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On a Random Number of Disorders

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ON A RANDOM NUMBER OF DISORDERS

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Abstract. We register a random sequence constructed based on Markov processes by switching between them. At two random moments \( \theta_1, \theta_2 \), where \( 0 \leq \theta_1 \leq \theta_2 \), the source of observations is changed. In effect the number of homogeneous segments is random. The transition probabilities of each process are known and \textit{a priori} distribution of the disorder moments is given. The various questions are formulated concerning the distribution changes in the model in the former research. The random number of distributional segments creates new problems in solutions with relation to analysis of the model with deterministic number of segments. Two cases are presented in details. In the first one the objectives is to stop on or between the disorder moments while in the second one our objective is to find the strategy which immediately detects the distribution changes. Both problems are reformulated to optimal stopping of the observed sequences. The detailed analysis of the problem is presented to show the form of optimal decision function.

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1. INTRODUCTION

Suppose that process \( X = \{X_n, n \in \mathbb{N}\}, \mathbb{N} = \{0, 1, 2, \ldots\} \), is observed sequentially. The process is obtained from three Markov processes by switching between them at two

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random moments of time, \( \theta_1 \) and \( \theta_2 \). Our objective is to detect these moments based on observation of \( X \).

Such model of data appears in many practical problem of the quality control (see [5], [2], [12]), traffic anomalies in networks [6], epidemiology [1]. In management of manufacture the plants which produce some details changes their parameters. It makes that the details change their quality. Production can be divided into three sorts. Assuming that at the beginning of production process the quality is highest, from some moment \( \theta_1 \) the products should be classified to lower sort and beginning with \( \theta_2 \) the details should be categorized as at lowest quality. The aim is to recognize the moments of these changes.

Shiryaev [13] has considered the disorder problem for independent random variables with one disorder where the mean distance between disorder time and the moment of its detection was minimized. The probability maximizing approach to the problem was used by [3] and the stopping time which is in a given neighborhood of the moment of disorder with maximal probability was found. The problem with two disorders was considered by Yoshida [17], the author [14, 15] and Sarnowski and the author [11]. In [17] the problem of optimal stopping the observation of process \( X \) so as to maximize the probability that the distance between \( \theta_i, i = 1, 2 \), and the moment of disorder will not exceed a given number (for each disorder independently). This question has been reformulated in [15] to simultaneous detection of both disorders under requirement that performance of procedure is globally measured for both detection and it has been extended to the case with unknown distribution between disorders (see [4]) in [11]. The methods of solution is based on reformulation of the question to the double optimal stopping problem (see [7], [9]) for markovian function of some statistics. In [14] the strategy which stops the process between the first and the second disorder with maximal probability has been constructed. The considerations are inspired by the problem regarding how can we protect ourselves against a second fault in a technological system after the occurrence of an initial fault or by the problem of detection the beginning and the end of an epidemic.

This paper is devoted to a generalization of the double disorder problem considered
both in [14] and [15] in which immediate switch from the first preliminary distribution to
the third one is possible with the positive probability that the random variables $\theta_1$ and $\theta_2$
are equal. It is also possible that we observe the homogeneous data without disorder when
both disorder moments are equal to 0. The extension leads to serious difficulties in the
construction of equivalent double optimal stopping models. The formulation of the problem
can be found in section 2. The main results are subject of sections 4 (see Theorem 4.1) and
5.

2. FORMULATION OF DETECTION PROBLEMS

Let $(X_n, n \in \mathbb{N})$ be an observable sequence of random variables defined on the space
$(\Omega, \mathcal{F}, P)$ with values in $(E, B)$, where $E$ is a subset of $\mathbb{R}$. On $(E, B)$ there are $\sigma$-additive
measures $\{\mu_x\}_{x \in E}$. Space $(\Omega, \mathcal{F}, P)$ supports variables $\theta_1, \theta_2$. They are $\mathcal{F}$-measurable vari-
ables with values in $\mathbb{N}$. We assume the following distributions:

\begin{align}
\mathbb{P}(&\theta_1 = j) = \mathbb{I}_{\{j=0\}}(j)\pi + \mathbb{I}_{\{j>0\}}(j)(1-\pi)p_1^{j-1}q_1, \\
\mathbb{P}(&\theta_2 = k \mid \theta_1 = j) = \mathbb{I}_{\{k=j\}}(k)\rho + \mathbb{I}_{\{k>j\}}(k)(1-\rho)p_2^{k-j-1}q_2
\end{align}

where $j = 0, 1, 2, \ldots, k = j, j+1, j+2, \ldots$. Additionally we consider Markov processes
$(X_i^n, \mathcal{G}_i^n, \mu_i^n)$ on $(\Omega, \mathcal{F}, P)$, $i = 0, 1, 2$, where $\sigma$-fields $\mathcal{G}_i^n$ are the smallest $\sigma$-fields for
which $(X^i), i = 0, 1, 2$, are adapted, respectively. Let us define process $(X_n, n \in \mathbb{N})$ in the
following way:

\begin{align}
X_n = X_0^0 \cdot \mathbb{I}_{\{\theta_1>n\}} + X_1^1 \cdot \mathbb{I}_{\{\theta_1 \leq n < \theta_2\}} + X_2^2 \cdot \mathbb{I}_{\{\theta_2 \leq n\}}.
\end{align}

We make inference based on the observable sequence $(X_n, n \in \mathbb{N})$ only. It should be em-
phasized that the sequence $(X_n, n \in \mathbb{N})$ is not markovian under admitted assumption as it
has been mentioned in [14], [16] and [6]. However, the sequence satisfies the Markov prop-
gy given $\theta_1$ and $\theta_2$ (see [15] and [8]). Thus for further consideration we define filtration
$\{\mathcal{F}_n\}_{n \in \mathbb{N}}$, where $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$, related to real observation. Variables $\theta_1, \theta_2$ are
not stopping times with respect to $\mathcal{F}_n$ and $\sigma$-fields $\mathcal{G}_n^\bullet$. Moreover, we assume that $\theta_1, \theta_2$ are
independent of \((X^i_n, n \in \mathbb{N})\). Measures \(\mu^i_x\) satisfy the relations: 
\[
\mu^i_x(dy) = f^i_x(y)\mu_x(dy),
\]
\(i = 0, 1, 2\), where the functions \(f^i_x(.)\) are different and \(\frac{f^i_x(y)}{f^{(i+1)\mod 3}_x(y)} < \infty\) for 
\(i = 0, 1, 2\) and all \(x, y \in \mathcal{E}\). We assume that the measures \(\mu^i_x, x \in \mathcal{E}\) are known in ad-
vance and we have that 
\[
P(X^i_1 \in A \mid X^i_0 = x) = \int_A f^i_x(y)\mu_x(dy) = \mu^i_x(A)
\]
for every 
\(A \in \mathcal{B}\) and \(i \in \{0, 1, 2\}\).

The model presented has the following heuristic justification: two disorders take place in the observed sequence \((X_n)\). They affect distributions by changing their parameters. Disorders occur at two random moments of time \(\theta_1\) and \(\theta_2\), \(\theta_1 \leq \theta_2\). They split the sequence of observations into segments, at most three ones. The first segment is described by \((X^0_n)\), the second one - for \(\theta_1 \leq n < \theta_2\) - by \((X^1_n)\). The third is given by \((X^2_n)\) and is observed when \(n \geq \theta_2\). When the first disorder takes the place there is a "switch" from the initial distribution to distribution with density \(f^i_x\) with respect of measure \(\mu_x\), where \(i = 1\) or \(i = 2\).

It depends on if \(\theta_1 < \theta_2\) or \(\theta_1 = \theta_2\). Next, if \(\theta_1 < \theta_2\), at the random time \(\theta_2\) the distribution of observations becomes \(\mu^2_x\). We assume that the variables \(\theta_1, \theta_2\) are unobservable.

Let \(\mathcal{S}\) denote the set of all stopping times with respect to the filtration \((\mathcal{F}_n), n = 0, 1, \ldots\) and \(\mathcal{T} = \{(\tau, \sigma) : \tau \leq \sigma, \tau, \sigma \in \mathcal{S}\}\). Two problems with three distributional segments are recalled to investigate them under weaker assumption that there are at most three homogeneous segments.

2.1. Detection of change. Our aim is to stop the observed sequence between the two disorders. This can be interpreted as a strategy for protecting against a second failure when the first has already happened. The mathematical model of this is to control the probability 
\[
P_x(\tau < \infty, \theta_1 \leq \tau < \theta_2) = \sup_{\tau \in \mathcal{T}} P_x(\tau < \infty, \theta_1 \leq \tau < \theta_2).
\]

2.2. Disorders detection. Our aim is to indicate the moments of switching with given precision \(d_1, d_2\) (Problem \(D_{d_1,d_2}\)). We want to determine a pair of stopping times \((\tau^*, \sigma^*) \in \mathcal{T}\) such that for every \(x \in \mathcal{E}\)
\[
P_x(|\tau^* - \theta_1| \leq d_1, |\sigma^* - \theta_2| \leq d_2) = \sup_{(\tau, \sigma) \in \mathcal{T}} P_x(|\tau - \theta_1| \leq d_1, |\sigma - \theta_2| \leq d_2).\]
The problem has been considered in [15] under natural simplification that there are three segments of data (i.e. there is $0 < \theta_1 < \theta_2$). In the section 5 the problem $D_{00}$ is analyzed.

