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On the Russell Problem

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Abstract. We give a partial answer to the Russell Conjecture about characterization of the affine space. We also characterize testing sets for properness and non-properness sets of polynomial mappings of \( k \)-uniruled varieties, where \( k \) is an algebraically closed field.

1. Introduction.

Let \( k \) be an uncountable algebraically closed field. Let \( K_n = \{ x \in k^n : x_1 \cdot \ldots \cdot x_n = 0 \} \) (i.e., \( K_n \) is the union of coordinate hyperplanes in \( k^n \)). Peter Russell stated the following:

Conjecture. Let \( k = \mathbb{C} \). Let \( X \) be an affine, smooth variety of dimension \( n \), which is contractible. Then \( X \) is isomorphic to \( k^n \) if and only if there is a closed embedding of \( K_n \) into \( X \).

In the paper [7] we have showed that the Russell Conjecture is true if \( X \) is additionally dominated by \( \mathbb{C}^n \). The Russell Conjecture suggests a certain characterization of the affine space \( X \) over any field. Here we generalize our result from [7] and we prove:

Theorem 1.1. Let \( X \) be a \( k \)-uniruled smooth affine variety of dimension \( n \). Assume that \( \text{Pic}(X) = 0 \) and \( H^0(X, O^*) = k \). If there is a closed embedding \( \iota : K_n \rightarrow X \), then \( X \cong k^n \).

Corollary 1.2. The Russell Conjecture holds for every \( \mathbb{C} \)-uniruled contractible (smooth) affine variety.

Let us recall that an affine variety \( X \) is \( k \)-uniruled if for a sufficiently general point \( x \) in \( X \) there is an affine parametric curve \( \phi_x : k \rightarrow X \) such that \( \phi_x(0) = x \).

In the paper we also study generically-finite polynomial mappings of affine \( k \)-uniruled varieties. We generalize some results from [7] and moreover we prove some of this result in more general setting. In particular we give a wide description of hypersurfaces which are testing sets in the case \( X \) is a \( k \)-uniruled affine variety. In particular we prove:

Theorem 1.3. Let \( X \) be a affine \( k \)-uniruled variety. Let \( S_1, \ldots, S_m \) be hypersurfaces in \( k^m \), which have no common points at infinity. Then \( S = \bigcup_{i=1}^m S_i \) is a testing set for polynomial mappings \( X \rightarrow k^m \).

For example the set \( S = \bigcup_{i=1}^m \{ x \in k^n : x_i = 0 \} \) is a testing set for polynomial mappings \( f : X \rightarrow k^m \) and we have the following statement:

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Corollary 1.4. Let $X$ be a affine $k$-uniruled variety. Let $f = (f_1, \ldots, f_m) : X \to k^m$ be a generically-finite polynomial mapping. If restrictions of $f$ to hypersurfaces $V(f_i) = \{x \in X : f_i(x) = 0\}$, $i = 1, \ldots, m$ are finite, then the mapping $f$ is finite, too.

We also continue the study the set of non-properness of a generically-finite polynomial mapping $f = (f_1, \ldots, f_n) : X \to Y$, where $X$ is an affine $k$–uniruled variety and $Y$ is an affine variety. Let us recall that $f$ is not proper at a point $y$ if there is no Zariski open neighborhood $U$ of $y$ such that $f^{-1}(\text{cl}(U))$ is proper. We prove:

Theorem 1.5. For a generically-finite dominant polynomial mapping $f : X \to Y$, where $X$ is a $k$–uniruled affine variety and $Y$ is an affine variety, the set $S_f$ is either empty or it is a $k$–uniruled hypersurface (in $Y$).

2. Preliminaries.

We assume that $k$ is an algebraically closed field. For simplicity we also assume that $k$ is uncountable. In this paper by a locally principal divisor on a variety $X$ we mean a Cartier divisor, which is locally given by polynomial equations. If $D$ is given by a system $\{U_\alpha, f_\alpha\}_{\alpha \in A}$, (where $f_\alpha \in k[U_\alpha]$), then by its support we mean a hypersurface $\text{supp}(D) := \bigcup_{\alpha \in A} \{x \in U_\alpha : f_\alpha(x) = 0\}$.

Definition 2.1. Let $X \subset k^n$ be a curve. We say that $X$ is an affine parametric curve if there is a surjective polynomial mapping $\phi : k \ni t \to \phi(t) \in X$.

In analogous way we say that a projective curve $X$ is parametric, if there is a surjective polynomial mapping $\phi : \mathbb{P}^1(k) \ni t \to \phi(t) \in X$.

Now let us recall some basic facts about $k$-uniruled varieties (see [7] and [10]).

Proposition 2.2. Let $k$ be an uncountable field. Let $X$ be an irreducible affine variety of dimension $\geq 1$. The following conditions are equivalent:

1) for every point $x \in X$ there is a polynomial affine curve in $X$ going through $x$;

2) there exists a Zariski-open, non-empty subset $U$ of $X$, such that for every point $x \in U$ there is a polynomial affine curve in $X$ going through $x$;

3) there exists an affine variety $W$ with $\dim W = \dim X - 1$ and a dominant polynomial mapping $\phi : W \times k \to X$.

We have the following definition of a $k$-uniruled variety.

Definition 2.3. An affine irreducible variety $X$ is called $k$-uniruled if it is of dimension $\geq 1$, and satisfies one of equivalent conditions 1) – 3) listed in Proposition 2.2.

Example 2.4. Let $H \subset k^n$ be an irreducible hypersurface of degree $d < n$. Then $H$ is a $k$-uniruled variety. In fact $H$ can be covered by lines.

Let us recall the following:

Definition 2.5. An irreducible algebraic variety $X$ we will call semi-affine if there exists a proper generically-finite polynomial mapping $X \to X'$, where $X'$ is an affine variety.

