INSTITUTE OF MATHEMATICS
of the
Polish Academy of Sciences

ul. Śniadeckich 8, P.O.B. 21, 00-956 Warszawa 10, Poland http://www.impan.pl

IM PAN Preprint 716 (2010)

Zbigniew Jelonek

On the Russell Problem

Presented by Zbigniew Jelonek

Published as manuscript

Received 17 March 2010

ON THE RUSSELL PROBLEM

ZBIGNIEW JELONEK

ABSTRACT. We give a partial answer to the Russell Conjecture about characterization of the affine space. We also characterize testing sets for properness and non-properness sets of polynomial mappings of $k-$uniruled varieties, where k is an algebraically closed field.

1. INTRODUCTION.

Let k be an uncountable algebraically closed field. Let $K_n := \{x \in k^n : x_1 \cdot \ldots \cdot x_n = 0\}$ (i.e., K_n is the union of coordinate hyperplanes in k^n). Peter Russell stated the following:

Conjecture. *Let $k = \mathbb{C}$. Let X be an affine, smooth variety of dimension n, which is contractible. Then X is isomorphic to k^n if and only if there is a closed embedding of K_n into X.*

In the paper [7] we have showed that the Russell Conjecture is true if X is additionally dominated by $\mathbb{C}^n$. The Russell Conjecture suggests a certain characterization of the affine space X over any field. Here we generalize our result from [7] and we prove:

Theorem 1.1. *Let X be a $k-$uniruled smooth affine variety of dimension n. Assume that $Pic(X) = 0$ and $H^0(X, \mathcal{O}^*) = k$. If there is a closed embedding $\iota : K_n \to X$, then $X \cong k^n$.*

Corollary 1.2. *The Russell Conjecture holds for every $\mathbb{C}-$uniruled contractible (smooth) affine variety.*

Let us recall that an affine variety X is $k-$uniruled if for a sufficiently general point x in X there is an affine parametric curve $\phi_x : k \to X$ such that $\phi_x(0) = x$.

In the paper we also study generically-finite polynomial mappings of affine $k-$uniruled varieties. We generalize some results from [7] and moreover we prove some of this result in more general setting. In particular we give a wide description of hypersurfaces which are testing sets in the case X is a $k-$uniruled affine variety. In particular we prove:

Theorem 1.3. *Let X be a affine k-uniruled variety. Let $S_1, \ldots, S_m$ be hypersurfaces in k^m, which have no common points at infinity. Then $S = \bigcup_{i=1}^m S_i$ is a testing set for polynomial mappings $X \to k^m$.*

For example the set $S = \bigcup_{i=1}^m \{x \in k^n : x_i = 0\}$ is a testing set for polynomial mappings $f : X \to k^m$ and we have the following statement:

Date: January 4, 2010.

1991 *Mathematics Subject Classification.* MSC Classification: 14 A.

Key words and phrases. affine space, testing set, the set of non-proper points.

The author was partially supported by the grant of Polish Ministry of Science, 20010-20013.

Corollary 1.4. *Let X be a affine k-uniruled variety. Let $f = (f_1, \ldots, f_m) : X \to k^m$ be a generically-finite polynomial mapping. If restrictions of f to hypersurfaces $V(f_i) = \{x \in X : f_i(x) = 0\}$, $i = 1, \ldots, m$ are finite, then the mapping f is finite, too.*

We also continue the study the set of non-properness of a generically-finite polynomial mapping $f = (f_1, \ldots, f_n) : X \to Y$, where X is an affine $k-$uniruled variety and Y is an affine variety. Let us recall that f is *not proper* at a point y if there is no Zariski open neighborhood U of y such that $f^{-1}(\mathrm{cl}(U))$ is proper. We prove:

Theorem 1.5. *For a generically-finite dominant polynomial mapping $f : X \to Y$, where X is a $k-$uniruled affine variety and Y is an affine variety, the set S_f is either empty or it is a $k-$uniruled hypersurface (in Y).*

2. Preliminaries.

We assume that k is an algebraically closed field. For simplicity we also assume that k is uncountable. In this paper by a locally principal divisor on a variety X we mean a Cartier divisor, which *is locally given by polynomial equations.* If D is given by a system $\{U_\alpha, f_\alpha\}_{\alpha \in A}$, (where $f_\alpha \in k[U_\alpha]$), then by its support we mean a hypersurface $\mid D \mid := \bigcup_{\alpha \in A} \{x \in U_\alpha : f_\alpha(x) = 0\}$.

Definition 2.1. Let $X \subset k^n$ be a curve. We say that X is an affine parametric curve if there is a surjective polynomial mapping $\phi : k \ni t \to \phi(t) \in X$.

In analogous way we say that a projective curve X is parametric, if there is a surjective polynomial mapping $\phi : \mathbb{P}^1(k) \ni t \to \phi(t) \in X$.

Now let us recall some basic facts abouts k-uniruled varieties (see [7] and [10]).

Proposition 2.2. *Let k be an uncountable field. Let X be an irreducible affine variety of dimension ≥ 1. The following conditions are equivalent:*

1) for every point $x \in X$ there is a polynomial affine curve in X going through x;

2) there exists a Zariski-open, non-empty subset U of X, such that for every point $x \in U$ there is a polynomial affine curve in X going through x;

3) there exists an affine variety W with $\dim W = \dim X - 1$ *and a dominant polynomial mapping $\phi : W \times k \to X$.*

We have the following definition of a *k-uniruled* variety .

Definition 2.3. An affine irreducible variety X is called *k-uniruled* if it is of dimension ≥ 1, and satisfies one of equivalent conditions $1) - 3)$ listed in Proposition 2.2.

Example 2.4. Let $H \subset k^n$ be an irreducible hypersurface of degree $d < n$. Then H is a k-uniruled variety. In fact H can be covered by lines.