3. ON SOME A POSTERIORI PROCESSES

The formulated problems will be translated to the optimal stopping problems for some Markov processes. The important part of the reformulation process is choice of the statistics describing knowledge of the decision maker. The a posteriori probabilities of some events play the crucial role. Let us define following a posteriori processes (cf. [17], [14]).

\begin{align*}
\Pi^i_n &= P_x(\theta_i \leq n|\mathcal{F}_n), \\
\Pi^{12}_n &= P_x(\theta_1 = \theta_2 > n|\mathcal{F}_n) = P_x(\theta_1 = \theta_2 > n|\mathcal{F}_{mn}), \\
\Pi_{mn} &= P_x(\theta_1 = m, \theta_2 > n|\mathcal{F}_{mn}),
\end{align*}

for $m, n = 1, 2, \ldots$, $m < n$, $i = 1, 2$. For recursive representation of (3.1)–(3.3) we need following functions:

\begin{align*}
\Pi^1(x, y, \alpha, \beta, \gamma) &= 1 - \frac{p_1(1 - \alpha)f_0^1(y)}{H(x, y, \alpha, \beta, \gamma)}, \\
\Pi^2(x, y, \alpha, \beta, \gamma) &= \frac{(q_2\alpha + p_2\beta + q_1\gamma)f_2^2(y)}{H(x, y, \alpha, \beta, \gamma)}, \\
\Pi^{12}(x, y, \alpha, \beta, \gamma) &= \frac{p_1\gamma f_0^1(y)}{H(x, y, \alpha, \beta, \gamma)}, \\
\Pi(x, y, \alpha, \beta, \gamma, \delta) &= \frac{p_2\delta f_1^1(y)}{H(x, y, \alpha, \beta, \gamma)}.
\end{align*}

where $H(x, y, \alpha, \beta, \gamma) = (1 - \alpha)p_1f_0^1(y) + [p_2(\alpha - \beta) + q_1(1 - \alpha - \gamma)]f_1^1(y) + [q_2\alpha + p_2\beta + q_1\gamma]f_2^2(y)$. In the sequel we adopt the following denotations

\begin{align*}
\vec{\alpha} &= (\alpha, \beta, \gamma) \\
\vec{\Pi}_n &= (\Pi^1_n, \Pi^2_n, \Pi^{12}_n).
\end{align*}

The basic formulae used in the transformation of the disorder problems to the stopping problems are given in the following
Lemma 3.1. For each $x \in E$ and each Borel function $u : \mathbb{R} \to \mathbb{R}$ the following formulae for $m, n = 1, 2, \ldots, m < n$ hold:

\[
\Pi_{n+1}^i = \Pi^i(X_n, X_{n+1}, \Pi_n^1, \Pi_n^2, \Pi_n^{12})
\]

(3.6)

\[
\Pi_{n+1}^2 = \Pi^2(X_n, X_{n+1}, \Pi_n^1, \Pi_n^2, \Pi_n^{12})
\]

(3.7)

\[
\Pi_{n+1}^{12} = \Pi^{12}(X_n, X_{n+1}, \Pi_n^1, \Pi_n^2, \Pi_n^{12})
\]

(3.8)

\[
\Pi_{m+1} = \Pi(X_n, X_{n+1}, \Pi_n^1, \Pi_n^2, \Pi_n^{12}, \Pi_{mn})
\]

(3.9)

with boundary condition $\Pi_0^1 = \pi$, $\Pi_0^2(x) = \pi \rho$, $\Pi_{mm} = (1 - \rho) \frac{q f_{X_{m-1}}(X_m)}{p f_{X_{m-1}}(X_m)} (1 - \Pi_m^1)$.

Proof. The case of (3.6), (3.7) and (3.9), when $0 < \theta_1 < \theta_2$, has been proved in [17] and [14]. Let us assume $0 / x \leq \theta_1 / x \leq \theta_2$ and suppose that $A_i \in F_i, i \leq n + 1$. Denote $D = \{ \omega : X_0 = x, X_i(\omega) \in A_i, 1 \leq i \leq n \}$.

Ad. (3.6) Let us consider the probability

\[
P_x(\theta_1 > n + 1 | X_i \in A_i, i \leq n + 1) = \frac{P_x(\theta_1 > n + 1, X_{n+1} \in A_{n+1} | D)}{P_x(X_{n+1} \in A_{n+1} | D)}
\]

This follows from Bayes’ formula. Let us transform the probability appearing in the numerator:

\[
P_x(\theta_1 > n + 1, X_{n+1} \in A_{n+1} | X_i \in A_i, i \leq n) = P_x(\theta_1 > n | X_i \in A_i, i \leq n) \cdot p_1 \cdot \mu_{X_n}^1(A_{n+1})
\]

Now we split the probability in the denominator into the following parts

\[
P_x(X_{n+1} \in A_{n+1} | D) = P_x(\theta_2 > \theta_1 > n, X_{n+1} \in A_{n+1} | D)
\]

(3.10)

\[
+ P_x(\theta_1 < n < \theta_2, X_{n+1} \in A_{n+1} | D)
\]

(3.11)

\[
+ P_x(n < \theta_1 = \theta_2, X_{n+1} \in A_{n+1} | D)
\]

(3.12)

\[
+ P_x(\theta_1 \leq \theta_2 < n, X_{n+1} \in A_{n+1} | D)
\]

(3.13)
In (3.10) we have:

\[ P_x(\theta_1 > n, X_{n+1} \in A_{n+1} \mid D) = P_x(\theta_1 > n, \theta_1 = n + 1, X_{n+1} \in A_{n+1} \mid D) \]

\[ + P_x(\theta_1 > n, \theta_1 \neq n + 1, X_{n+1} \in A_{n+1} \mid D) \]

\[ = P_x(\theta_1 > n \mid D)]\mu^0_{X_n}(A_{n+1})p_1 + q_1\mu^1_{X_n}(A_{n+1}] \]

In (3.11) we get:

\[ P_x(\theta_1 \leq n < \theta_2, X_{n+1} \in A_{n+1} \mid D) = P_x(\theta_1 \leq n < \theta_2, \theta_2 = n + 1, X_{n+1} \in A_{n+1} \mid D) \]

\[ + P_x(\theta_1 \leq n < \theta_2, \theta_2 \neq n + 1, X_{n+1} \in A_{n+1} \mid D) \]

\[ = (P_x(\theta_1 \leq n \mid D) - P_x(\theta_2 \leq n \mid D)) \times [q_2\mu^2_{X_n}(A_{n+1}) + p_2\mu^1_{X_n}(A_{n+1})] \]

In (3.13) the conditional probability is equal

\[ P_x(\theta_1 = \theta_2 > n, X_{n+1} \in A_{n+1} \mid D) = P_x(\theta_1 = \theta_2 > n, \theta_2 = n + 1, X_{n+1} \in A_{n+1} \mid D) \]

\[ + P_x(\theta_1 = \theta_2 > n, \theta_2 \neq n + 1, X_{n+1} \in A_{n+1} \mid D) \]

\[ = P_x(\theta_1 = \theta_2 > n \mid D)[q_1\mu^2_{X_n}(A_{n+1}) + p_1\mu^0_{X_n}(A_{n+1})] \]

In (3.12) this part has form:

\[ P_x(\theta_2 \leq n, X_{n+1} \in A_{n+1} \mid D) = P_x(\theta_2 \leq n \mid D)\mu^2_{X_n}(A_{n+1}) \]

Thus, taking into account (3.1) we have \( \Pi^1_{n+1} = 1 - P_x(\theta_1 > n + 1 \mid \mathcal{F}_{n+1}) \) and by (3.10)-(3.13) we get

\[ \Pi^1_{n+1} = 1 - [(1 - \Pi^1_n)p_1] \mathbf{H}^{-1}(X_n, X_{n+1}, \overline{\Pi}_n) \]

Using (3.1), it can be seen that (3.6) is satisfied.
Ad. (3.7) Applying similar reasoning and transformations to the process $\Pi_{n+1}^2$ we get:

$$\Pi_{n+1}^2 = P_x(\theta_2 \leq n + 1 \mid F_{n+1}) = \frac{P_x(\theta_2 \leq n + 1, X_{n+1} \in A \mid F_n)}{P_x(X_{n+1} \in A \mid F_n)}$$

which leads to formula (3.7).