We say that a semi-affine variety $X$ is $k$-uniruled if there is a dominant generically finite morphism $f : H \times k \to X$, where $H$ is an affine variety. Of course we can assume that $H$ is smooth.
We also have to recall some facts about sets of non-properness of polynomial mappings (see [7] and [8]).

**Definition 2.6.** Let \( f : X \rightarrow Y \) be a polynomial map. We say that \( f \) is proper at a point \( y \in Y \) if there exists an open neighborhood \( U \) of \( y \) such that \( \text{res}_{f^{-1}(U)} f : f^{-1}(U) \rightarrow U \) is a finite map.

We have the following important theorem (for a proof see [7]):

**Theorem 2.7.** Let \( f : X \rightarrow Y \) be a dominant polynomial map of irreducible varieties of the same dimension. Assume that \( X \) is semi-affine and \( Y \) affine. Then the set \( S_f \) of points at which \( f \) is not proper is either empty or it is a hypersurface.

**Remark 2.8.** The proof given in [7] is over \( k = \mathbb{C} \), however essentially it works for arbitrary field, some obvious modification we leave to the reader.

3. The case of surfaces.

Our next aim is to give a characterization of the testing sets, as well as the characterization of the set of non-proper points for a dominant map \( f : X \rightarrow k^m \), where \( X \) is a affine \( k \)-uniruled surface. In fact we will do it in a more general setting.

**Definition 3.1.** Let \( X, Y \) be algebraic varieties and \( f : X \rightarrow Y \) be a polynomial dominant map. By a compactification of \( f \) we mean a variety \( X \) and a map \( f : X \rightarrow Y \), such that

1) \( f \) is proper,
2) \( X \subset \overline{X} \),
3) \( \text{res}_X f = f \).

We have the following easy proposition:

**Proposition 3.2.** Let \( X, Y \) be algebraic varieties and \( f : X \rightarrow Y \) be a polynomial dominant map. Then \( f \) has a compactification. Moreover, if \( X \) is normal we can choose \( \overline{X} \) to be normal, too. If \( X \) is semi-affine, then \( \overline{X} \) is also semi-affine.

**Proof.** It is enough to take \( \overline{X} := \text{closure of } \text{graph}(f) \subset X' \times Y \), (where \( X' \) is a completion of \( X \)), and to take as \( \overline{f} \) the canonical projection. If \( X \) is normal we can additionally take the normalization of \( \overline{X} \). \( \square \)

**Remark 3.3.** Assume that \( X \) is a smooth surface. Since we can resolve singularities of a surface (see [1]), we can always assume that \( \overline{X} \) is smooth, too.

Let a map \( \overline{f} \) be a compactification of some dominant map \( f : X \rightarrow Y \), where \( X \) is a semi-affine variety and \( Y \) is an affine variety. By the lemma below the subvariety \( D := \overline{X} \setminus X \) is a hypersurface. Let \( D_1 \cup \ldots \cup D_r \) be a decomposition of \( D \) into irreducible components. We call a component \( D_i \) horizontal if \( \dim \overline{f(D_i)} = \dim D_i \), otherwise we call it vertical.

**Lemma 3.4.** Let \( V \) be an algebraic variety which contains a semi-affine variety \( X \) as an open dense subset. Then the subvariety \( D := V \setminus X \) is a hypersurface. Moreover, if \( V \) is complete of dimension \( n \geq 2 \), then \( D \) is connected.

Let \( X \) be a smooth projective surface and let \( D = \sum_{i=1}^n D_i \) be a simple normal crossing (s.n.c) divisor on \( X \) (here we consider only reduced divisors). Let \( \text{graph}(D) \) be a graph of \( D \), i.e., a graph with one vertex \( Q_i \) for each irreducible component \( D_i \) of \( D \), and one edge between \( Q_i \) and \( Q_j \) for each point of intersection of \( D_i \) and \( D_j \).
Definition 3.5. Let $D$ be a simple normal crossing divisor on a smooth surface $X$. We say that $D$ is a tree if $\text{graph}(D)$ is connected and acyclic.

We have the following fact which is obvious from graph theory:

Proposition 3.6. Let $X$ be a smooth projective surface and let divisor $D \subset X$ be a tree. Assume that $D', D'' \subset D$ are connected divisors without common components. Then $D'$ and $D''$ have at most one common point.

Now we can prove:

Theorem 3.7. Let $X, Y$ be algebraic surfaces, $X$ is normal, semi-affine and $Y$ is affine. Assume, that $X$ contains a smooth cylinder $H = \Gamma \times k$ as an open, dense subset. Let $f : X \to Y$ be a dominant polynomial map and $\overline{f} : \overline{X} \to Y$ be a compactification of $f$. Let $Q := X \setminus X$. Then

1) every horizontal component of $Q$ is an affine parametric curve,
2) every vertical component of $Q$ is a projective parametric curve,
3) if $H_1, H_2$ are horizontal components, then $H_1 \cap H_2 = \emptyset$,
4) every connected vertical curve meets at most one horizontal component.

Proof. We can assume that $X$ and $\overline{X}$ are smooth. Let $\widetilde{X}$ be a smooth completion of $\overline{X}$. We can assume that the mapping $\overline{f} : \overline{X} \to Y$ has an extension to a morphism $\overline{f}' : \widetilde{X} \to Y$, where $\overline{Y}$ is a projective closure of $Y$. In particular $\overline{f}'^{-1}(\overline{Y} \setminus Y) = \widetilde{X} \setminus \overline{X}$.

