F_n(\cdot), Du_n(s,\cdot)\rangle\big)(y)| ds \\ &\le C\|f\|_0 \frac{1}{\min(t^\gamma,1)} |x-y|, \quad x,y \in H, \quad n \in \mathbb{N}. \end{aligned} \tag{5.27}$$

*II Step. For any $f : H \to \mathbb{R}$ which is Lipschitz continuous and bounded (with Lipschitz constant indicated by $Lip(f)$), we have:*

$$\lim_{n\to\infty} P_t^n f(x) = P_t f(x), \quad x \in H, \quad t \ge 0.$$

Recall that $P_t^n f(x) = \mathbb{E}[f(\pi_n(X_t^{\pi_n x}))]$ (see (5.20)).
Note that, for any compact set $K \subset H$, we have $\lim_{n\to\infty} \sup_{x\in K} |\pi_n x - x| = 0$.

Moreover (see (5.7)) we know that, for any $t \ge 0$, the mapping:

$$x \mapsto X_t^x \quad \text{is Lipschitz continuous from} \quad H \quad \text{into} \quad H.$$

For any compact set $K \subset H$, we get

$$\begin{aligned} \sup_{x\in K} |\mathbb{E}[f(\pi_n(X_t^{\pi_n x}))] - \mathbb{E}[f(X_t^x)]| &\le Lip(f)\mathbb{E}\big[\sup_{x\in K} |\pi_n(X_t^{\pi_n x})) - X_t^x|\big] \\ &\le Lip(f)\mathbb{E}\big[\sup_{x\in K}\big(|\pi_n(X_t^{\pi_n x})) - \pi_n(X_t^x)| + |\pi_n(X_t^x) - X_t^x|\big)\big] \\ &\le Lip(f)\mathbb{E}\big[\sup_{x\in K}\big(|X_t^{\pi_n x} - X_t^x| + |\pi_n(X_t^x) - X_t^x|\big)\big] \\ &\le Lip(f) c_t\, \mathbb{E}\big[\sup_{x\in K}\big(|\pi_n x - x| + |\pi_n(X_t^x) - X_t^x|\big)\big]. \end{aligned}$$

Passing to the limit, as $n \to \infty$ (using that, $\mathbb{P}$-a.s., the image of $K$ under the mapping $x \mapsto X_t^x$ is again a compact set of $H$) we get the assertion by the dominated convergence theorem.

*III Step.* Passing to the limit as $n \to \infty$ in (5.27) we get the assertion (5.13) when $f$ is in addition Lipschitz continuous and bounded. Since any $f \in UC_b(H)$ can be approximated in a uniform way on $H$ by a sequence $(f_n)$ of Lipschitz continuous and bounded functions, we have the assertion even for $f \in UC_b(H)$.

To finish we use an argument from the proof of [5, Theorem 9.28].

It holds, for any $t > 0$,

$$Var\big[p_t(x,\cdot) - (p_t(y,\cdot)\big] = \sup_{f \in B_b(H),\, \|f\|_0 \le 1} |P_t f(x) - P_t f(y)|$$

$$= \sup_{f \in UC_b(H),\, \|f\|_0 \le 1} |P_t f(x) - P_t f(y)| \le 2C \, \|f\|_0 \frac{1}{\min(t^\gamma, 1)} |x - y|,$$

where $p_t(x,\cdot)$ is the kernel of $P_t$ and $Var$ denotes the total variation. This completes the proof. □

**Acknowledgment** The authors wish to thank professor Kwapien for providing the proof of part (ii) in Theorem 4.6, which significantly improved our initial result.

# References


[1] Albeverio S., Wu J.L., Zhang T.S., Parabolic SPDEs driven by Poisson white noise, Stochastic Process. Appl. 74 (1998), no. 1, 2136

[2] Applebaum D., Lévy processes and stochastic calculus, Cambridge Studies in Advanced Mathematics, 93. Cambridge University Press, Cambridge, 2004.

[3] Bogachev V. I., Röckner M., Schmuland B., Generalized Mehler semigroups and applications, Probab. Theory Related Fields 105 (1996), 193-225.

[4] Chojnowska-Mikhalik A., On Processes of Ornstein-Uhlenbeck type in Hilbert space, Stochastics 21 (1987), 251-286.

[5] Da Prato G., Zabczyk J., Stochastic equations in infinite dimensions. Encyclopedia of Mathematics and its Applications, 44, Cambridge University Press, Cambridge, 1992.

[6] Da Prato G., Zabczyk J., Ergodicity for infinite-dimensional systems. London Mathematical Society Lecture Note Series, 229, Cambridge University Press, Cambridge, 1996.

[7] Dawson D.A., Li Z., Schmuland B., Sun W., Generalized Mehler semigroups and catalytic branching processes with immigration, Potential Anal. 21 (2004), no. 1, 75-97.

[8] Engel K., Nagel R., One-parameter Semigroups for Linear Evolution Equations, Springer Graduate Texts in Mathematics 194, 2000.

[9] Ethier S.N., Kurtz T.G., Markov Processes: Characterization and Convergence, John Wiley, 1986.

[10] Feller W., An introduction to probability theory and its applications, Vol. II. Second edition, John Wiley & Sons, Inc., New York-London-Sydney, 1971.

[11] Fuhrman M., Röckner M., Generalized Mehler semigroups: the non-Gaussian case, Potential Anal. 12 (2000), no. 1, 1-47.

[12] Kallenberg O. Foundations of modern probability. Second edition. Probability and its Applications (New York), Springer-Verlag, New York, 2002.

[13] Kwapień S., Woyczyński W.A., Random series and stochastic integrals: single and multiple. Probability and its Applications. Birkhuser Boston, Inc., Boston, MA, 1992.

[14] Parthasarathy K.R., Probability measures on metric spaces, Academic Press, New York and London, 1967.

[15] Peszat S., Zabczyk J., Stochastic Partial Differential Equations with Lévy noise, Cambridge, 2007.

[16] Priola E., Zabczyk J., Harmonic functions for generalized Mehler semigroups. Stochastic partial differential equations and applications-VII, 243-256, Lect. Notes Pure Appl. Math., 245, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[17] Priola E., Zabczyk J., Densities for Ornstein-Uhlenbeck processes with jumps, to appear in Bullettin of the London Mathematical Society.

[18] Priola E., Zabczyk J., Liouville theorems for non local opertors, J. Funct. Anal. 216 (2004), 455-490.

[19] Röckner M., Wang F.Y., Harnack and Functional Inequalities for Generalised Mehler Semigroups, J. Funct. Anal. 203 (2003), 237-261.

[20] Rosiński J., Woyczyński W.A., On It stochastic integration with respect to $p$-stable motion: inner clock, integrability of sample paths, double and multiple integrals. Ann. Probab. 14(1986), 271-286.

[21] Sato K.I., Lévy processes and infinite divisible distributions, Cambridge University Press, 1999.

[22] Triebel H., Interpolation theory, function spaces, differential operators, Second edition, Johann Ambrosius Barth, Heidelberg, 1995.

[23] Zolotarev V. M., One-dimensional stable distributions. Translated from the Russian by H. H. McFaden. Translation edited by Ben Silver. Translations of Mathematical Monographs, 65. American Mathematical Society, Providence, RI, 1986.

[24] Zinn, J. Admissible translates of stable measures, Studia Math. 54 (1976), 245-257.