**Remark 3.1.** Let us assume that the considered Markov processes have the finite state space and $\vec{x}_n = (x_0, x_1, \ldots, x_n)$ is given. In this case the formula (3.9) follows from the Bayes formula:

$$P_x(\theta_1 = j, \theta_2 = k \mid F_n) = \begin{cases} 
  p^\theta_{jk} \prod_{s=1}^n f^0_{x_{s-1}}(x_s) S_n^{-1}(\vec{x}_n) & \text{if } j > n, \\
  p^\theta_{jk} \prod_{s=1}^{j-1} f^0_{x_{s-1}}(x_s) \times \prod_{t=j}^n f^1_{x_{t-1}}(x_t) (S_n^{-1}(\vec{x}_n))^{-1} & \text{if } j \leq n < k, \\
  p^\theta_{jk} \prod_{s=1}^n f^0_{x_{s-1}}(x_s) \prod_{t=j}^{k-1} f^1_{x_{t-1}}(x_t) \times \prod_{u=k}^n f^2_{x_{u-1}}(x_u) S_n^{-1}(\vec{x}_n) & \text{if } k \leq n,
\end{cases}$$

where $p^\theta_{jk} = P(\theta_1 = j, \theta_2 = k)$, $S_0(x_0) = 1$ and for $n \geq 1$

$$S_n(\vec{x}_n) = (1 - \pi)(1 - \rho) \sum_{j=1}^{n-1} \sum_{k=j+1}^n \{p^j_1 q^k_1 \rho^j_1 \rho^k_1 \prod_{s=1}^j f^0_{x_{s-1}}(x_s) \prod_{t=j}^{k-1} f^1_{x_{t-1}}(x_t) \prod_{u=k}^n f^2_{x_{u-1}}(x_u)\} + (1 - \pi)\rho \sum_{j=1}^n \{p^j_1 q^j_1 \prod_{s=1}^j f^0_{x_{s-1}}(x_s) \prod_{t=j}^n f^1_{x_{t-1}}(x_t)\}$$

$$+ (1 - \pi)(1 - \rho) \sum_{j=1}^n \{p^j_1 q^j_1 \rho^{n-j} \prod_{s=1}^j f^0_{x_{s-1}}(x_s) \prod_{t=j}^n f^1_{x_{t-1}}(x_t)\} + (1 - \pi)p^n_1 \prod_{s=1}^n f^0_{x_{s-1}}(x_s).$$

Moreover

$$\Pi_{m,n+1}(x) = p_2 f^2_{x_n}(X_{n+1}) \Pi_{m,n}(x) S_n(\vec{x}_{n+1}) S_{n+1}^{-1}(\vec{x}_n)$$
and $S_{n+1}(\bar{x}_{n+1}) = H(X_n, X_{n+1}, \overline{\Pi}_n)S_n(\bar{x}_n)$. Hence
\[
\Pi_{mn+1}(x) = \frac{p_2 f_{X_n}^1(X_{n+1}) \Pi_{mn}(x)}{H(X_n, X_{n+1}, \overline{\Pi}_n)}.
\]

**Lemma 3.2.** For each $x \in E$ and each Borel function $u : \mathbb{R} \rightarrow \mathbb{R}$ the following equations are fulfilled

\begin{align}
\tag{3.14} E_x(u(X_{n+1})(1 - \Pi^1_{n+1}) | \mathcal{F}_n) &= (1 - \Pi^1_n - \Pi^2_n)P_1 \int_E u(y) f^0_{X_n}(y) \mu_{X_n}(dy), \\
\tag{3.15} E_x(u(X_{n+1})(\Pi^1_{n+1} - \Pi^2_{n+1}) | \mathcal{F}_n) &= [q_1(1 - \Pi^1_n - \Pi^2_n) + p_2(\Pi^1_n - \Pi^0_n)] \int_E u(x) f^1_{X_n}(y) \mu_{X_n}(dy), \\
\tag{3.16} E_x(u(X_{n+1})\Pi^2_{n+1}) | \mathcal{F}_n) &= [q_2 \Pi^1_n + p_2 \Pi^2_n + q_1 \Pi^2_n] \int_E u(y) f^2_{X_n}(y) \mu_{X_n}(dy), \\
\tag{3.17} E_x(u(X_{n+1})\Pi^2_{n+1}) | \mathcal{F}_n) &= [p_1 \Pi^2_{n+1}] \int_E u(y) f^0_{X_n}(y) \mu_{X_n}(dy), \\
\tag{3.18} E_x(u(X_{n+1}) | \mathcal{F}_n) &= \int_E u(y) H(X_n, y, \overline{\Pi}_n(x)) \mu_{X_n}(dy).
\end{align}

**Proof.** The relations (3.14)-(3.17) are consequence of suitable division of $\Omega$ defined by $(\theta_1, \theta_2)$ and properties established in Lemma 6.2. Let us prove the equation (3.16).

To do this we need to define first $\sigma$-field $\mathcal{F}_n = \sigma(\theta_1, \theta_2, X_0, ..., X_n)$. Notice that $\mathcal{F}_n \subset \mathcal{F}_n$.

We have:

\[
\begin{align*}
E_x(u(X_{n+1}) \Pi^2_{n+1}) | \mathcal{F}_n) &= E_x(u(X_{n+1})E_x(\Pi^2_{\{\theta_2 \leq n+1\}} | \mathcal{F}_{n+1}) | \mathcal{F}_n) \\
&= E_x(u(X_{n+1})I_{\{\theta_2 \leq n+1\}} | \mathcal{F}_n) = E_x(E_x(u(X_{n+1})I_{\{\theta_2 \leq n+1\}} | \mathcal{F}_n) | \mathcal{F}_n) \\
&= E_x\left(\int_E u(y) P_x(dy | \mathcal{F}_n, \theta_2 \leq n + 1) P_x(\theta_2 \leq n + 1 | \mathcal{F}_n) \right) \\
&= \int_E u(y) f^2_{X_n}(y) \mu_{X_n}(dy) P_x(\theta_2 \leq n + 1 | \mathcal{F}_n) \\
&= \int_E u(y) f^2_{X_n}(y) \mu_{X_n}(dy) (P_x(\theta_2 = n + 1, \theta_1 \leq n < \theta_2 | \mathcal{F}_n) + P_x(\theta_2 \leq n | \mathcal{F}_n)) \\
&= L^{6.2} (q_2 \Pi^1_n + p_2 \Pi^2_n + q_1 \Pi^2_n) \int_E u(y) f^2_{X_n}(y) \mu_{X_n}(dy)
\end{align*}
\]

We used the properties of conditional expectation here. Similar transformations give us equations (3.14), (3.17) and (3.15). The sum of (3.14)-(3.17) gives (3.18). This proves Lemma 3.2.
4. DETECTION OF NEW HOMOGENEOUS SEGMENT

4.1. Equivalent optimal stopping problem. For \( X_0 = x \) let us define: \( Z_n = P_x(\theta_1 \leq n < \theta_2 \mid F_n) \) for \( n = 0, 1, 2, \ldots \). We have

\[
Z_n = P_x(\theta_1 \leq n < \theta_2 \mid F_n) = \Pi_n^1 - \Pi_n^2
\]

\( Y_n = \text{esssup}_{\tau \in \mathcal{T}, \tau \geq n} P_x(\theta_1 \leq \tau < \theta_2 \mid F_n) \) for \( n = 0, 1, 2, \ldots \) and

\[
\tau_0 = \inf\{n : Z_n = Y_n\}
\]

Notice that, if \( Z_\infty = 0 \), then \( Z_\tau = P_x(\theta_1 \leq \tau < \theta_2 \mid F_\tau) \) for \( \tau \in \mathcal{T} \). Since \( F_n \subseteq F_\tau \) (when \( n \leq \tau \)) we have

\[
Y_n = \text{ess sup}_{\tau \geq n} P_x(\theta_1 \leq \tau < \theta_2 \mid F_n) = \text{ess sup}_{\tau \geq n} E_x(\mathbb{I}_{\{\theta_1 \leq \tau < \theta_2\}} \mid F_n)
\]

\[
= \text{ess sup}_{\tau \geq n} E_x(Z_\tau \mid F_n)
\]

**Lemma 4.1.** The stopping time \( \tau_0 \) defined by formula (4.2) is the solution of problem (2.4).

**Proof.** From the theorems presented in [3] it is enough to show that \( \lim_{n \to \infty} Z_n = 0 \).

For all natural numbers \( n, k \), where \( n \geq k \) for each \( x \in E \) we have:

\[
Z_n = E_x(\mathbb{I}_{\{\theta_1 \leq n < \theta_2\}} \mid F_n) \leq E_x(\sup_{j \geq k} \mathbb{I}_{\{\theta_1 \leq j < \theta_2\}} \mid F_n)
\]

From Levy’s theorem \( \limsup_{n \to \infty} Z_n \leq E_x(\sup_{j \geq k} \mathbb{I}_{\{\theta_1 \leq j < \theta_2\}} \mid F_\infty) \) where \( F_\infty = \sigma(\bigcup_{n=1}^\infty F_n) \). It is true that: \( \limsup_{j \geq k, k \to \infty} \mathbb{I}_{\{\theta_1 \leq j < \theta_2\}} = 0 \) a.s. and by the dominated convergence theorem we get

\[
\lim_{k \to \infty} E_x(\sup_{j \geq k} \mathbb{I}_{\{\theta_1 \leq j < \theta_2\}} \mid F_\infty) = 0 \ a.s.
\]

what ends the proof of the Lemma.
The reduction of the disorder problem to optimal stopping of Markov sequence is the consequence of the following lemma.

**Lemma 4.2.** System \( X^x = \{X_n^x\} \), where \( X_n^x = (X_{n-1}^x, X_n, \Pi_1^n, \Pi_2^n, \Pi_1^{12}_n) \) forms a family of random Markov functions.

**Proof.** Define a function:

\[
\varphi(x_1, x_2, \alpha; z) = (x_2, z, \Pi^1(x_2, z, \alpha), \Pi^2(x_2, z, \alpha), \Pi^{12}(x_2, z, \alpha))
\]

Observe that

\[
X_n^x = \varphi(X_{n-2}^x, X_{n-1}, \Pi_{n-1}^x; X_n) = \varphi(X_{n-1}^x; X_n)
\]

Hence \( X_n^x \) can be interpreted as function of previous state \( X_{n-1}^x \) and random variable \( X_n \). Moreover, applying (3.18), we get that conditional distribution of \( X_n \) given \( \sigma \)-field \( F_{n-1} \) depends only on \( X_{n-1}^x \). According to [13] (pp. 102-103) system \( X^x \) is a family of random Markov functions.