The inclusion $\iota : \Gamma \times k \to X$ induces the birational mapping $\phi : \Gamma \times \mathbb{P}^1(k) \to \widetilde{X}$, (here $\Gamma$ is a smooth completion of $\Gamma$). Note that the divisor $D = \Gamma \times \infty + \sum_{i=1}^l \{a_i\} \times \mathbb{P}^1$ is a tree. Now we have the following picture:

Here mappings $f_1$ and $f_2$ are compositions of blowing-up’s. Note that the divisor $D' = f_1^*(D)$ is a tree. Let $\Gamma \times \infty'$ denote a proper transform of $\Gamma \times \infty$. It is an easy observation that $f_2(\Gamma \times \infty') \subset \widetilde{X} \setminus \overline{X}$. The curve $L = \widetilde{X} \setminus \overline{X}$ is a complement of a semi-affine variety hence it is connected (for details see [7], Lemma 4.5). So also the curve $L' = f_2^{-1}(L) \subset D'$ is connected. Now by Proposition 3.6 we have that every irreducible curve $Z \subset D'$ which does not belong to $L'$ has at most one common point with $L'$. Let $S \subset Q$ be a horizontal component. There is a curve $Z \subset D'$, which has exactly one common point with $L'$ such that $S = f_2(Z \setminus L)$. Moreover $Z$ is different from $\Gamma \times \infty'$, hence $Z \setminus L = k$. Now let
Let $H_1, H_2$ be horizontal components. Take $Z_1 = f_2^{-1}(H_1), Z_2 = f_2^{-1}(H_2)$. The curves $Z_1, Z_2$ are connected and they have common points with $L'$. Since $D'$ is the tree, we have $(Z_1 \setminus L') \cap (Z_2 \setminus L') = \emptyset$. Consequently $H_1 \cap H_2 = \emptyset$. In a similar way we can prove 4). This completes the proof. \hfill \Box

**Theorem 3.8.** Let $X, Y$ be algebraic surfaces, where $X$ is semi-affine and $Y$ is affine. Let $f : X \to Y$ be a polynomial dominant map. Let us assume that $X$ contains a smooth cylinder $H = \Gamma \times k$, as an open, dense subset. The set $S_f$ of points at which $f$ is not proper consists of a finite number (possibly 0) of affine parametric curves.

**Proof.** Taking a normalization we can assume that $X$ is normal. Let $\overline{f} : \overline{X} \to Y$ be a normal compactification of $f$. By Theorem 2.7 the set $S_f$ is a curve. Moreover, it is easy to see that $S_f = \overline{f}(\overline{X} \setminus X)$. Thus in fact we have $S_f = \overline{f}(R)$, where $R$ is a union of horizontal components of $\overline{X} \setminus X$. Now the conclusion holds by Theorem 3.7. \hfill \Box

**Corollary 3.9.** Let $X, Y$ be affine algebraic surfaces and let $X$ be $k$–uniruled. Let $f : X \to Y$ be a polynomial dominant map. Then the set $S_f$ of points at which $f$ is not proper consists of a finite number (possibly 0) of affine parametric curves.

**Proof.** Since $X$ is $k$–uniruled, we have a dominant mapping $\phi : \Gamma \times k \to X$. We can assume that the curve $\Gamma$ is smooth. Let $\phi : Z \to X$ be a compactification of $\phi$ and take $g = f \circ \phi$. Then $S_f = S_g$. Now the conclusion holds by Theorem 3.8. \hfill \Box

We state now the following basic definition:

**Definition 3.10.** Let $X, Y$ be algebraic varieties, where $X$ is semi-affine and $Y$ is affine. Let $S$ be a closed subset of $Y$. We will call $S$ a testing set for properness of polynomial mappings $f : X \to Y$ (briefly a testing set) if for every generically-finite polynomial mapping $f : X \to Y$, if $\text{res}_{f^{-1}(S)}f : f^{-1}(S) \ni x \to f(x) \in S$ is proper then $f$ is proper, too.

The following fact will be frequently used

**Lemma 3.11.** Let $X, Y$ be algebraic varieties, where $X$ is semi-affine and $Y$ is affine. Let $f : X \to Y$ be a generically finite dominant mapping. Assume that $T = \bigcup_{j=1}^{m} T_j$ is a connected hypersurface in $Y$, with irreducible components $T_j$, which is a support of a locally principal divisor. Moreover, assume that $\text{res}_{f^{-1}(T)}f : f^{-1}(T) \to T$ is a proper mapping. If for every $j = 1, \ldots, m$ the mapping $f$ is proper at some point $y_j \in T_j$, then it is proper at every point $y \in T$.

**Proof.** We can assume that $X$ is normal. Let $\overline{f} : \overline{X} \to Y$ be a normal compactification of $f$ and denote $D := \overline{X} \setminus X$. By the Stein Factorization Theorem (see e.g. [4]) there exist a variety $W$, and regular surjective mappings $p : \overline{X} \to W$, $q : W \to Y$, such that $f = q \circ p$ and $p$ has only connected fibers (in particular being generically finite) it is a birational mapping and $q$ is finite.

Now assume on the contrary, that the mapping $f$ is not proper at a point $y \in T_j \subset T$. We will show that this assumption leads to a contradiction.

First of all, since $\text{res}_{f^{-1}(T)}f : f^{-1}(T) \to T$ is a proper mapping we have $\text{cl}(f^{-1}(T)) \cap D = \emptyset$, i.e., the set $f^{-1}(T)$ is closed in $\overline{X}$. Moreover, since the mapping $f$ is proper at points $y_j \in T_j$ there is no horizontal components over $T_j$, $j = 1, \ldots, m$. 