Let us recall the following:

Definition 2.5. *An irreducible algebraic variety X we will call semi-affine if there exists a proper generically-finite polynomial mapping $X \to X'$, where X' is an affine variety.*

We say that a semi-affine variety X is k-uniruled if there is a dominant generically finite morphism $f : H \times k \to X$, where H is an affine variety. Of course we can assume that H is smooth.

We also have to recall some facts about sets of non-properness of polynomial mappings (see [7] and [8]).

Definition 2.6. Let $f : X \to Y$ be a polynomial map. We say that f *is proper* at a point $y \in Y$ if there exists an open neighborhood U of y such that $\mathrm{res}_{f^{-1}(U)} f : f^{-1}(U) \to U$ is a finite map.

We have the following important theorem (for a proof see [7]):

Theorem 2.7. *Let $f : X \to Y$ be a dominant polynomial map of irreducible varieties of the same dimension. Assume that X is semi-affine and Y affine. Then the set S_f of points at which f is not proper is either empty or it is a hypersurface.*

Remark 2.8. The proof given in [7] is over $k = \mathbb{C}$, however essentially it works for arbitrary field, some obvious modification we leave to the reader.

3. The case of surfaces.

Our next aim is to give a characterization of the testing sets, as well as the characterization of the set of non-proper points for a dominant map $f : X \to k^m$, where X is a affine $k-$uniruled surface. In fact we will do it in a more general setting.

Definition 3.1. *Let X, Y be algebraic varieties and $f : X \to Y$ be a polynomial dominant map. By a compactification of f we mean a variety $\overline{X}$ and a map $\overline{f} : \overline{X} \to Y$, such that*

1) $\overline{f}$ is proper,

2) $X \subset \overline{X}$,

3) $res_X \overline{f} = f$.

We have the following easy proposition:

Proposition 3.2. *Let X, Y be algebraic varieties and $f : X \to Y$ be a polynomial dominant map. Then f has a compactification. Moreover, if X is normal we can choose $\overline{X}$ to be normal, too. If X is semi-affine, then $\overline{X}$ is also semi-affine.*

Proof. It is enough to take $\overline{X} := closure\ of\ graph(f) \subset X' \times Y$, (where X' is a completion of X), and to take as $\overline{f}$ the canonical projection. If X is normal we can additionally take the normalization of $\overline{X}$. □

Remark 3.3. Assume that X is a smooth surface. Since we can resolve singularities of a surface (see [1]), we can always assume that $\overline{X}$ is smooth, too.

Let a map $\overline{f}$ be a compactification of some dominant map $f : X \to Y$, where X is a semi-affine variety and Y is an affine variety. By the lemma below the subvariety $D := \overline{X} \setminus X$ is a hypersurface. Let $D_1 \cup \ldots \cup D_r$ be a decomposition of D into irreducible components. We call a component D_i horizontal if dim $\overline{f}(D_i)$ =dim D_i, otherwise we call it vertical.

Lemma 3.4. *Let V be an algebraic variety which contains a semi-affine variety X as an open dense subset. Then the subvariety $D := V \setminus X$ is a hypersurface. Moreover, if V is complete of dimension $n \geq 2$, then D is connected.*

Let X be a smooth projective surface and let $D = \sum_{i=1}^{n} D_i$ be a simple normal crossing (s.n.c) divisor on X (here we consider only reduced divisors). Let $graph(D)$ be a graph of D, i.e., a graph with one vertex Q_i for each irreducible component D_i of D, and one edge between Q_i and Q_j for each point of intersection of D_i and D_j.

Definition 3.5. *Let D be a simple normal crossing divisor on a smooth surface X. We say that D is a tree if graph(D) is connected and acyclic.*

We have the following fact which is obvious from graph theory:

Proposition 3.6. *Let X be a smooth projective surface and let divisor $D \subset X$ be a tree. Assume that $D', D'' \subset D$ are connected divisors without common components. Then D' and D'' have at most one common point.*

Now we can prove:

Theorem 3.7. *Let X, Y be algebraic surfaces, X is normal, semi-affine and Y is affine. Assume, that X contains a smooth cylinder $H = \Gamma \times k$ as an open, dense subset. Let $f : X \to Y$ be a dominant polynomial map and $\overline{f} : \overline{X} \to Y$ be a compactification of f. Let $Q := \overline{X} \setminus X$. Then*

1) every horizontal component of Q is an affine parametric curve,

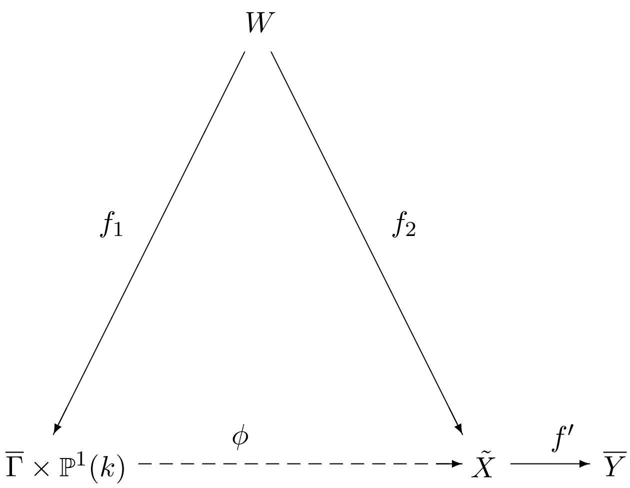
2) every vertical component of Q is a projective parametric curve,

3) if H_1, H_2 are horizontal components, then $H_1 \cap H_2 = \emptyset$,

4) every connected vertical curve meets at most one horizontal component.