This fact implies that we can reduce initial problem (2.4) to the problem of optimal stopping five-dimensional process \( (X_{n-1}^x, X_n, \Pi_1^n, \Pi_2^n, \Pi_1^{12}_n) \) with reward

\[
h(x_1, x_2, \alpha) = \alpha - \beta
\]

The reward function results from equation (4.1). Thanks to Lemma 4.2 we construct the solution using standard tools of optimal stopping theory (cf [13]), as we do below.

Let us define two operators for any Borel function \( v : \mathbb{E}^2 \times [0, 1]^3 \rightarrow [0, 1] \) and the set \( D = \{\omega : X_{n-1} = y, X_n = z, \Pi_1^n = \alpha, \Pi_2^n = \beta, \Pi_1^{12}_n = \gamma\} \):

\[
T_x v(y, z, \alpha) = E_x(v(X_n, X_{n+1}, \Pi_{n+1}^x) | D)
\]

\[
Q_x v(y, z, \alpha) = \max\{v(y, z, \alpha), T_x v(y, z, \alpha)\}
\]
From well-known theorems of optimal stopping theory ([13]), we infer that the solution of the problem (2.4) is the Markov time $\tau_0$:

$$
\tau_0 = \inf\{n \geq 0 : h(X_n, X_{n+1}, \Pi_{n+1}) \geq h^*(X_n, X_{n+1}, \Pi_{n+1})\}
$$

where:

$$
h^*(y, z, \vec{\alpha}) = \lim_{k \to \infty} Q_x^k h(y, z, \vec{\alpha})
$$

Of course

$$
Q_x^v(y, z, \vec{\alpha}) = \max\{Q_x^{k-1} v, T_x Q_x^{k-1} v\} = \max\{v, T_x Q_x^{k-1} v\}
$$

To obtain a clearer formula for $\tau_0$, we formulate (cf (3.5) and (3.4)):

**Theorem 4.1.** (a) The solution of problem (2.4) is given by:

$$(4.5) \quad \tau^* = \inf\{n \geq 0 : (X_n, X_{n+1}, \Pi_{n+1}) \in B^*\}
$$

Set $B^*$ is of the form:

$$
B^* = \{(y, z, \vec{\alpha}) : (\alpha - \beta) \geq (1 - \alpha) \times \left[ p_1 \int_{\mathcal{E}} R^*(y, u, \Pi_1(y, u, \vec{\alpha})) f^0_y(u) \mu_y(du) \right. \\
+ \left. q_1 \int_{\mathcal{E}} S^*(y, u, \Pi_1(y, u, \vec{\alpha})) f^1_y(u) \mu_y(du) \right] \\
+ \left. (\alpha - \beta) p_2 \int_{\mathcal{E}} S^*(y, u, \Pi_1(y, u, \vec{\alpha})) f^1_y(u) \mu_y(du) \right\}
$$

Where:

$$
R^*(y, z, \vec{\alpha}) = \lim_{k \to \infty} R_x^k(y, z, \vec{\alpha}) , \quad S^*(y, z, \vec{\alpha}) = \lim_{k \to \infty} S_x^k(y, z, \vec{\alpha})
$$
Functions $R^k$ and $S^k$ are defined recursively: $R^1(y, z, \vec{\alpha}) = 0$, $S^1(y, z, \vec{\alpha}) = 1$ and

\begin{align}
R^{k+1}(y, z, \vec{\alpha}) &= (1 - \mathbb{1}_{R^k}(y, z, \vec{\alpha})) \\
&\quad \times \left( p_1 \int_{E} R^k(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})) f^0_y(u) \mu_y(du) \\
&\quad + q_1 \int_{E} S^k(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})) f^1_y(u) \mu_y(du) \right), \\
S^{k+1}(y, z, \vec{\alpha}) &= \mathbb{1}_{R^k}(y, z, \vec{\alpha}) + (1 - \mathbb{1}_{R^k}(y, z, \vec{\alpha})) \\
&\quad \times p_2 \int_{E} S^k(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})) f^1_y(u) \mu_y(du),
\end{align}

where the set $\mathcal{R}_k$ is:

\[
\mathcal{R}_k = \left\{ (y, z, \vec{\alpha}) : h(y, z, \vec{\alpha}) \geq T_x Q^{k-1}_x h(y, z, \vec{\alpha}) \right\} \\
= \left\{ (y, z, \vec{\alpha}) : (\alpha - \beta) \geq (1 - \alpha) \right\} \\
&\quad \times \left[ p_1 \int_{E} R^k(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})) f^0_y(u) \mu_y(du) \\
&\quad + q_1 \int_{E} S^k(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})) f^1_y(u) \mu_y(du) \right] \\
&\quad + (\alpha - \beta) p_2 \int_{E} S^k(y, u, \overrightarrow{\Pi}_1(y, u, \vec{\alpha})) f^1_y(u) \mu_y(du) \right\}
\]

(b) The value problem. The optimal value for (2.4) is given by the formula

\[
V(\tau^*) = p_1 \int_{E} R^*(x, u, \overrightarrow{\Pi}_1(x, u, \pi, \rho \pi, \rho(1 - \pi)))) f^0_x(u) \mu_x(du) \\
&+ q_1 \int_{E} S^*(x, u, \overrightarrow{\Pi}_1(x, u, \pi, \rho \pi, \rho(1 - \pi)))) f^1_x(u) \mu_x(du).
\]

**Proof.** Part (a) results from Lemma 3.2 - the problem reduces to the problem of optimal stopping of the Markov process $(X_{n-1}, X_n, \Pi^1_n, \Pi^2_n, \Pi^{12}_n)$ with payoff function $h(y, z, \vec{\alpha}) = \alpha - \beta$. Given (3.15) with the function $u$ equal to unity we get on
\( D = \{ \omega : X_{n-1} = y, X_n = z, \Pi_n^1 = \alpha, \Pi_n^2 = \beta, \Pi_n^{12} = \gamma \} \):

\[
T_x h(y, z, \alpha) = E_x (\Pi_{n+1}^1 - \Pi_{n+1}^2 \mid F_n) \bigg|_D
= \left[ (\Pi_n^1 - \Pi_n^2)p_2 \int_{\mathbb{E}} f_{X_n}^1(u)\mu_{X_n}(du) + (1 - \Pi_n^1)q_1 \int_{\mathbb{E}} f_{X_n}^0(u)\mu_{X_n}(du) \right] \bigg|_D
= (1 - \alpha)q_1 + (\alpha - \beta)p_2
\]

From the definition of \( R^1 \) and \( S^1 \) it is clear that

\[
h(y, z, \alpha) = \alpha - \beta = (1 - \alpha)R^1(y, z, \alpha) + (\alpha - \beta)S^1(y, z, \alpha)
\]

Also \( R_1 = \{(y, z, \alpha) : h(y, z, \alpha) \geq T_x h(y, z, \alpha)\} \). From the definition of \( Q_x \) and the facts above we obtain

\[
Q_x h(y, z, \alpha) = (1 - \alpha)R^2(y, z, \alpha) + (\alpha - \beta)S^2(y, z, \alpha)
\]

where \( R^2(y, z, \alpha) = q_1(1 - \mathbb{I}_{R_1}(y, z, \alpha)) \) and \( S^2(y, z, \alpha) = p_2 + (1 - p_2)\mathbb{I}_{R_1}(y, z, \alpha) \).

Suppose the following induction hypothesis holds

\[
Q_x^{k-1} h(y, z, \alpha) = (1 - \alpha)R^k(y, z, \alpha) + (\alpha - \beta)S^k(y, z, \alpha)
\]

where \( R^k \) and \( S^k \) are given by equations (4.6), (4.7), respectively. We will show

\[
Q_x^k h(y, z, \alpha) = (1 - \alpha)R^{k+1}(y, z, \alpha) + (\alpha - \beta)S^{k+1}(y, z, \alpha)
\]

From the induction assumption and equations (3.14), (3.15) we obtain:

\[
T_x b Q_x^{k-1} h(y, z, \alpha) = T_x (1 - \alpha)R^k(y, z, \alpha) + T_x (\alpha - \beta)S^k(y, z, \alpha)
= (1 - \alpha)p_1 \int_{\mathbb{E}} R^k(y, u, \Pi_1(y, u, \alpha))f_{y}^0(u)\mu_y(du)
+ [(1 - \alpha)q_1 + (\alpha - \beta)p_2] \int_{\mathbb{E}} S^k(y, u, \Pi_1(y, u, \alpha))f_{y}^1(u)\mu_y(du)
= (1 - \alpha) \left[ p_1 \int_{\mathbb{E}} R^k(y, u, \Pi_1(y, u, \alpha))f_{y}^0(u)\mu_y(du) + q_1 \int_{\mathbb{E}} S^k(y, u, \Pi_1(y, u, \alpha)) \right] \times f_{y}^1(u)\mu_y(du)
+ (\alpha - \beta)p_2 \int_{\mathbb{E}} S^k(y, u, \Pi_1(y, u, \alpha))f_{y}^1(u)\mu_y(du)
\]
Notice that
\[(1 - \alpha)R^{k+1}(y, z, \bar{\alpha}) + (\alpha - \beta)S^{k+1}(y, z, \bar{\alpha})\]
is equal \(\alpha - \beta = h(y, z, \bar{\alpha})\) for \((y, z, \bar{\alpha}) \in \mathcal{R}_k\) and, taking into account (4.8), it is equal \(T_xQ_x^h(y, z, \bar{\alpha}) = Q_x^h(y, z, \bar{\alpha})\) for \((y, z, \bar{\alpha}) \notin \mathcal{R}_k\), where \(\mathcal{R}_k\) is given by (4.8). Finally we get
\[
Q_x^h(y, z, \bar{\alpha}) = (1 - \alpha)R^{k+1}(y, z, \bar{\alpha}) + (\alpha - \beta)S^{k+1}(y, z, \bar{\alpha})
\]
This proves (4.6) and (4.7). Using the monotone convergence theorem and theorems of optimal stopping theory ([13]) we conclude that the optimal stopping time \(\tau^*\) is given by (cf 4.5).