There are two cases possible:

a) the set $\overline{f}^{-1}(y)$ is finite,
b) the set $\overline{f}^{-1}(y)$ is infinite.

ad a) We have that there is a point $b \in \overline{f}^{-1}(y) \cap D$. Let $T$ be a support of a divisor $T'$. Consider the locally-principal divisor $Z := \overline{f}^{-1}(T') \cap (X \setminus f^{-1}(T))$. It has the support in $D$ and it has only horizontal components which go through $b$. One of them lies over some $T_j$, which is a contradiction.

ad b) We will show that this case also is impossible. Indeed let $b \in q^{-1}(y)$ be a point in $W$ such that $p^{-1}(b)$ is infinite. Let $R$ be an irreducible component of the hypersurface $q^{-1}(T)$ which contains the point $b$. The variety $p^{-1}(R)$ is connected and contains the connected set $p^{-1}(b)$. Moreover, it is contained in $\overline{f}^{-1}(T)$. Since $f^{-1}(T)$ is disjoint from $D$ and since $p^{-1}(b)$ must be in $D$, we have that $p^{-1}(R)$ is also in $D$. But $p^{-1}(R)$ contains a horizontal component which lies over $R$ and consequently over some $T_j$. This is a contradiction. □

**Corollary 3.12.** Let $X, Y$ be algebraic varieties, where $X$ is semi-affine and $Y$ is affine. Let $f : X \to Y$ be a generically finite dominant mapping. Assume that $T$ is a connected hypersurface in $Y$, such that every irreducible component of $T$ is a support of a locally principal divisor. Moreover, assume that $\text{res}_{f^{-1}(T)}f : f^{-1}(T) \to T$ is a proper mapping. If the mapping $f$ is proper at some point $y \in T$, then it is proper at every point $y \in T$.

**Theorem 3.13.** Let $X, Y$ be algebraic surfaces, where $X$ is semi-affine and $Y$ is affine. Let $X$ contain a smooth cylinder $H = \Gamma \times k$ as an open, dense subset. Assume that $T = \bigcup_{j=1}^{r} T_j$ is a curve in $Y$ such that

1) every $T_j$ is a support of some locally principal divisor,
2) if $S \subset T$ is an irreducible component of some $T_j$ which is an affine parametric curve, then for some $T_k$ we have $S \subset T_k$ and $S \cap T_k \neq \emptyset$,
3) for every affine parametric curve $\Gamma \subset Y$ we have $\Gamma \cap T \neq \emptyset$.

Then $T$ is a testing set for properness of polynomial mappings $f : X \to Y$.

**Proof.** Let $f : X \to Y$ be a generically-finite polynomial mapping and $\text{res}_{f^{-1}(T)}f : f^{-1}(T) \ni x \to f(x) \in T$ be a proper mapping. We have to show that $f$ is proper, too.

Taking the normalization we can assume that $X$ is normal. Let $\overline{f} : \overline{X} \to Y$ be a normal compactification of $f$ and denote $D := \overline{X} \setminus X$. By the Stein Factorization Theorem there exist a normal surface $W$, and regular surjective mappings $p : \overline{X} \to W$, $q : W \to Y$, such that $f = q \circ p$ and $p$ has only connected fibers (in particular, being generically finite it is a birational mapping) and $q$ is finite. We have:

**Lemma 3.14.** Let $X, Y, f$ be as above. Assume that $S, T \subset Y$ are curves, $S$ is irreducible and $T$ is the support of a locally principal divisor. Moreover, assume that the mapping $\text{res}_{f^{-1}(S \cup T)}f : f^{-1}(S \cup T) \ni x \to f(x) \in S \cup T$ is proper. If $S \cap T$ has an isolated point, then the mapping $f$ is proper at some point $y \in S$.

**Proof.** Let us assume the contrary, i.e., that $S \subset S_f$. Hence there is a horizontal curve $S' \subset \overline{X} \setminus X$ such that $\overline{f}(S') = S$. Let $a$ be an isolated point of the intersection $S \cap T$ and $b \in S'$ be a point such that $\overline{f}(b) = a$.

There are two cases possible:
i) the point \( b \) is an isolated component of the set \( \mathcal{F}^{-1}(a) \),

ii) the point \( b \) is not an isolated component of the set \( \mathcal{F}^{-1}(a) \),

ad i) Let us note that by our assumptions the set \( f^{-1}(S \cup T) \) is closed in \( \mathbb{X} \). Let \( T' \) be a divisor with support \( |T'|=T \) and consider a locally-principal divisor \( T'' := \mathcal{F}(T') \cap (\mathbb{X} \setminus f^{-1}(S \cup T)) \). It has support in \( D \) and cuts \( S' \) in \( b \). Let us denote a component of \( T'' \) which contain the point \( b \) by \( R \). By i) the component \( R \) is horizontal. Since \( a \) is an isolated component of the intersection \( S \cap T \), we have that \( R \neq S' \), which contradicts Theorem 4.6.

ad ii) We will show that this case is impossible. Indeed, let \( c \in q^{-1}(a) \) be a point in \( W \) such that \( p^{-1}(c) \) is infinite and \( b \in p^{-1}(c) \). Let \( R \) be an irreducible component of divisor \( \mathcal{F}(T') \) which contains the point \( c \). The curve \( p^{-1}(R) \) is connected and contains the curve \( p^{-1}(c) \). Moreover, it is contained in \( \mathcal{F}^{-1}(T) \). Since \( f^{-1}(T) \) is disjoint from \( D \) and since \( p^{-1}(c) \) must be in \( D \), we have that \( p^{-1}(R) \) is also in \( D \). But the curve \( p^{-1}(R) \) contains a horizontal component \( H \) which lies over \( R \). Moreover, since \( a \) is an isolated component of the intersection \( S \cap T \), we have that \( H \neq S' \). It means that the connected vertical curve \( p^{-1}(c) \) meets two different horizontal components and this is a contradiction. \( \Box \)

We now return to the proof of Theorem 3.13. By Lemma 3.11, Lemma 3.14 and Theorem 3.8 we easily see that for every \( y \in T \), the mapping \( f \) is proper at \( y \). Finally if \( S_f \) denotes the set of points at which the mapping \( f \) is not proper, we have that \( S_f \cap T = \emptyset \). By Theorem 3.8 and 3), this implies that \( S_f = \emptyset \), i.e., the mapping \( f \) is proper. \( \Box \)

The theorem above can be slightly generalized:

\textbf{Corollary 3.15.} Let \( X, Y \) be algebraic surfaces, where \( X \) is semi-affine and \( k-\)uniruled and \( Y \) is affine. Assume that \( T = \bigcup_{j=1}^m T_j \) is a curve in \( Y \) such that

1) every \( T_j \) is a support of some locally principal divisor,

2) if \( S \subset T \) is an irreducible component of some \( T_j \) which is an affine parametric curve, then for some \( T_k \) we have \( S \not\subset T_k \) and \( S \cap T_k \neq \emptyset \),

3) for every affine parametric curve \( \Gamma \subset Y \) we have \( \Gamma \cap T \neq \emptyset \).