Proof. We can assume that X and $\overline{X}$ are smooth. Let $\tilde{X}$ be a smooth completion of $\overline{X}$. We can assume that the mapping $\overline{f} : \overline{X} \to Y$ has an extension to a morphism $f' : \tilde{X} \to \overline{Y}$, where $\overline{Y}$ is a projective closure of Y. In particular $f'^{-1}(\overline{Y} \setminus Y) = \tilde{X} \setminus \overline{X}$.

The inclusion $\iota : \Gamma \times k \to X$ induces the birational mapping $\phi : \overline{\Gamma} \times \mathbb{P}^1(k) \to \tilde{X}$, (here $\overline{\Gamma}$ is a smooth completion of Γ). Note that the divisor $D = \overline{\Gamma} \times \infty + \sum_{i=1}^{l} \{a_i\} \times \mathbb{P}^1$ is a tree. Now we have the following picture:



Here mappings f_1 and f_2 are compositions of blowing-up's. Note that the divisor $D' = f_1^*(D)$ is a tree. Let $\overline{\Gamma} \times \infty'$ denote a proper transform of $\overline{\Gamma} \times \infty$. It is an easy observation that $f_2(\overline{\Gamma} \times \infty') \subset \tilde{X} \setminus \overline{X}$. The curve $L = \tilde{X} \setminus \overline{X}$ is a complement of a semi-affine variety hence it is connected (for details see [7], Lemma 4.5). So also the curve $L' = f_2^{-1}(L) \subset D'$ is connected. Now by Proposition 3.6 we have that every irreducible curve $Z \subset D'$ which does not belong to L' has at most one common point with L'. Let $S \subset Q$ be a horizontal component. There is a curve $Z \subset D'$, which has exactly one common point with L' such that $S = f_2(Z \setminus L)$. Moreover Z is different from $\overline{\Gamma} \times \infty'$, hence $Z \setminus L = k$. Now let

S be a vertical component. Now the curve Z which lies over S is disjoint from L' and $S = f_2(Z) = f_2(\mathbb{P}^1(k))$.

Let H_1, H_2 be horizontal components. Take $Z_1 = f_2^{-1}(\overline{H_1}), Z_2 = f_2^{-1}(\overline{H_2})$. The curves Z_1, Z_2 are connected and they have common points with L'. Since D' is the tree, we have $(Z_1 \setminus L') \cap (Z_2 \setminus L') = \emptyset$. Consequently $H_1 \cap H_2 = \emptyset$. In a similar way we can prove 4).This completes the proof. □

Theorem 3.8. *Let X, Y be algebraic surfaces, where X is semi-affine and Y is affine. Let $f : X \to Y$ be a polynomial dominant map. Let us assume that X contains a smooth cylinder $H = \Gamma \times k$, as an open, dense subset. The set S_f of points at which f is not proper consists of a finite number (possibly 0) of affine parametric curves.*

Proof. Taking a normalization we can assume that X is normal. Let $\overline{f} : \overline{X} \to Y$ be a normal compactification of f. By Theorem 2.7 the set S_f is a curve. Moreover, it is easy to see that $S_f = \overline{f}(\overline{X} \setminus X)$. Thus in fact we have $S_f = \overline{f}(R)$, where R is a union of horizontal components of $\overline{X} \setminus X$. Now the conclusion holds by Theorem 3.7. □

Corollary 3.9. *Let X, Y be affine algebraic surfaces and let X be $k-$uniruled. Let $f : X \to Y$ be a polynomial dominant map. Then the set S_f of points at which f is not proper consists of a finite number (possibly 0) of affine parametric curves.*

Proof. Since X is $k-$uniruled, we have a dominant mapping $\phi : \Gamma \times k \to X$. We can assume that the curve Γ is smooth. Let $\overline{\phi} : Z \to X$ be a compactification of ϕ and take $g = f \circ \overline{\phi}$. Then $S_f = S_g$. Now the conclusion holds by Theorem 3.8. □

We state now the following basic definition:

Definition 3.10. *Let X, Y be algebraic varieties, where X is semi-affine and Y is affine. Let S be a closed subset of Y. We will call S a testing set for properness of polynomial mappings $f : X \to Y$ (briefly a testing set) if for every generically-finite polynomial mapping $f : X \to Y$, if $res_{f^{-1}(S)} f : f^{-1}(S) \ni x \to f(x) \in S$ is proper then f is proper, too.*

The following fact will be frequently used

Lemma 3.11. *Let X, Y be algebraic varieties, where X is semi-affine and Y is affine. Let $f : X \to Y$ be a generically finite dominant mapping. Assume that $T = \bigcup_{j=1}^m T_i$ is a connected hypersurface in Y, with irreducible components T_i, which is a support of a locally principal divisor. Moreover, assume that $res_{f^{-1}(T)} f : f^{-1}(T) \to T$ is a proper mapping. If for every $j = 1, \ldots, m$ the mapping f is proper at some point $y_j \in T_j$, then it is proper at every point $y \in T$.*

Proof. We can assume that X is normal. Let $\overline{f} : \overline{X} \to Y$ be a normal compactification of f and denote $D := \overline{X} \setminus X$. By the Stein Factorization Theorem (see e.g. [4]) there exist a variety W, and regular surjective mappings $p : \overline{X} \to W$, $q : W \to Y$, such that $f = q \circ p$ and p has only connected fibers (in particular being generically finite it is a birational mapping) and q is finite.

Now assume on the contrary, that the mapping f is not proper at a point $y \in T_i \subset T$. We will show that this assumption leads to a contradiction.

First of all, since $res_{f^{-1}(T)} f : f^{-1}(T) \to T$ is a proper mapping we have $\mathrm{cl}(f^{-1}(T)) \cap D = \emptyset$, i.e., the set $f^{-1}(T)$ is closed in $\overline{X}$. Moreover, since the mapping f is proper at points $y_j \in T_j$ there is no horizontal components over $T_j,\ j = 1, \ldots, m$.