**Proof.** Part (b). First, notice that \(\Pi_1, \Pi_2\) and \(\Pi_{12}\) are given by (3.6)-(3.8) and the boundary condition formulated in Lemma 3.1. Under the assumption \(\tau^* < \infty\) a.s. we get:

\[
P_x(\tau^* < \infty, \theta_1 \leq \tau^* < \theta_2) = \sup_{\tau} EZ_{\tau}
\]

\[
= E \max \{h(x, X_1, \bar{\Pi}_1), T_xh^*(x, X_1, \bar{\Pi}_1)\} = E \lim_{k \to \infty} Q_x^k(h(x, X_1, \bar{\Pi}_1))
\]

\[
= E \left[ (1 - \Pi_1^1)R^*(x, X_1, \bar{\Pi}_1) + (\Pi_1^1 - \Pi_2^1)S^*(x, X_1, \bar{\Pi}_1) \right]
\]

\[
= \int_{E} R^*(x, u, \bar{\Pi}_1(x, u, \pi, \rho \pi, \rho(1 - \pi))) f_x^0(u) \mu_x(du) + \int_{E} S^*(x, u, \bar{\Pi}_1(x, u, \pi, \rho \pi, \rho(1 - \pi))) f_x^1(u) \mu_x(du)
\]

We used Lemma 3.2 here and simple calculations for \(\Pi_1^1, \Pi_2^1\) and \(\Pi_{12}^1\). This ends the proof.

**4.2. Remarks.** It is notable that the solution of formulated problem depends only on two-dimensional vector of posterior processes because \(\Pi_{12}^n = \rho(1 - \Pi^n_1)\). The formulas obtained are very general and for this reason - quite complicated. We simplify the model by assuming that \(P(\theta_1 > 0) = 1\) and \(P(\theta_2 > \theta_1) = 1\). However, it seems that some further simplifications can be made in special cases. Further research should be carried out in this direction. From a practical point of view, computer algorithms are necessary to construct \(B^*\) – the set in which it is optimally to stop our observable sequence.
5. IMMEDIATE DETECTION OF THE FIRST AND THE SECOND DISORDER

5.1. Equivalent double optimal stopping problem. Let us consider the problem $D_{00}$ formulated in (2.5). A compound stopping variable is a pair $(\tau, \sigma)$ of stopping times such that $0 \leq \tau \leq \sigma$ a.e. The aim is to find a compound stopping variable $(\tau^*, \sigma^*)$ such that

\begin{equation}
\mathbb{P}_x((\theta_1, \theta_2) = (\tau^*, \sigma^*)) = \sup_{(\tau, \sigma) \in \mathcal{T}} \mathbb{P}_x((\theta_1, \theta_2) = (\tau, \sigma)).
\end{equation}

Denote $T_m = \{ (\tau, \sigma) \in \mathcal{T} : \tau \geq m \}, \mathcal{T}_{mn} = \{ (\tau, \sigma) \in \mathcal{T} : \tau = m, \sigma \geq n \}$ and $S_m = \{ \tau \in \mathcal{S} : \tau \geq m \}$. Let us denote $\mathcal{F}_{mn} = \mathcal{F}_n, m, n \in \mathbb{N}, m \leq n$. We define two-parameter stochastic sequence $\xi(x) = \{ \xi_{mn}, m, n \in \mathbb{N}, m < n, x \in \mathcal{E} \}$, where

$\xi_{mn} = \mathbb{P}_x(\theta_1 = m, \theta_2 = n \mid \mathcal{F}_{mn})$.

We can consider for every $x \in \mathcal{E}, m, n \in \mathbb{N}, m < n$, the optimal stopping problem of $\xi(x)$ on $\mathcal{T}_{mn}^+ = \{ (\tau, \sigma) \in \mathcal{T}_{mn} : \tau < \sigma \}$. A compound stopping variable $(\tau^*, \sigma^*)$ is said to be optimal in $\mathcal{T}_{mn}^+$ (or $\mathcal{T}_{mn}^+\mathcal{T}_{mn}$) if

\begin{equation}
\mathbb{E}_x \xi_{\tau^* \sigma^*} = \sup_{(\tau, \sigma) \in \mathcal{T}_{mn}^+} \mathbb{E}_x \xi_{\tau \sigma}
\end{equation}

(or $\mathbb{E}_x \xi_{\tau^* \sigma^*} = \sup_{(\tau, \sigma) \in \mathcal{T}_{mn}^+} \mathbb{E}_x \xi_{\tau \sigma}$). Let us define

\begin{equation}
\eta_{mn} = \text{ess sup}_{(\tau, \sigma) \in \mathcal{T}_{mn}^+} \mathbb{E}_x (\xi_{\tau \sigma} \mid \mathcal{F}_{mn}).
\end{equation}

If we put $\xi_{m\infty} = 0$, then

$\eta_{mn} = \text{ess sup}_{(\tau, \sigma) \in \mathcal{T}_{mn}^+} \mathbb{P}_x(\theta_1 = \tau, \theta_2 = \sigma \mid \mathcal{F}_{mn})$.

From the theory of optimal stopping for double indexed processes (cf. [7],[10]) the sequence $\eta_{mn}$ satisfies

$\eta_{mn} = \max\{ \xi_{mn}, \mathbb{E}(\eta_{mn+1} \mid \mathcal{F}_{mn}) \}$.

Moreover, if $\sigma_n^* = \inf\{ n > m : \eta_{mn} = \xi_{mn} \},$ then $(m, \sigma_n^*)$ is optimal in $\mathcal{T}_{mn}^+$ and

$\eta_{mn} = \mathbb{E}_x (\xi_{mn} \sigma_n^* \mid \mathcal{F}_{mn})$ a.e.. The case when there are no segment with distribution $f_2(y)$...
appears with probability \( \rho \). It will be taken into account. Define
\[
\hat{\eta}_{mn} = \max\{\xi_{mn}, \mathbb{E}(\eta_{m,n+1}|\mathcal{F}_{mn})\} \text{ for } n \geq m.
\]
if \( \hat{\sigma}^*_m = \inf\{n \geq m : \hat{\eta}_{mn} = \xi_{mn}\} \), then \((m, \hat{\sigma}^*_m)\) is optimal in \( T_{mn} \) and \( \hat{\eta}_{nm} = \mathbb{E}_x(\xi_m\sigma_m^*|\mathcal{F}_{mn}) \) a.e.. For further consideration denote
\[
(5.4) \quad \eta_m = \mathbb{E}_x(\eta_{mm+1}|\mathcal{F}_m).
\]

**Lemma 5.1.** Stopping time \( \sigma^*_m \) is optimal for every stopping problem \((5.3)\).

**Proof.** It suffices to prove \( \lim_{n \to \infty} \xi_{mn} = 0 \) (cf. [3]). We have for \( m,n,k \in \mathbb{N}, n > k > m \) and every \( x \in \mathbb{E} \)
\[
\mathbb{E}_x(\mathbb{I}_{\{\theta_1=m,\theta_2=n\}}|\mathcal{F}_{mn}) = \mathbb{E}_x(\sup_{j \geq k} \mathbb{I}_{\{\theta_1=m,\theta_2=j\}}|\mathcal{F}_m),
\]
where \( \mathbb{I}_A \) is the characteristic function of the set \( A \). By Levy’s theorem
\[
\lim_{n \to \infty} \mathbb{E}_x(\mathbb{I}_{\{\theta_1=m,\theta_2=n\}}) \leq \mathbb{E}_x(\sup_{j \geq k} \mathbb{I}_{\{\theta_1=m,\theta_2=j\}}),
\]
where \( \mathcal{F}_\infty = \mathcal{F}_{n\infty} = \sigma(\bigcup_{n=1}^{\infty} \mathcal{F}_n) \). We have \( \lim_{k \to \infty} \sup_j \mathbb{I}_{\{\theta_1=m,\theta_2=j\}} = 0 \) a.e. and by dominated convergence theorem
\[
\lim_{k \to \infty} \mathbb{E}_x(\sup_{j \geq k} \mathbb{I}_{\{\theta_1=m,\theta_2=j\}}) = 0.
\]

What is left is to consider the optimal stopping problem for \( (\eta_{mn})_{m=0,n=m}^{\infty,\infty} \) on \((T_{mn})_{m=0,n=m}^{\infty,\infty}\). Let us define
\[
(5.5) \quad V_m = \text{ess sup}_{\tau \in S_m} \mathbb{E}_x(\eta_\tau|\mathcal{F}_m).
\]
Then \( V_m = \max\{\eta_m, \mathbb{E}_x(V_{m+1}|\mathcal{F}_m)\} \) a.e. and we define \( \tau^*_n = \inf\{k \geq n : V_k = \eta_k\} \).