Then \( T \) is a testing set for properness of polynomial mappings \( f : X \to Y \).

\textbf{Proof.} Let \( H = \Gamma \times k \) be a smooth cylinder. Let \( g : H \to X \) be a dominant mapping. Take a compactification \( \overline{g} : \overline{H} \to X \). Since \( \overline{g} \) is a proper generically-finite mapping and \( X \) is semi-affine, we have that \( \overline{H} \) is a semi-affine surface. A mapping \( f : X \to Y \) is proper if and only if the mapping \( F := f \circ \overline{g} \) is proper. Since the mapping \( \overline{g} \) is proper, we have that \( \text{res}_{f^{-1}(T)} f : f^{-1}(T) \ni x \to f(x) \in T \) is a proper mapping if and only if \( \text{res}_{F^{-1}(T)} F : F^{-1}(T) \ni x \to F(x) \in T \) is a proper mapping. Now the proof reduces to the proof of Theorem 3.13. \( \Box \)

\textbf{Corollary 3.16.} Let \( T_1, \ldots, T_m \) be hypersurfaces in \( k^m \) which have no common points at infinity. Let \( X \) be a semi-affine and \( k- \)uniruled surface. Then \( T = \bigcup_{i=1}^m T_i \) is a testing set for polynomial mappings \( f : X \to k^m \).

\textbf{Proof.} First take \( T_i = \{ x : x_i = 0 \} \) and \( T = \bigcup_{i=1}^m T_i \). We have to show that if \( f : X \to k^m \) is a generically-finite polynomial mapping and \( \text{res}_{f^{-1}(T)} f : f^{-1}(T) \ni x \to f(x) \in T \) is a proper mapping then \( f \) is a proper mapping, too. Let us take \( Y = \text{cl}(f(X)) \) and take \( T_i = T_i \cap Y, T' = \bigcup_{i=1}^m T_i' \). We can assume that \( Y \not\subset T_i \) for \( i = 1, \ldots, m \). Hence all \( T_i' \) are
locally principal divisors on \( Y \). We will show that \( T' \) satisfies all assumptions of Corollary 3.15. It satisfies 1),3), because \( T = \bigcup_{i=1}^{m} T_i \) satisfies 1),3) in \( k^m \).

It also satisfies 2). Indeed, let \( \Gamma \subset T' \) be an affine parametric curve, we have to show that \( \Gamma \) meets another component of \( T' \). We can assume that \( \Gamma \subset T_1, \ldots , T_n \) but \( \Gamma \not\subset T_{s+1}, \ldots , T_m \) (where \( s < m \), because the intersection of all \( T_i \) is one point). If we put \( Z = \bigcap_{i=1}^{n} T_i \) and \( Z_i = T_i \cap Z \) for \( i > s \), then \( Z_i \) are coordinate hyperplanes in \( Z \cong k^{m-s} \).

It means that for at least one index ”\( j > s \)" we have \( \Gamma \cap Z_j \neq \emptyset \). Hence \( \Gamma \) has common points with the curve \( T_j \cap Y = T'_j \). By construction all components of \( T'_j \) are different from \( \Gamma \).

The general case can be easily deduced from the particular one. Indeed, let \( T_i = V(g_i) \) for some reduced polynomial \( g_i \in k[y_1, \ldots , y_m] \), \( i = 1, \ldots , m \). By the assumption, the mapping \( G := (g_1, \ldots , g_m) : k^m \to k^m \) is finite. Now it is enough to consider the mapping \( f' = G \circ f \) and to use the first part of our proof. \( \square \)

In particular the set \( S = \bigcup_{i=1}^{m} \{ x : x_i = 0 \} \) is a testing set for polynomial mappings \( f : X \to k^m \) and we have the following statement.

**Corollary 3.17.** Let \( X \) be a semi-affine and \( k \)-uniruled surface. Let \( f = (f_1, \ldots , f_m) : X \to k^m \) be a generically-finite polynomial mapping. If the restrictions of \( f \) to curves \( V(f_i) \), \( i = 1, \ldots , m \) are proper, then the mapping \( f \) is also proper.

4. Geometric characterization of \( S_f \).

Now we pass to the general situation.

**Theorem 4.1.** Let \( f : X \to Y \) be a dominant polynomial map of \( n \)-dimensional varieties, where \( X \) is semi-affine, \( k \)-uniruled and \( Y \) is affine. Then the set \( S_f \) of points at which \( f \) is not proper is either empty or it is a \( k \)-uniruled hypersurface.

**Proof.** As usual we can assume that \( X \) contains a smooth affine cylinder \( H = \Gamma \times k \) as an open, dense subset. Let \( \overline{f} : \overline{X} \to Y \) be a compactification of the mapping \( f \).