There are two cases possible:

a) the set $\overline{f}^{-1}(y)$ is finite,

b) the set $\overline{f}^{-1}(y)$ is infinite.

ad a) We have that there is a point $b \in \overline{f}^{-1}(y) \cap D$. Let T be a support of a divisor T'. Consider the locally-principal divisor $Z := \overline{f}^{*}(T') \cap (\overline{X} \setminus f^{-1}(T))$. It has the support in D and it has only horizontal components which go through b. One of them lies over some T_j, which is a contradiction.

ad b) We will show that this case also is impossible. Indeed let $b \in q^{-1}(y)$ be a point in W such that $p^{-1}(b)$ is infinite. Let R be an irreducible component of the hypersurface $q^{-1}(T)$ which contains the point b. The variety $p^{-1}(R)$ is connected and contains the connected set $p^{-1}(b)$. Moreover, it is contained in $\overline{f}^{-1}(T)$. Since $f^{-1}(T)$ is disjoint from D and since $p^{-1}(b)$ must be in D, we have that $p^{-1}(R)$ is also in D. But $p^{-1}(R)$ contains a horizontal component which lies over R and consequently over some T_j. This is a contradiction. □

Corollary 3.12. *Let X, Y be algebraic varieties, where X is semi-affine and Y is affine. Let $f : X \to Y$ be a generically finite dominant mapping. Assume that T is a connected hypersurface in Y, such that every irreducible component of T is a support of a locally principal divisor. Moreover, assume that $res_{f^{-1}(T)} f : f^{-1}(T) \to T$ is a proper mapping. If the mapping f is proper at some point $y_1 \in T$, then it is proper at every point $y \in T$.*

Theorem 3.13. *Let X, Y be algebraic surfaces, where X is semi-affine and Y is affine. Let X contain a smooth cylinder $H = \Gamma \times k$ as an open, dense subset. Assume that $T = \bigcup_{j=1}^{r} T_j$ is a curve in Y such that*

1) every T_j is a support of some locally principal divisor,

2) if $S \subset T$ is an irreducible component of some T_j which is an affine parametric curve, then for some T_k we have $S \not\subset T_k$ and $S \cap T_k \neq \emptyset$,

3) for every affine parametric curve $\Gamma \subset Y$ we have $\Gamma \cap T \neq \emptyset$.

Then T is a testing set for properness of polynomial mappings $f : X \to Y$.

Proof. Let $f : X \to Y$ be a generically-finite polynomial mapping and $res_{f^{-1}(T)} f : f^{-1}(T) \ni x \to f(x) \in T$ be a proper mapping. We have to show that f is proper, too.

Taking the normalization we can assume that X is normal. Let $\overline{f} : \overline{X} \to Y$ be a normal compactification of f and denote $D := \overline{X} \setminus X$. By the Stein Factorization Theorem there exist a normal surface W, and regular surjective mappings $p : \overline{X} \to W$, $q : W \to Y$, such that $f = q \circ p$ and p has only connected fibers (in particular, being generically finite it is a birational mapping) and q is finite. We have:

Lemma 3.14. *Let X, Y, f be as above. Assume that $S, T \subset Y$ are curves, S is irreducible and T is the support of a locally principal divisor. Moreover, assume that the mapping $res_{f^{-1}(S \cup T)} f : f^{-1}(S \cup T) \ni x \to f(x) \in S \cup T$ is proper. If $S \cap T$ has an isolated point, then the mapping f is proper at some point $y \in S$.*

Proof. Let us assume the contrary, i.e., that $S \subset S_f$. Hence there is a horizontal curve $S' \subset \overline{X} \setminus X$ such that $\overline{f}(S') = S$. Let a be an isolated point of the intersection $S \cap T$ and $b \in S'$ be a point such that $\overline{f}(b) = a$.

There are two cases possible:

i) the point b is an isolated component of the set $\overline{f}^{-1}(a)$,

ii) the point b is not an isolated component of the set $\overline{f}^{-1}(a)$,

ad i) Let us note that by our assumptions the set $f^{-1}(S \cup T)$ is closed in $\overline{X}$. Let T' be a divisor with support $\mid T' \mid = T$ and consider a locally-principal divisor $T'' := \overline{f}^*(T') \cap (\overline{X} \setminus f^{-1}(S \cup T))$. It has support in D and cuts S' in b. Let us denote a component of T" which contain the point b by R. By i) the component R is horizontal. Since a is an isolated component of the intersection $S \cap T$, we have that $R \neq S'$, which contradicts Theorem 4.6.

ad ii) We will show that this case is impossible. Indeed, let $c \in q^{-1}(a)$ be a point in W such that $p^{-1}(c)$ is infinite and $b \in p^{-1}(c)$. Let R be an irreducible component of divisor $\overline{q}^*(T')$ which contains the point c. The curve $p^{-1}(R)$ is connected and contains the curve $p^{-1}(c)$. Moreover, it is contained in $\overline{f}^{-1}(T)$. Since $f^{-1}(T)$ is disjoint from D and since $p^{-1}(c)$ must be in D, we have that $p^{-1}(R)$ is also in D. But the curve $p^{-1}(R)$ contains a horizontal component H which lies over R. Moreover, since a is an isolated component of the intersection $S \cap T$, we have that $H \neq S'$. It means that the connected vertical curve $p^{-1}(c)$ meets two different horizontal components and this is a contradiction. □

We now return to the proof of Theorem 3.13. By Lemma 3.11, Lemma 3.14 and Theorem 3.8 we easily see that for every $y \in T$, the mapping f is proper at y. Finally if S_f denotes the set of points at which the mapping f is not proper, we have that $S_f \cap T = \emptyset$. By Theorem 3.8 and 3), this implies that $S_f = \emptyset$, i.e., the mapping f is proper. □

The theorem above can be slightly generalized:

Corollary 3.15. *Let X, Y be algebraic surfaces, where X is semi-affine and $k-$uniruled and Y is affine. Assume that $T = \bigcup_{j=1}^{r} T_j$ is a curve in Y such that*

1) every T_j is a support of some locally principal divisor,

2) if $S \subset T$ is an irreducible component of some T_j which is an affine parametric curve, then for some T_k we have $S \not\subset T_k$ and $S \cap T_k \neq \emptyset$,

3) for every affine parametric curve $\Gamma \subset Y$ we have $\Gamma \cap T \neq \emptyset$.