**Lemma 5.2.** The strategy \( \tau^*_0 \) is the optimal strategy of the first stop.
To show that $\tau_0^*$ is the optimal first stop strategy we prove that $P_x(\tau_0^* < \infty) = 1$. To this end, we argue in the usual manner i.e. we show $\lim_{m \to \infty} \eta_m = 0$.

We have

$$\eta_m = E_x(\xi_{m} \sigma_m^* | F_m)$$

$$\leq E_x(\sup_j I_{\theta_1 = j, \theta_2 = \sigma_j^*} | F_m).$$

Similarly as in proof of Lemma 5.1 we have got

$$\lim_{m \to \infty} \eta_m(x) \leq E_x(\sup_j \{\theta_1 = j, \theta_2 = \sigma_j^*\} | F_\infty).$$

Since

$$\lim_{k \to \infty} \lim_{j \to k} I_{\theta_1 = k, \theta_2 = \sigma_j^*} = \lim_{k \to \infty} \sup \{\theta_1 = k\} = 0,$$

it follows that

$$\lim_{m \to \infty} \eta_m(x) \leq \lim_{k \to \infty} E_x(\sup_j \{\theta_1 = j, \theta_2 = \sigma_j^*\} | F_\infty) = 0.$$

Lemmas 5.1 and 5.2 describe the method of solving the “disorder problem” formulated in Section 2 (see (5.1)).

5.2. Solution of the equivalent double stopping problem. For the sake of simplicity we shall confine ourselves to the case $d_1 = d_2 = 0$. It will be easily seen how to generalize the solution of the problem to solve $D_{d_1, d_2}$ for $d_1 > 0$ or $d_2 > 0$. First of all we construct multidimensional Markov chains such that $\xi_{mn}$ and $\eta_m$ will be the functions of their states. By consideration of the section 3 concerning a posteriori processes we get $\xi_{00} = \pi_0$ and
for \( m < n \)

\[
\xi_{m,n}^{x} = P_x(\theta_1 = m, \theta_2 = n | F_m) \\
= \frac{(1 - \pi)(1 - \rho)}{p_1^{n-1}p_2^{m-1}q_2 \prod_{x=1}^{m-1} f_{x-1}^0 (x) \prod_{t=j}^{n-1} f_{x-1}^1 (x) f_{x}^2 (X_n)} \\
= \frac{q_2}{p_2} \Pi_{m,n}(x) \frac{f_{x}^2 (X_n)}{f_{x-1}^2 (X_n)}
\]

and for \( n = m \), by Lemma 6.2

\[
(5.6) \quad \xi_{m,m} = P_x(\theta_1 = m, \theta_2 = m | F_m) = \frac{q_1}{p_1} \frac{f_{m-1}^0 (X_m)}{f_{m-1}^2 (X_m)} (1 - \Pi_m).
\]

We can observe that \((X_n, X_{n+1}, \Pi_{n+1}, \Pi_{m+1})\) for \( n = m + 1, m + 2, \ldots \) is a function of \((X_{n-1}, X_n, \Pi_{n}, \Pi_{m,n})\) and \(X_{n+1}\). Besides the conditional distribution of \(X_{n+1}\) given \(F_n\) (cf. (3.18)) depends on \(X_n, \Pi_n(x)\) and \(\Pi_m(x)\) only. These facts imply that \(\{(X_n, X_{n+1}, \Pi_{n+1}, \Pi_{m,n})\}_{n=m+1}^{\infty}\) form a homogeneous Markov process (see Chapter 2.15 of [12]). This allows us to reduce the problem (5.3) for each \(m\) to the optimal stopping problem of the Markov process \(Z_m(x) = \{(X_{n-1}, X_n, \Pi_{n}, \Pi_{m,n})\}, m, n \in \mathbb{N}, m < n, x \in \mathbb{E}\) with the reward function \(h(t, u, \alpha, \delta) = \frac{q_2}{p_2} \delta \frac{f_{2}^2(u)}{f_{1}^2(u)}\).

**Lemma 5.3.** A solution of the optimal stopping problem (5.3) for \(m = 1, 2, \ldots \) has a form

\[
(5.7) \quad \sigma^*_m = \inf \{n > m : \frac{f_{x}^2 (X_n)}{f_{x-1}^2 (X_n)} \geq R^*(X_n)\}
\]

where \(R^*(t) = p_2 \int_E r^*(t, s) f_{1}^1(s) \mu(s) ds\). The function \(r^* = \lim_{n \to \infty} r_n\), where \(r_0(t, u) = \frac{f_{2}^2(u)}{f_{1}^2(u)}\).

\[
(5.8) \quad r_{n+1}(t, u) = \max \{\frac{f_{2}^2(u)}{f_{1}^2(u)}, p_2 \int_E r_n(u, s) f_{u}^1(s) \mu_u(s) ds\}.
\]

So \(r^*(t, u)\) satisfies the equation

\[
(5.9) \quad r^*(t, u) = \max \{\frac{f_{2}^2(u)}{f_{1}^2(u)}, p_2 \int_E r^*(u, s) f_{u}^1(s) \mu_u(ds)\}.
\]
The value of the problem

\begin{equation}
\eta_m = \mathbb{E}_x(\eta_{m+1}|\mathcal{F}_m) = \frac{q_1}{p_1} \frac{f_{X_{m-1}}^1(X_m)}{f_{X_{m-1}}^0(X_m)}(1 - \Pi_m^1)R^*_\rho(X_{m-1}, X_m),
\end{equation}

where

\begin{equation}
R^*_\rho(t, u) = \max \{ \rho \frac{f_2^2(u)}{f_1^1(u)}, \frac{q_2}{p_2}(1 - \rho)R^*(u) \}.
\end{equation}

**Proof.** For any Borel function \( u : \mathbb{E} \times \mathbb{E} \times [0, 1] \rightarrow [0, 1] \) and \( D = \{ \omega : X_{n-1} = t, X_n = u, \Pi_n^1(x) = \alpha, \Pi_n^2(x) = \beta, \Pi_m(x) = \delta \} \) let us define two operators

\[
T_x u(t, u, \vec{\alpha}, \delta) = \mathbb{E}_x(u(X_n, X_{n+1}, \Pi_{n+1}^1(x), \Pi_{m+1}^1(x))|D)
\]

and

\[
Q_x u(t, u, \vec{\alpha}, \delta) = \max \{ u(t, u, \vec{\alpha}, \delta), T_x u(t, u, \vec{\alpha}, \delta) \}.
\]

On the bases of the well-known theorem from the theory of optimal stopping (see [13], [10]) we conclude that the solution of (5.3) is a Markov time

\[
\sigma^*_m = \inf \{ n > m : h(X_{n-1}, X_n, \Pi_n^1, \Pi_m) = h^*(X_{n-1}, X_n, \Pi_n^1, \Pi_m) \},
\]

where \( h^* = \lim_{k \to \infty} Q^*_x h(t, u, \vec{\alpha}, \delta) \). By (3.9) and (3.18) on \( D = \{ \omega : X_{n-1} = t, X_n = u, \Pi_n^1 = \alpha, \Pi_n^2 = \beta, \Pi_m = \delta \} \) we have

\[
T_x h(t, u, \vec{\alpha}, \delta) = \mathbb{E}_x(\frac{q_2}{p_2} \Pi_{m+1} f_{X_{n+1}}^2(X_n) f_{X_{n+1}}^0(X_n)|D)
\]

\[
= \frac{q_2}{p_2} \delta \mathbb{E}\left( \frac{f_{X_n+1}^1(X_n)}{H(u, X_{n+1}, \vec{\alpha}) f_{X_n+1}^1(X_n)} | \mathcal{F}_n \right) D
\]

\[
= q_2 \delta \int \frac{f_{X_n+1}^2(s)}{H(u, \alpha, s)} H(u, s, \vec{\alpha}) \mu_u(ds) = q_2 \delta
\]