Let \( y_0 \in S_f \). There is a curve \( \Lambda \subset X \) such that the mapping \( f|_{\Lambda} \) is not proper at \( y_0 \). Moreover we can assume that \( \Lambda \cap H \neq \emptyset \). Consequently we can assume that \( \Lambda \subset H \). Let \( \pi : H \ni (\gamma, t) \to \gamma \in \Gamma \) and \( \Lambda' = \pi(\Lambda) \). We can assume that \( \Lambda' \) is a curve. Hence the curve \( \Lambda \) is contained in a cylindrical surface \( S = \Lambda' \times k \subset H \). Let \( S' \) be a closure of \( S \) in \( X \). Put \( f' = f|_{S'} \). Then \( S_f' \subset S_f \). Since \( y_0 \in S_{f'} \) by a construction and the set \( S_{f'} \) is a union of parametric curves the proof is complete. \( \square \)

We can apply our result to find out something about geometrical properties of the set \( Y \setminus f(X) \). The following corollary is an easy consequence of Theorem 4.1:

**Corollary 4.2.** Let \( f : X \to Y \) be a dominant polynomial map of \( n \)-dimensional varieties, where \( X \) is semi-affine, \( k \)-uniruled and \( Y \) is affine. Every \( n-1 \)-dimensional component \( C \) of the set \( \text{cl}(Y \setminus f(X)) \) is a \( k \)-uniruled hypersurface. In particular, for every point \( x \in C \) there is an affine parametric curve in \( C \) through \( x \).

5. Testing sets.

Our aim in this section is to generalize Theorem 3.13 to higher dimensions. First we will prove the following variant of Lemma 3.14:
Lemma 5.1. Let $X$ be a semi-affine surface and let $Y$ be an affine surface. Assume, that $X$ contains a smooth cylinder $H = \Gamma \times k$ as an open, dense subset. Let $f : X \to Y$ be a generically-finite polynomial mapping. Assume, that $T_i$, $i = 1, \ldots, m$ are locally principal divisors in $Y$ and the mapping $\text{res}_{f^{-1}(T)} : f^{-1}(T) \ni x \to f(x) \in T_i$, where $T = \bigcup_{i=1}^{m} | T_j |$, is proper. Then $f$ is proper at every isolated point of the intersection $\bigcap_{i=1}^{m} | T_j |$.

Proof. As usual, we can assume that $X$ is normal. Let $\overline{f} : \overline{X} \to Y$ be a normal compactification of $f$ and denote $D := \overline{X} \setminus X$. By the Stein Factorization Theorem there exist a normal surface $W$, and regular surjective mappings $p : \overline{X} \to W$, $q : W \to Y$, such that $f = q \circ p$ and $p$ has only connected fibers (in particular being generically finite it is a birational mapping) and $q$ is finite.

Let $a$ be an isolated component of $\bigcap_{j=1}^{m} | T_j |$. There are two cases possible:

i) the set $\overline{f}^{-1}(a)$ is finite,

ii) the set $\overline{f}^{-1}(a)$ is infinite.

ad i) It is enough to show that $\overline{f}^{-1}(a) \cap D = \emptyset$. Assume on the contrary, that there is a point $b \in \overline{f}^{-1}(a) \cap D$. Let us note that by our assumptions the set $f^{-1}(| T |)$ is closed in $\overline{X}$. We can consider locally-principal divisors $D_i := \overline{f}^{-1}(T_i) \cap (\overline{X} \setminus f^{-1}(T))$, $i = 1, \ldots, m$. They have supports in $D$ and meets in $b$. Let us denote a component of $| D_i |$ which contains the point $b$, by $R_i$, $i = 1, \ldots, m$. By i) the components $R_i$, $i = 1, \ldots, m$ are horizontal. Since $a$ is an isolated component of the intersection $\bigcap_{j=1}^{m} | T_j |$, we see that $R_i \neq R_j$, for some $i \neq j$, which contradicts Theorem 3.7..

ad ii) We will show that this case is impossible. Indeed let $b \in q^{-1}(a)$ be a point in $W$ such that $p^{-1}(b)$ is infinite. Let $R_i$, $i = 1, \ldots, m$ be irreducible components of divisors $q^*(T_i)$ which contain the point $b$. The curves $p^{-1}(R_i)$, $i = 1, \ldots, m$ are connected and contain the curve $p^{-1}(b)$. Moreover, they are contained in $\overline{f}^{-1}(T)$. Since $f^{-1}(T)$ is disjoint from $D$ and since $p^{-1}(b)$ must be in $D$, we have that $p^{-1}(R_i)$, $i = 1, \ldots, m$ are also in $D$. But the curves $p^{-1}(R_i)$ contain horizontal components $H_i$ which are over $R_i$. Moreover, since $a$ is an isolated component of the intersection $\bigcap_{j=1}^{m} | T_j |$, we see that $H_i \neq H_j$, for some $i \neq j$. This means that a connected vertical curve $p^{-1}(b)$ meets two different horizontal components, which is a contradiction. \[\square\]

Now we are in a position to prove the following:

Theorem 5.2. Let $X, Y$ be irreducible $n$–dimensional varieties, where $X$ is semi-affine and $k$-uniruled and $Y$ is affine. Let $T$ be a hypersurface on $Y$ such that

1) every irreducible component of $T$ is a support of some locally principal divisor,

2) if $T' \subset T$ is a connected component of $T$ which is $k$-uniruled then $T'$ contains irreducible components $T_1', \ldots, T_r'$ such that the intersection $\bigcap_{i=1}^{r} T_i'$ has a point as an isolated component,

3) for every affine $k$-uniruled hypersurface $\Gamma \subset Y$ we have $\Gamma \cap T \neq \emptyset$.

Then $T$ is a testing set for polynomial mappings $f : X \to Y$. Moreover, if every irreducible component of $T$ is not $C$-uniruled, then we can change the assumption 1) to the weaker assumption that $T$ is a support of a locally principal divisor.

Proof. As usual we can assume that $X$ is normal and $X$ contains a smooth affine cylinder $H = \Gamma \times k$ as an open, dense subset.
Let \( f : X \to Y \) be a generically-finite polynomial mapping and \( \text{res}_{f^{-1}(T)} f : f^{-1}(T) \ni x \to f(x) \in T \) be a proper mapping. We have to show that \( f \) is proper, too. Let \( \overline{f} : \overline{X} \to Y \) be a compactification of \( f \) and denote \( D := \overline{X} \setminus X \).