Then T is a testing set for properness of polynomial mappings $f : X \to Y$.

Proof. Let $H = \Gamma \times k$ be a smooth cylinder. Let $g : H \to X$ be a dominant mapping. Take a compactification $\overline{g} : \overline{H} \to X$. Since $\overline{g}$ is a proper generically-finite mapping and X is semi-affine, we have that $\overline{H}$ is a semi-affine surface. A mapping $f : X \to Y$ is proper if and only if the mapping $F := f \circ \overline{g}$ is proper. Since the mapping $\overline{g}$ is proper, we have that $res_{f^{-1}(T)} f : f^{-1}(T) \ni x \to f(x) \in T$ is a proper mapping if and only if $res_{F^{-1}(T)} F : F^{-1}(T) \ni x \to F(x) \in T$ is a proper mapping. Now the proof reduces to the proof of Theorem 3.13. □

Corollary 3.16. *Let $T_1, \ldots, T_m$ be hypersurfaces in k^m which have no common points at infinity. Let X be a semi-affine and $k-$ uniruled surface. Then $T = \bigcup_{i=1}^{m} T_i$ is a testing set for polynomial mappings $f : X \to k^m$.*

Proof. First take $T_i = \{x : x_i = 0\}$ and $T = \bigcup_{i=1}^{m} T_i$. We have to show that if $f : X \to k^m$ is a generically-finite polynomial mapping and $res_{f^{-1}(T)} f : f^{-1}(T) \ni x \to f(x) \in T$ is a proper mapping then f is a proper mapping, too. Let us take $Y = \mathrm{cl}(f(X))$ and take ${T'}_i = T_i \cap Y$, $T' = \bigcup_{i=1}^{m} T'_i$. We can assume that $Y \not\subset T_i$ for $i = 1, \ldots, m$. Hence all T'_i are locally principal divisors on Y. We will show that T' satisfies all assumptions of Corollary 3.15. It satisfies 1),3), because $T = \bigcup_{i=1}^m T_i$ satisfies 1),3) in k^m.

It also satisfies 2). Indeed, let $\Gamma \subset T'$ be an affine parametric curve, we have to show that Γ meets another component of T'. We can assume that $\Gamma \subset T_1, \ldots, T_s$ but $\Gamma \not\subset T_{s+1}, \ldots, T_m$ (where $s < m$, because the intersection of all T_i is one point). If we put $Z = \bigcap_{i=1}^s T_i$ and $Z_i = T_i \cap Z$ for $i > s$, then Z_i are coordinate hyperplanes in $Z \cong k^{m-s}$. It means that for at least one index "$j > s$" we have $\Gamma \cap Z_j \neq \emptyset$. Hence Γ has common points with the curve $T_j \cap Y = T'_j$. By construction all components of T'_j are different from Γ.

The general case can be easily deduced from the particular one. Indeed, let $T_i = V(g_i)$ for some reduced polynomial $g_i \in k[y_1, \ldots, y_m]$, $i = 1, \ldots, m$. By the assumption, the mapping $G := (g_1, \ldots, g_m) : k^m \to k^m$ is finite. Now it is enough to consider the mapping $f' = G \circ f$ and to use the first part of our proof. □

In particular the set $S = \bigcup_{i=1}^m \{x : x_i = 0\}$ is a testing set for polynomial mappings $f : X \to k^m$ and we have the following statement

Corollary 3.17. *Let X be a semi-affine and $k-$uniruled surface. Let $f = (f_1, \ldots, f_m) : X \to k^m$ be a generically-finite polynomial mapping. If the restrictions of f to curves $V(f_i)$, $i = 1, \ldots, m$ are proper, then the mapping f is also proper.*

4. Geometric characterization of S_f.

Now we pass to the general situation.

Theorem 4.1. *Let $f : X \to Y$ be a dominant polynomial map of $n-$dimensional varieties, where X is semi-affine, $k-$uniruled and Y is affine. Then the set S_f of points at which f is not proper is either empty or it is a k-uniruled hypersurface.*

Proof. As usual we can assume that X contains a smooth affine cylinder $H = \Gamma \times k$ as an open, dense subset. Let $\overline{f} : \overline{X} \to Y$ be a compactification of the mapping f.

Let $y_0 \in S_f$. There is a curve $\Lambda \subset X$ such that the mapping $f|_\Lambda$ is not proper at y_0. Moreover we can assume that $\Lambda \cap H \neq \emptyset$. Consequently we can assume that $\Lambda \subset H$. Let $\pi : H \ni (\gamma, t) \to \gamma \in \Gamma$ and $\Lambda' = \pi(\Lambda)$. We can assume that Λ' is a curve. Hence the curve Λ is contained in a cylindrical surface $S = \Lambda' \times k \subset H$. Let S' be a closure of S in X. Put $f' = f|_{S'}$. Then $S_{f'} \subset S_f$. Since $y_0 \in S_{f'}$ by a construction and the set $S_{f'}$ is a union of parametric curves the proof is complete. □

We can apply our result to find out something about geometrical properties of the set $Y \setminus f(X)$. The following corollary is an easy consequence of Theorem 4.1:

Corollary 4.2. *Let $f : X \to Y$ be a dominant polynomial map of $n-$dimensional varieties, where X is semi-affine, $k-$uniruled and Y is affine. Every $n-1$-dimensional component C of the set $\mathrm{cl}(Y \setminus f(X))$ is a k-uniruled hypersurface. In particular, for every point $x \in C$ there is an affine parametric curve in C through x.*

5. Testing sets.

Our aim in this section is to generalize Theorem 3.13 to higher dimensions. First we will prove the following variant of Lemma 3.14:

Lemma 5.1. *Let X be a semi-affine surface and let Y be an affine surface. Assume, that X contains a smooth cylinder $H = \Gamma \times k$ as an open, dense subset. Let $f : X \to Y$ be a generically-finite polynomial mapping. Assume, that $T_i,\ i = 1, \ldots, m$ are locally principal divisors in Y and the mapping $res_{f^{-1}(T)} f : f^{-1}(T) \ni x \to f(x) \in T$, where $T = \bigcup_{j=1}^{m} \mid T_j \mid$, is proper. Then f is proper at every isolated point of the intersection $\bigcap_{j=1}^{m} \mid T_j \mid$.*

Proof. As usual, we can assume that X is normal. Let $\overline{f} : \overline{X} \to Y$ be a normal compactification of f and denote $D := \overline{X} \setminus X$. By the Stein Factorization Theorem there exist a normal surface W, and regular surjective mappings $p : \overline{X} \to W$, $q : W \to Y$, such that $f = q \circ p$ and p has only connected fibers (in particular being generically finite it is a birational mapping) and q is finite.

Let a be an isolated component of $\bigcap_{j=1}^{m} \mid T_j \mid$. There are two cases possible:

i) the set $\overline{f}^{-1}(a)$ is finite,

ii) the set $\overline{f}^{-1}(a)$ is infinite.

ad i) It is enough to show that $\overline{f}^{-1}(a) \cap D = \emptyset$. Assume on the contrary, that there is a point $b \in \overline{f}^{-1}(a) \cap D$. Let us note that by our assumptions the set $f^{-1}(\mid T \mid)$ is closed in $\overline{X}$. We can consider locally-principal divisors $D_i := \overline{f}^{*}(T_i) \cap (\overline{X} \setminus f^{-1}(T)),\ i = 1, \ldots, m$. They have supports in D and meets in b. Let us denote a component of $\mid D_i \mid$ which contains the point b, by $R_i,\ i = 1, \ldots, m$. By i) the components $R_i,\ i = 1, \ldots, m$ are horizontal. Since a is an isolated component of the intersection $\bigcap_{j=1}^{m} \mid T_j \mid$, we see that $R_i \neq R_j$, for some $i \neq j$, which contradicts Theorem 3.7.

ad ii) We will show that this case is impossible. Indeed let $b \in q^{-1}(a)$ be a point in W such that $p^{-1}(b)$ is infinite. Let $R_i,\ i = 1, \ldots, m$ be irreducible components of divisors $\overline{q}^{*}(T_i)$ which contain the point b. The curves $p^{-1}(R_i),\ i = 1, \ldots, m$ are connected and contain the curve $p^{-1}(b)$. Moreover, they are contained in $\overline{f}^{-1}(T)$. Since $f^{-1}(T)$ is disjoint from D and since $p^{-1}(b)$ must be in D, we have that $p^{-1}(R_i),\ i = 1, \ldots, m$ are also in D. But the curves $p^{-1}(R_i)$ contain horizontal components H_i which are over R_i. Moreover, since a is an isolated component of the intersection $\bigcap_{j=1}^{m} \mid T_j \mid$, we see that $H_i \neq H_j$, for some $i \neq j$. This means that a connected vertical curve $p^{-1}(b)$ meets two different horizontal components, which is a contradiction. □

Now we are in a position to prove the following:

Theorem 5.2. *Let X, Y be irreducible $n-$dimensional varieties, where X is semi-affine and k-uniruled and Y is affine. Let T be a hypersurface on Y such that*

1) every irreducible component of T is a support of some locally principal divisor,

2) if $T' \subset T$ is a connected component of T which is k-uniruled then T' contains irreducible components $T'_1, \ldots, T'_r$ such that the intersection $\bigcap_{i=1}^{r} T'_i$ has a point as an isolated component,

3) for every affine k-uniruled hypersurface $\Gamma \subset Y$ we have $\Gamma \cap T \neq \emptyset$.

Then T is a testing set for polynomial mappings $f : X \to Y$. Moreover, if every irreducible component of T is not $\mathbb{C}$-uniruled, then we can change the assumption 1) to the weaker assumption that T is a support of a locally principal divisor.

Proof. As usual we can assume that X is normal and X contains a smooth affine cylinder $H = \Gamma \times k$ as an open, dense subset.

Let $f : X \to Y$ be a generically-finite polynomial mapping and $res_{f^{-1}(T)} f : f^{-1}(T) \ni x \to f(x) \in T$ be a proper mapping. We have to show that f is proper, too. Let $\overline{f} : \overline{X} \to Y$ be a compactification of f and denote $D := \overline{X} \setminus X$.

We have:

Lemma 5.3. *Let f, X, Y, T' be as above. Then f is proper at every isolated point of the intersection $\bigcap_{i=1}^{r} T'_i$.*

Proof. Let a be an isolated component of $\bigcap_{i=1}^{r} T'_i$. Let us assume that the mapping f is not proper at the point a and take a point $c \in \overline{f}^{-1}(a) \cap D$. There is an irreducible curve $\Lambda \subset \overline{X}$ which contains the point c and $\Lambda' := \Lambda \cap H \neq \emptyset$. Moreover, we can assume that Λ' contains a point b which is smooth with respect to f. As in previous proofs we can assume that Λ' is contained in a cylindrical surface $S = G \times k \subset H$. Let S' be a closure of S in X. Put $f' = f|_{S'}$. Since $b \in \Lambda'$ and f is smooth at the point b we have that the mapping f' is generically-finite. Denote $Y' := \mathrm{cl}(f'(S'))$. The variety Y' is an affine surface. By the choice of the point c and the curve Λ the mapping f' is not proper at the point $a \in Y'$.