and

\begin{equation}
Q_x h(t, u, \vec{\alpha}, \delta) = \frac{q_2}{p_2} \delta \max \{ \frac{f_2^2(u)}{f_1^1(u)}, p_2 \}.
\end{equation}
Let us define \( r_0(t, u) = 1 \) and
\[
r_{n+1}(t, u) = \max\{ \frac{f^2(u)}{f^1(u)} p_2 \int r_n(u, s) f^1(s) \mu_u(ds) \}.
\]
We show that
\[
Q^\ell_x h(t, u, \vec{\alpha}, \delta) = \frac{q_2}{p_2} \delta r^\ell(t, u)
\]
for \( \ell = 1, 2, \ldots \). We have by (5.12) \( Q_x h = \frac{q_2}{p_2} \gamma r_1 \) and assume (5.13) for \( \ell \leq k \). By (3.18) on \( D = \{ \omega : X_{n-1} = t, X_n = u, \Pi_1^n = \alpha, \Pi_2^n = \beta, \Pi_{12}^n = \gamma, \Pi_{mn} = \delta \} \) we have got
\[
T_2 Q^k_x h(t, u, \vec{\alpha}, \delta) = \mathbb{E}_x(\frac{q_2}{p_2} \Pi_{mk+1} r_k(X_n, X_{n+1})|D)
\]
\[
= \frac{q_2}{p_2} \delta p_2 \int r_k(u, s) f^1(s) \mu_u(ds).
\]
It is easy to show (see [13]) that
\[
Q^{k+1}_x = \max\{ h, T_2 Q^k_x h \}, \text{ for } k = 1, 2, \ldots.
\]
Hence we have got \( Q^{k+1}_x \) and (5.13) is proved for \( \ell = 1, 2, \ldots \). This gives
\[
h^*(t, u, \vec{\alpha}, \delta) = \frac{q_2}{p_2} \delta \lim_{k \to \infty} r_k(t, u) = \frac{q_2}{p_2} \delta r^*(t, u)
\]
and
\[
\eta_{mn} = \text{ess sup}_{(r, \sigma) \in F_{mn}} \mathbb{E}_x(\xi_{r, \sigma}|F_{mn}) = h^*(X_{n-1}, X_n, \bar{\Pi}_n, \Pi_{mn}).
\]
We have by (5.14) and (3.9)
\[
T_x h^*(t, u, \vec{\alpha}, \delta) = \frac{q_2}{p_2} \delta p_2 \int r^*(u, s) f^1(s) \mu_u(ds) = \frac{q_2}{p_2} \delta R^*(u)
\]
and \( \sigma^*_m \) has form (5.7). By (5.4), (5.6) and (3.18) we obtain
\[
\eta_m = \max\{ \xi_{mn}, \mathbb{E}(\eta_{mn+1}|F_m) \} = f(X_{m-1}, X_m, \bar{\Pi}_m, \Pi_{mn})
\]
\[
= \max\{ \frac{q_1}{p_1} \int f^2(X_m) f^1(X_{m-1}) (1 - \Pi_1^m), \frac{q_2}{p_2} (1 - \Pi_{mn}) R^*(X_m) \}
\]
\[
= \frac{q_1}{p_1} \int f^1(X_m) f^1(X_{m-1}) (1 - \Pi_1^m) \frac{R^*_m(X_{m-1}, X_m)}{1 - \Pi_{mn}}.
\]
**Remark 5.1.** Based on the results of Lemma 5.3 and properties of the a posteriori process $\Pi_{nm}$ we have that the expected value of success for the second stop when the observer stops immediately at $n = 0$ is $\pi \rho$ and when at least one observation has been made

$$E(\eta_1|F_0) = \frac{q_1}{p_1} E((1 - \Pi_1^1)\frac{f_1(x)}{f_2(x)} R_\rho(x, X_1)|F_0) = \frac{q_1}{p_1} (1 - \pi)p_1 \int_E f_x^1(u) R^*_\rho(x, u)\mu_x(du).$$

As a consequence we have optimal second moment

$$\hat{\sigma}_0^* = \begin{cases} 0 & \text{if } \pi \rho > q_1(1 - \pi) \int_E f_x^1(u) R_\rho(x, u)\mu_x(du), \\ \sigma_0^* & \text{otherwise.} \end{cases}$$

By lemmas 5.3 and 3.1 (formula (3.9)) the optimal stopping problem (5.5) has been transformed to the optimal stopping problem for the homogeneous Markov process

$$W = \{(X_{m-1}, X_m, \pi_{lm}, \Pi_{lm}^1, \Pi_{lm}), m \in \mathbb{N}, x \in E\}$$

with the reward function

$$f(t, u, \alpha) = \frac{q_1}{p_1} f_t^1(u) \left(1 - \alpha\right) R^*_\rho(t, u).$$

**Theorem 5.1.** A solution of the optimal stopping problem (5.5) for $n = 1, 2, \ldots$ has a form

$$v^*_{n+1}(t, u) = \max\left\{f_t^2(u) F_t^1(u) R^*_\rho(t, u), p_1 \int_E v^*(u, s) f_s^0(s)\mu_u(ds)\right\}.\tag{5.18}$$

So $v^*(t, u)$ satisfies the equation

$$v^*(t, u) = \max\left\{f_t^2(u) F_t^1(u) R^*_\rho(t, u), p_1 \int_E v^*(u, s) f_s^1(s)\mu_u(ds)\right\}.$$\tag{5.19}

The value of the problem $V_n = v^*(X_{n-1}, X_n)$. 
PROOF. For any Borel function \( u : \mathbb{E} \times \mathbb{E} \times [0, 1]^3 \to [0, 1] \) and \( D = \{ \omega : X_{n-1} = t, X_n = u, \Pi_n^1(x) = \alpha, \Pi_n^2(x) = \beta, \Pi_n^3 = \gamma \} \) let us define two operators

\[
T_x u(t, u, \bar{\alpha}) = \mathbf{E}_x(u(X_n, X_{n+1}, \bar{\Pi}_{n+1}) | D)
\]

and \( Q_x u(t, u, \bar{\alpha}) = \max\{u(t, u, \bar{\alpha}), T_x u(t, u, \bar{\alpha})\} \). Similarly as in the proof of Lemma 5.3 we conclude that the solution of (5.5) is a Markov time

\[
\tau_m^x = \inf\{n > m : f(X_{n-1}, X_n, \bar{\Pi}_n) = f^*(X_{n-1}, X_n, \bar{\Pi}_n)\},
\]

where \( f^* = \lim_{k \to \infty} Q_k^x f(t, u, \bar{\alpha}) \). By (3.18) and (5.16) on \( D = \{ \omega : X_{n-1} = t, X_n = u, \Pi_n^1 = \alpha, \Pi_n^2 = \beta, \Pi_n^3 = \gamma \} \) we have

\[
T_x f(t, u, \bar{\alpha}) = \mathbf{E}_x\left( \frac{q_1}{p_1} (1 - \Pi_n^1) \frac{f^1_{X_n}(X_{n+1})}{f^0_{X_n}(X_{n+1})} R^*_\rho(X_n, X_{n+1}) | D \right)
\]

\[
= \frac{q_1}{p_1} (1 - \alpha) p_1 \mathbf{E}\left( \frac{f^0_{X_n}(X_{n+1})}{H(u, X_{n+1}, \alpha, \beta)} \frac{f^1_{X_n}(X_{n+1})}{f^0_{X_n}(X_{n+1})} R^*_\rho(X_n, X_{n+1}) | \mathcal{F}_n \right) | D
\]

(3.18)

\[
= \frac{q_1}{p_1} (1 - \alpha) p_1 \int E(\frac{f^1_s(u)}{H(u, s, \alpha, \beta)} H(u, s, \alpha, \beta) R^*_\rho(u, s) \mu_u(ds))
\]

\[
= \frac{q_1}{p_1} (1 - \alpha) p_1 \int E R^*_\rho(u, s) f^1_{X_n}(s) \mu_u(ds)
\]

and

\[
(5.20) Q_x f(t, u, \bar{\alpha}) = \frac{q_1}{p_1} (1 - \alpha) \max\{\frac{f^1_{X_n}(u)}{f^0_{X_n}(u)} R^*_\rho(t, u), p_1 \int E \right R^*_\rho(u, s) f^1_{X_n}(s) \mu_u(ds)\}
\]

Let us define \( v_1(t, u) = \max\{\frac{f^1_{X_n}(u)}{f^0_{X_n}(u)} R^*_\rho(t, u), p_1 \int E \right R^*_\rho(u, s) f^1_{X_n}(s) \mu_u(ds)\} \) and

\[
v_{n+1}(t, u) = \max\{\frac{f^1_{X_n}(u)}{f^0_{X_n}(u)} R^*_\rho(t, u), p_1 \int v_n(u, s) f^0_{X_n}(s) \mu_u(ds)\}.
\]

We show that

\[
(5.21) Q^f_x f(t, u, \bar{\alpha}) = \frac{q_1}{p_1} (1 - \alpha) v_1(t, u)
\]
for \( \ell = 1, 2, \ldots \). We have by (5.20) \( Q_x f(t, u, \vec{\alpha}) = \frac{q_1}{p_1} (1 - \alpha) v_k(t, u) \) and assume (5.21) for \( \ell \leq k \). By (3.18) on \( D = \{ \omega : X_{n-1} = t, X_n = u, \Pi_n^1 = \alpha, \Pi_n^2 = \beta, \Pi_n^{12} = \gamma \} \) we have got

\[
T_x Q^k x f(t, u, \vec{\alpha}) = \mathbb{E}_x \left( \frac{q_1}{p_1} (1 - \Pi_{k+1}^1) v_k(X_n, X_{n+1}) \right) | D \\
= \frac{q_1}{p_1} (1 - \alpha) p_1 \int v_k(u, s) f^0_u(s) \mu_u(ds).
\]

Hence we have got \( Q^k x f(t, u, \vec{\alpha}) = \frac{q_1}{p_1} (1 - \alpha) v_k(t, u) \) and (5.21) is proved for \( \ell = 1, 2, \ldots \). This gives

\[ f^*(t, u, \vec{\alpha}) = \frac{q_1}{p_1} (1 - \alpha) \lim_{k \to \infty} v_k(t, u) = \frac{q_1}{p_1} \alpha v^*(t, u) \]

and

\[ V_m = \frac{q_1}{p_1} (1 - \Pi_m^1) v^*(X_{m-1}, X_m). \]