We have:

**Lemma 5.3.** Let \( f, X, Y, T' \) be as above. Then \( f \) is proper at every isolated point of the intersection \( \bigcap_{i=1}^{r} T'_i \).

*Proof.* Let \( a \) be an isolated component of \( \bigcap_{i=1}^{r} T'_i \). Let us assume that the mapping \( f \) is not proper at the point \( a \) and take a point \( c \in f^{-1}(a) \cap D \). There is an irreducible curve \( \Lambda \subset \overline{X} \) which contains the point \( c \) and \( \Lambda' := \Lambda \cap H \neq \emptyset \). Moreover, we can assume that \( \Lambda' \) contains a point \( b \) which is smooth with respect to \( f \). As in previous proofs we can assume that \( \Lambda' \) is contained in a cylindrical surface \( S = G \times k \subset H \). Let \( S' \) be a closure of \( S \) in \( X \). Put \( f' = f|_{S'} \). Since \( b \in \Lambda' \) and \( f \) is smooth at the point \( b \) we have that the mapping \( f' \) is generically-finite. Denote \( Y' := \text{cl}(f'(S')) \). The variety \( Y' \) is an affine surface. By the choice of the point \( c \) and the curve \( \Lambda \) the mapping \( f' \) is not proper at the point \( a \in Y' \).

Let \( T_i \) be locally principal divisors with support \( T'_i \), \( i = 1, \ldots, r \). We can consider divisors \( R_i := v^i(T_i) \), where \( v : Y' \to Y \) is an inclusion. We have \( a \in \bigcap | R_i | \) and the point \( a \) is an isolated point of this intersection. Moreover, the mapping \( f' \) is proper on the preimage of the set \( \bigcup | R_i | \). By Lemma 5.1 it follows that the mapping \( f' \) is proper at the point \( a \), which is a contradiction. Hence our assumption that the mapping \( f \) is not proper at the point \( a \) is false. \( \square \)

We now return to the proof of Theorem 5.2. By Lemma 5.3 and Theorem 4.1 we can easily see that for every \( y \in T \) the mapping \( f \) is proper at \( y \). Finally, if \( S_f \) denotes the set of points at which the mapping \( f \) is not proper we see that \( S_f \cap T = \emptyset \). By Theorem 4.1 and 3) it follows that \( S_f = \emptyset \), i.e., the mapping \( f \) is proper. \( \square \)

**Corollary 5.4.** Let \( X \) be a semi-affine and \( k \)-uniruled \( n \)-dimensional variety. Assume that \( T \) is a hypersurface in \( k^n \) such that

1) if \( T' \subset T \) is a connected component of \( T \) which is \( k \)-uniruled then \( T' \) contains irreducible components \( T'_1, \ldots, T'_r \) such that the intersection \( \bigcap_{i=1}^{r} T'_i \) has a point as an isolated component,

2) for every affine \( k \)-uniruled hypersurface \( \Gamma \subset k^n \) we have \( \Gamma \cap T \neq \emptyset \).

Then \( T \) is a testing set for polynomial mappings \( f : X \to k^n \).

A simple application of Theorem 5.2 is that if \( T_1, \ldots, T_n \) are hypersurfaces in \( k^n \) without common points at infinity, then the set \( T = \bigcup_{i=1}^{n} T_i \) is a testing set for polynomial mappings \( f : X \to k^n \) (where \( X \) is a semi-affine, \( k \)-uniruled variety). In fact we can easily generalize this as follows:

**Proposition 5.5.** Let \( T_1, \ldots, T_m \) be hypersurfaces in \( k^m \) which have no common points at infinity. Let \( X \) be a semi-affine and \( k \)-uniruled \( n \)-dimensional variety. Then the set \( T = \bigcup_{i=1}^{m} T_i \) is a testing set for polynomial mappings \( f : X \to k^m \).

*Proof.* Let \( f : X \to k^n \) be a dominant mapping which is proper over \( T \). Assume that \( f \) is not proper. We can assume that \( X \) contains an affine cylinder \( H = \Gamma \times k \) as an open dense subset. As in previous proofs we can construct a cylindrical surface \( S = G \times k \subset H \), such that the mapping \( f \) is not proper on \( S' = \text{cl}(S) \subset X \). This contradicts Corollary 3.16. \( \square \)
In particular the set \( S = \bigcup_{i=1}^{m} \{ x : x_i = 0 \} \) is a testing set for polynomial mappings \( f : X \to k^m \) and so we have:

**Corollary 5.6.** Let \( X \) be a semi-affine and \( k \)-uniruled \( n \)-dimensional variety. Let \( f = (f_1, \ldots, f_m) : X \to k^n \) be a generically-finite polynomial mapping. If the mappings \( \text{res}_{V(f_i)} f \), \( i = 1, \ldots, m \) are proper, then the mapping \( f \) is proper, too.

### 6. The Russell Problem.

Now we pass to the application of Theorem 5.2. Let \( K_n := \{ x \in k^n : x_1 \cdot \ldots \cdot x_n = 0 \} \) (i.e., \( K_n \) is the union of coordinate hyperplanes in \( k^n \)). Peter Russell stated the following:

**Conjecture.** Let \( k = \mathbb{C} \). Let \( X \) be an affine, smooth variety of dimension \( n \), which is contractible. Then \( X \) is isomorphic to \( k^n \) if and only if there is a closed embedding of \( K_n \) into \( X \).

In the paper [7] we have showed that the Russell Conjecture is true if \( X \) is additionally dominated by \( \mathbb{C}^n \). The Russell Conjecture suggests a certain characterization of the affine space \( X \) over any field. Here we generalize our result from [7] and we prove:

**Theorem 6.1.** Let \( X \) be a \( k \)-uniruled smooth affine variety of dimension \( n \). Assume that \( \text{Pic}(X) = 0 \) and \( H^0(X, O^*) = k \). If there is a closed embedding \( \iota : K_n \to X \), then \( X \cong k^n \).