Let T_i be locally principal divisors with support T'_i, $i = 1, \ldots, r$. We can consider divisors $R_i := \iota^*(T_i)$, where $\iota : Y' \to Y$ is an inclusion. We have $a \in \bigcap \mid R_i \mid$ and the point a is an isolated point of this intersection. Moreover, the mapping f' is proper on the preimage of the set $\bigcup \mid R_i \mid$. By Lemma 5.1 it follows that the mapping f' is proper at the point a, which is a contradiction. Hence our assumption that the mapping f is not proper at the point a is false. □

We now return to the proof of Theorem 5.2. By Lemma 5.3 and Theorem 4.1 we can easily see that for every $y \in T$ the mapping f is proper at y. Finally, if S_f denotes the set of points at which the mapping f is not proper we see that $S_f \cap T = \emptyset$. By Theorem 4.1 and 3) it follows that $S_f = \emptyset$, i.e., the mapping f is proper. □

Corollary 5.4. *Let X be a semi-affine and k-uniruled $n-$dimensional variety. Assume that T is a hypersurface in k^n such that*

1) if $T' \subset T$ is a connected component of T which is $k-$ uniruled then T' contains irreducible components $T'_1, \ldots, T'_r$ such that the intersection $\bigcap_{i=1}^{r} T'_i$ has a point as an isolated component,

2) for every affine k-uniruled hypersurface $\Gamma \subset k^n$ we have $\Gamma \cap T \neq \emptyset$.

Then T is a testing set for polynomial mappings $f : X \to k^n$.

A simple application of Theorem 5.2 is that if $T_1, \ldots, T_n$ are hypersurfaces in k^n without common points at infinity, then the set $T = \bigcup_{i=1}^{n} T_i$ is a testing set for polynomial mappings $f : X \to k^n$ (where X is a semi-affine, $k-$uniruled variety). In fact we can easily generalize this as follows:

Proposition 5.5. *Let $T_1, \ldots, T_m$ be hypersurfaces in k^m which have no common points at infinity. Let X be a semi-affine and $k-$uniruled $n-$dimensional variety. Then the set $T = \bigcup_{i=1}^{m} T_i$ is a testing set for polynomial mappings $f : X \to k^m$.*

Proof. Let $f : X \to k^n$ be a dominant mapping which is proper over T. Assume that f is not proper. We can assume that X contains an affine cylinder $H = \Gamma \times k$ as an open dense subset. As in previous proofs we can construct a cylindrical surface $S = G \times k \subset H$, such that the mapping f is not proper on $S' = \mathrm{cl}(S) \subset X$. This contradicts Corollary 3.16. □

In particular the set $S = \bigcup_{i=1}^{m}\{x : x_i = 0\}$ is a testing set for polynomial mappings $f : X \to k^m$ and so we have:

Corollary 5.6. *Let X be a semi-affine and $k-$uniruled $n-$dimensional variety. Let $f = (f_1, \ldots, f_m) : X \to k^m$ be a generically-finite polynomial mapping. If the mappings $res_{V(f_i)} f,\ i = 1, \ldots, m$ are proper, then the mapping f is proper, too.*

6. The Russell Problem.

Now we pass to the application of Theorem 5.2. Let $K_n := \{x \in k^n : x_1 \cdot \ldots \cdot x_n = 0\}$ (i.e., K_n is the union of coordinate hyperplanes in k^n). Peter Russell stated the following:

Conjecture. *Let $k = \mathbb{C}$. Let X be an affine, smooth variety of dimension n, which is contractible. Then X is isomorphic to k^n if and only if there is a closed embedding of K_n into X.*

In the paper [7] we have showed that the Russell Conjecture is true if X is additionally dominated by $\mathbb{C}^n$. The Russell Conjecture suggests a certain characterization of the affine space X over any field. Here we generalize our result from [7] and we prove:

Theorem 6.1. *Let X be a $k-$uniruled smooth affine variety of dimension n. Assume that $Pic(X) = 0$ and $H^0(X, \mathcal{O}^*) = k$. If there is a closed embedding $\iota : K_n \to X$, then $X \cong k^n$. More precisely, every closed embedding $\psi : K_n \to X$ can be extended to an isomorphism $\Psi : k^n \to X$.*

Proof. Let $\psi : K_n \to X$ be a closed embedding, and let $\Gamma_i := \psi(\{x : x_i = 0\})$. Moreover, denote $K'_n := \psi(K_n)$ and denote the point $\psi(0)$ by a.

Take $\pi_i := \{x \in k^n : x_i = 0\}$. Since $Pic(X) = 0$ there are irreducible polynomials $h_j, j = 1, \ldots, n$ such that $\Gamma_j = \{x \in X : h_j(x) = 0\}, j = 1, \ldots, n$. We see the following:

Lemma 6.2. *The restriction of the mapping $H = (h_1, \ldots, h_n) : X \to k^n$ to the set K'_n is an isomorphism. Moreover, $H^{-1}(K_n) = K'_n$.*

Proof. Let $\Gamma_{ij} = \Gamma_i \cap \Gamma_j$. Let $f_2 = 0, \ldots, f_n = 0$, be irreducible equations of the sets $\Gamma_{12}, \ldots, \Gamma_{1n}$ in the coordinate ring $k[\Gamma_1]$.