We have

\[
T_x f^*(t, u, \vec{\alpha}) = \frac{q_1}{p_1} (1 - \alpha) p_1 \int v^*(u, s) f^0_u(s) \mu_u(ds).
\]

Define \( B^* = \{ (t, u, \vec{\alpha}) : \frac{f^1(u)}{f^0(u)} R^*_n(t, u) \geq p_1 \int \mathbb{E}_x v^*(u, s) f^0_u(s) \mu_u(ds) \} \) then \( \tau_n^* \) for \( n \geq 1 \) has a form (5.17). The value of the problem (5.2), (5.5) and (2.5) is equal

\[ v_0(x) = \max \{ \pi, \mathbb{E}_x(V_1 | F_0) \} = \max \{ \pi, \frac{q_1}{p_1} (1 - \pi) p_1 \int v^*(u, s) f^0_u(s) \mu_u(ds) \} \]

and

\[
\hat{\tau}_0^* = \begin{cases} 0 & \text{if } \pi \geq q_1 (1 - \pi) \int v^*(u, s) f^0_u(s) \mu_u(ds), \\ \tau_0^* & \text{otherwise}. \end{cases}
\]

Based on Lemmas 5.3 and 5.1 the solution of the problem \( D_{00} \) can be formulated as follows.
Theorem 5.2. A compound stopping time $(\tau^*, \sigma_m^*)$, where $\sigma_m^*$ is given by (5.7) and $\tau^* = \hat{\tau}_0^*$ is given by (5.17) is a solution of the problem $D_{00}$. The value of the problem

$$P_x(\tau^* < \sigma^* < \infty, \theta_1 = \tau^*, \theta_2 = \sigma^*_m) = \max\{\pi, q_1(1 - \pi) \int_E v^*(u, s)f_u^0(s)\mu_u(ds)\}.$$ 

Remark 5.2. The problem can be extended to optimal detection of more than two successive disorders. The distribution of $\theta_1, \theta_2$ may be more general. The general a priori distributions of disorder moments leads to more complicated formulae, since the corresponding Markov chains are not homogeneous.

6. APPENDICES

APPENDIX 1 — USEFUL RELATIONS

6.1. Conditional probability of various event defined by disorder moments. According to definition of $\Pi_n^1, \Pi_n^2, \Pi_n^{12}$ we get

Lemma 6.1. For the model described in the section 2 the following formulae are valid.

1. $P_x(\theta_2 \geq n > \theta_1|F_n) = \Pi_n^1 - \Pi_n^2$;
2. $P_x(\theta_2 > \theta_1 > n|F_n) = 1 - \Pi_n^1 - \Pi_n^{12}$.

Proof.

1. Let $\theta_1 \leq \theta_2$. Since $\{\omega : \theta_2 \leq n\} \subset \{\omega : \theta_1 \leq n\}$ it follows that $P_x(\{\omega : \theta_1 \leq n < \theta_n\}|F_n) = P_x(\{\omega : \theta_1 \leq n\} \setminus \{\omega : \theta_2 \leq n\}|F_n) = \Pi_n^1 - \Pi_n^2$.

2. We have

$$\Omega = \{\omega : n < \theta_1 < \theta_2\} \cup \{\omega : \theta_1 \leq n < \theta_2\}$$

$$\cup \{\omega : \theta_1 \leq \theta_2 \leq n\} \cup \{\omega : \theta_1 = \theta_2 > n\}$$

hence $1 = P_x(\omega : n < \theta_1 < \theta_2|F_n) + (\Pi_n^1 - \Pi_n^2) + \Pi_n^2 + \Pi_n^{12}$ and

$$P_x(\omega : n < \theta_1 < \theta_2|F_n) = 1 - \Pi_n^1 - \Pi_n^{12}.$$
6.2. Some recursive formulae. In derivation of the formulae in Theorem 3.1 the form of the distribution of some random vectors is taken into account.

**Lemma 6.2.** For the model described in the section 2 the following formulae are valid.

1. \( P_x(\theta_2 = \theta_1 > n + 1 | \mathcal{F}_n) = p_1 \Pi_{n}^{12} = p_1 (1 - \Pi_n^1); \)
2. \( P_x(\theta_2 > \theta_1 > n + 1 | \mathcal{F}_n) = p_1 (1 - \Pi_n^1 - \Pi_n^{12}); \)
3. \( P_x(\theta_1 < n + 1 | \mathcal{F}_n) = P_x(\theta_1 < n + 1 < \theta_2 | \mathcal{F}_n) + P_x(\theta_2 < n + 1 | \mathcal{F}_n); \)
4. \( P_x(\theta_1 < n + 1 < \theta_2 | \mathcal{F}_n) = q_1 (1 - \Pi_n^1 - \Pi_n^{12}) + p_2 (\Pi_n^1 - \Pi_n^2); \)
5. \( P_x(\theta_2 < n + 1 | \mathcal{F}_n) = q_2 \Pi_n^1 + p_2 \Pi_n^2 + q_1 \Pi_n^{12}. \)

**Proof.**

1. On the set \( D = \{ \omega : X_0 = x, X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n \} \) we have

\[
\begin{align*}
P_x(\theta_2 = \theta_1 > n + 1 | D) &= \frac{\rho (1 - \pi) \sum_{j=n+2}^{\infty} p_1^{j-1} q_1 \int_{\times_{i=1}^{n+1} A_i} \Pi_{i=1}^{n+1} f_{x_{i-1}}^0 (x_i) dx_1 \ldots dx_n}{P(D)} \\
&= p_1 \frac{\rho (1 - \pi) p_1^n \int_{\times_{i=1}^{n+1} A_i} \Pi_{i=1}^{n} f_{x_{i-1}}^0 (x_i) dx_1 \ldots dx_n}{P(D)} = p_1 \Pi_{n}^{12},
\end{align*}
\]

\[
\begin{align*}
P_x(\theta_1 > n | D) &= \frac{(1 - \pi) \sum_{j=n+1}^{\infty} p_1^{j-1} q_1 \int_{\times_{i=1}^{n} A_i} \Pi_{i=1}^{n} f_{x_{i-1}}^0 (x_i) dx_1 \ldots dx_n}{P(D)} \\
&= \frac{(1 - \pi) p_1^n \int_{\times_{i=1}^{n} A_i} \Pi_{i=1}^{n} f_{x_{i-1}}^0 (x_i) dx_1 \ldots dx_n}{P(D)} = \frac{1}{\rho} \Pi_n^{12}.
\end{align*}
\]

2. Similarly as above we get

\[
\begin{align*}
P_x(\theta_2 > \theta_1 > n + 1 | D) &= \frac{\rho (1 - \pi) p_1^n p_2 \int_{\times_{i=1}^{n} A_i} \Pi_{i=1}^{n} f_{x_{i-1}}^0 (x_i) dx_1 \ldots dx_n}{P(D)} \\
&= p_1 P_x(\theta_2 > \theta_1 > n + 1 | D) \overset{L.6.1}{=} p_1 (1 - \Pi_n^1 - \Pi_n^{12}).
\end{align*}
\]

3. It is obvious by assumption \( \theta_1 \leq \theta_2. \)
4. On the set $D$ we have

\[
P_x(\theta_1 \leq n + 1 < \theta_2 | F_n) = \frac{\sum_{j=0}^{n+1} P(\omega : \theta_1 = j) \sum_{k=0}^{\infty} (1 - \rho)p_2^{k-j}q_2}{P(D)}
\times \int \prod_{s=1}^{n} f_{x_{s-1}}(x_{s}) \prod_{r=j}^{n-1} f_{x_{r-1}}(x_{r}) dx_{1} \ldots dx_{n}
\left(1 - \pi\right)p_1^n q_1 (1 - \rho)p_2 + p_2 \sum_{j=0}^{n} P(\omega : \theta_1 = j)p_2^{n+1-j}
= \frac{\sum_{j=0}^{n+1} P(\omega : \theta_1 = j) \sum_{k=0}^{\infty} (1 - \rho)p_2^{k-j}q_2}{P(D)}
\times \int \prod_{s=1}^{n} f_{x_{s-1}}(x_{s}) \prod_{r=j}^{n-1} f_{x_{r-1}}(x_{r}) dx_{1} \ldots dx_{n}
\left(1 - \pi\right)p_1^n q_1 (1 - \Pi_{n}^{12}) + p_2 (\Pi_{n}^{1} - \Pi_{n}^{2}).
\]

(L.6.1)

5. If we substitute $n$ by $n + 1$ in (6.1) than we obtain

\[
P_x(\theta_2 \leq n + 1 | F_n) = 1 - P_x(n + 1 < \theta_1 = \theta_2 | F_n)
\]

\[-P_x(n + 1 < \theta_1 < \theta_2 | F_n) - P_x(\theta_1 \leq n + 1 < \theta_2 | F_n)
= 1 - p_1 \Pi_{n}^{12} - p_1 (1 - \Pi_{n}^{1} - \Pi_{n}^{12}) - q_1 (1 - \Pi_{n}^{1} - \Pi_{n}^{12})
+ p_2 (\Pi_{n}^{2} - \Pi_{n}^{1}) = q_2 \Pi_{n}^{1} + p_2 \Pi_{n}^{2} + q_1 \Pi_{n}^{12}.
\]

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