More precisely, every closed embedding \( \psi : K_n \to X \) can be extended to an isomorphism \( \Psi : k^n \to X \).

**Proof.** Let \( \psi : K_n \to X \) be a closed embedding, and let \( \Gamma_i := \psi(\{ x : x_i = 0 \}) \). Moreover, denote \( K'_n := \psi(K_n) \) and denote the point \( \psi(0) \) by \( a \).

Take \( \pi_i := \{ x \in k^n : x_i = 0 \} \). Since \( \text{Pic}(X) = 0 \) there are irreducible polynomials \( h_j, j = 1, \ldots, n \) such that \( \Gamma_j = \{ x \in X : h_j(x) = 0 \}, j = 1, \ldots, n \). We see the following:

**Lemma 6.2.** The restriction of the mapping \( H = (h_1, \ldots, h_n) : X \to k^n \) to the set \( K'_n \) is an isomorphism. Moreover, \( H^{-1}(K_n) = K'_n \).

**Proof.** Let \( \Gamma_{ij} = \Gamma_i \cap \Gamma_j \). Let \( f_2 = 0, \ldots, f_n = 0 \), be irreducible equations of the sets \( \Gamma_{i2}, \ldots, \Gamma_{in} \) in the coordinate ring \( k[\Gamma_i] \).

Consider \( \tilde{h}_2 \). We have \( \{ x \in \Gamma_1 : \tilde{h}_2(x) = 0 \} = \Gamma_{12} \), hence from the Hilbert Nullstellensatz there exist an integer \( r \geq 1 \) and \( e_2 \in H^0(X, O^*) \) such that \( \tilde{h}_2 = e_2(f_2)^r \). By the assumption \( e_2 \in k \). Since polynomials \( h_1, \ldots, h_n \) give a local system of coordinates at the point \( a \), we must have \( r = 1 \) and \( \tilde{h}_2 = c_2 f_2, c_2 \neq 0 \). In a similar way \( h_j = c_j f_j, c_j \neq 0, j > 2 \).

By the symmetry we see that the polynomials \( \tilde{h}_j = \text{res}_{\Gamma_i} h_j, j \neq i \), are generators of the ideals \( I(\Gamma_{ij}) \) in the ring \( k[\Gamma_i] \).

Now, let \( \lambda = \psi^{-1} : K'_n \to K_n \) and let us consider a mapping \( \varepsilon^1 := \text{res}_{\Gamma_1} \lambda : \Gamma_1 \to \pi_1 \). We know that this mapping is polynomial, and moreover \( \varepsilon^1 = (0, e_2, \ldots, e_n) \). We see that \( \{ x \in \Gamma_1 : e_i(x) = 0 \} = \Gamma_{1i} \). Since \( e_i \) is an isomorphism, the polynomials \( e_i, i = 2, \ldots, n \), are irreducible in the ring \( k[\Gamma_1] \). Since \( \{ x \in \Gamma_1 : \varepsilon_i h_i(x) = 0 \} = \Gamma_{1i} \), there exist non-zero constants \( \kappa_{1i} \) such that \( \varepsilon_i = \kappa_{1i} \tilde{h}_i, i = 2, \ldots, n \). Hence \( \varepsilon^1 \) has coordinates \((0, \kappa_{12} \tilde{h}_2, \ldots, \kappa_{1n} \tilde{h}_n)\). In a similar way the mapping \( \varepsilon^k := \text{res}_{\Gamma_k} \lambda : \Gamma_k \to \pi_k \) has coordinates \((\kappa_{k1} \tilde{h}_1, \ldots, \kappa_{kk-1} \tilde{h}_{k-1}, 0, \kappa_{kk+1} \tilde{h}_{k+1}, \ldots, \kappa_{kn} \tilde{h}_n)\). To end the proof of our lemma it is enough to show that for every \( k, l \neq j \) we have \( \kappa_{kj} = \kappa_{lj} \). Indeed, in this case the
mapping $\lambda$ is the restriction to $K'_n$ of the mapping $\Lambda = (\kappa_1 h_1, \ldots, \kappa_n h_n)$, hence also the mapping $H = (h_1, \ldots, h_n)$ in the restriction to $K'_n$ is an embedding.

Since $\Gamma_k \cap \Gamma_l \not\subset \Gamma_j$, there exists a point $c \in (\Gamma_k \cap \Gamma_l) \setminus \Gamma_j$. Thus $\lambda(c) \notin \pi_j$ (i.e., $h_j(c) \neq 0$) and $\lambda(c) = \varepsilon^k(c) = (\ldots, \kappa_{kj} h_j(c), \ldots) = (\ldots, \kappa_{lj} h_j(c), \ldots) = \varepsilon^l(c)$, hence $\kappa_{kj} h_j(c) = \kappa_{lj} h_j(c)$ and $\kappa_{kj} = \kappa_{lj}$. Moreover, by the construction of $H$ we have $H^{-1}(K_n) = K'_n$. \hfill \Box

We now complete the proof of Theorem 6.1. By the lemma above the mapping $H$ in the restriction to the set $H^{-1}(K_n)$ is proper, hence by Corollary 5.6 the mapping $H$ is proper. Since $X$ is affine it means that the mapping $H$ is finite. Since $(d_0 \psi)^{-1}$ is an isomorphism, we also have that the mapping $d_a H : T_aX \to T_0 k^n$ is an isomorphism. In particular the mapping $H$ is separable and it is non-ramified at the point $a$. But $H^{-1}(0) = a$ and consequently $\deg H = 1$ (see e.g. [2]). This means that the mapping $H$ is birational. Finally, it is isomorphism by the Zariski Main Theorem. Now, if we take $\Psi := (\kappa_1 h_1, \ldots, \kappa_n h_n)^{-1} : k^n \to X$, then $\Psi$ is an isomorphism and $\text{res}_{K_n} \Psi = \psi$. \hfill \Box

References


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