Consider $\hat{h}_2$. We have $\{x \in \Gamma_1 : \hat{h}_2(x) = 0\} = \Gamma_{12}$, hence from the Hilbert Nullstellensatz there exist an integer $r \geq 1$ and $c_2 \in H^0(X, \mathcal{O}^*)$ such that $\hat{h}_2 = c_2(f_2)^r$. By the assumption $c_2 \in k$. Since polynomials $h_1, \ldots, h_n$ give a local system of coordinates at the point a, we must have $r = 1$ and $\hat{h}_2 = c_2 f_2, c_2 \neq 0$. In a similar way $h_j = c_j f_j, c_j \neq 0$, for $j > 2$.

By the symmetry we see that the polynomials $\hat{h}_j = res_{\Gamma_i} h_j, j \neq i$, are generators of the ideals $I(\Gamma_{ij})$ in the ring $k[\Gamma_i]$.

Now, let $\lambda = \psi^{-1} : K'_n \to K_n$ and let us consider a mapping $\varepsilon^1 := res_{\Gamma_1}\lambda : \Gamma_1 \to \pi_1$. We know that this mapping is polynomial, and moreover $\varepsilon^1 = (0, \varepsilon_2, \ldots, \varepsilon_n)$. We see that $\{x \in \Gamma_1 : \varepsilon_i(x) = 0\} = \Gamma_{1i}$. Since ε^1 is an isomorphism, the polynomials $\varepsilon_i, i = 2, \ldots, n$, are irreducible in the ring $k[\Gamma_1]$. Since $\{x \in \Gamma_1 : \hat{h}_i(x) = 0\} = \Gamma_{1i}$, there exist non-zero constants κ_{1i} such that $\varepsilon_i = \kappa_{1i}\hat{h}_i, i = 2, \ldots, n$. Hence ε^1 has coordinates $(0, \kappa_{12}\hat{h}_2, \ldots, \kappa_{1n}\hat{h}_n)$. In a similar way the mapping $\varepsilon^k := res_{\Gamma_k}\lambda : \Gamma_k \to \pi_k$ has coordinates $(\kappa_{k1}\hat{h}_1, \ldots, \kappa_{kk-1}\hat{h}_{k-1}, 0, \kappa_{kk+1}\hat{h}_{k+1}, \ldots, \kappa_{kn}\hat{h}_n)$. To end the proof of our lemma it is enough to show that for every $k, l \neq j$ we have $\kappa_{kj} = \kappa_{lj} (:= \kappa_j)$. Indeed, in this case the mapping λ is the restriction to K'_n of the mapping $\Lambda = (\kappa_1 h_1, \ldots, \kappa_n h_n)$, hence also the mapping $H = (h_1, \ldots, h_n)$ in the restriction to K'_n is an embedding.

Since $\Gamma_k \cap \Gamma_l \not\subset \Gamma_j$, there exists a point $c \in (\Gamma_k \cap \Gamma_l) \setminus \Gamma_j$. Thus $\lambda(c) \notin \pi_j$ (i.e., $h_j(c) \neq 0$) and $\lambda(c) = \varepsilon^k(c) = (\ldots, \kappa_{kj} h_j(c), \ldots) = (\ldots, \kappa_{lj} h_j(c), \ldots) = \varepsilon^l(c)$, hence $\kappa_{kj} h_j(c) = \kappa_{lj} h_j(c)$ and $\kappa_{kj} = \kappa_{lj}$. Moreover, by the construction of H we have $H^{-1}(K_n) = K'_n$. □

We now complete the proof of Theorem 6.1. By the lemma above the mapping H in the restriction to the set $H^{-1}(K_n)$ is proper, hence by Corollary 5.6 the mapping H is proper. Since X is affine it means that the mapping H is finite. Since $(d_0\psi)^{-1}$ is an isomorphism, we also have that the mapping $d_a H : T_a X \to T_0 k^n$ is an isomorphism. In particular the mapping H is separable and it is non-ramified at the point a. But $H^{-1}(0) = a$ and consequently $\deg H = 1$ (see e.g. [2]). This means that the mapping H is birational. Finally, it is isomorphism by the Zariski Main Theorem. Now, if we take $\Psi := (\kappa_1 h_1, \ldots, \kappa_n h_n)^{-1} : k^n \to X$, then Ψ is an isomorphism and $res_{K_n} \Psi = \psi$. □

References

[1] Abhyankar, S, Resolution of Singularities of Embedded Algebraic Surfaces, Springer Monographs in Marhematics, (1998).
[2] Fulton, W, Intersection Theory, Springer 1998.
[3] Hartshorne, R, Ample Subvarieties of Algebraic Varieties, Springer Verlag 1986.
[4] Iitaka, S, Algebraic Geometry, Springer Verlag 1982.
[5] Jelonek, Z, The set of points at which a polynomial map is not proper, **Ann. Polon. Math., 58**, (1993), 259-266.
[6] Jelonek, Z, A hypersurface which has the Abhyankar-Moh Property, **Math. Ann. 308**, 1997, 73-84.
[7] Jelonek, Z, Testing sets for properness of polynomial mappings, **Math. Ann. 315**, 1999, 1-35.
[8] Jelonek, Z, Topological characterization of finite mappings, **Bull. Acad. Polon. Sci. Math. 49** , 279-283, (2001),
[9] Mumford, D, Algebraic Geometry I, Springer Berlin 1976.
[10] Stasica, A, Geometry of the Jelonek set, J. Pure Appl. Algebra 137, 49-55, 1999.
[11] Shafarevich, I, R, Basic Algebraic Geometry, Springer 1974.

(Z. Jelonek) Instytut Matematyki, Polska Akademia Nauk, Św. Tomasza 30, 31-027 Kraków, Poland

E-mail address: `najelone@cyf-kr.edu.pl`