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Spectral Analysis of Differential and Difference Operators

Warsaw, 2007

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INSTITUTE OF MATHEMATICS. POLISH ACADEMY OF SCIENCES

Stefan Banach International Mathematical Center

SPECTRAL ANALYSIS OF DIFFERENTIAL AND DIFFERENCE OPERATORS

Research Seminar, Warsaw July 7th–14th, 2007.

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Preface

The ninth meeting in Warsaw on spectral properties of difference and differential operators was held in July 2007. This meeting was devoted to the presentation of the recent results of the participants but also to the continuation of collaboration between some of them. There were 15 lectures delivered during the meeting. In particular the following topics were discussed: direct and inverse spectral problems of Sturm-Liouville operators and Jacobi matrices, coherent states method for counting function, selfadjoint subspace of Boltzman operator, inverses of generators of C_0 semigroups, spectral concentrations of one-dimensional Schrödinger operators, Schrödinger and Laplace operators on metric graphs.

The extended abstracts contained in this book were sent by some participants of the meeting. The complete list of the lectures presented during the meeting is included.

Lectures

- D. J. Gilbert (Dublin)
  Spectral concentration and resonances for one-dimensional Schrödinger operators with integrable potentials
- O. Gomilko (Turun, Warsaw)
  Inverse operator of the generator of the bounded C_0-semigroup
- A. V. Kiselev (St. Petersburg)
  Annihilation phenomenon in self-adjoint operators: a criterion of pure singularity of the spectrum
- P. Kurasov (Lund, St. Petersburg, Stockholm)
  Schrödinger operators on graphs and geometry
- A. Luger (Lund)
  A singular Sturm-Liouville operator: Boundary conditions in the limit point case
- W. Motyka (Kraków)
  Mixed spectrum of periodically modulated Jacobi matrices
- M. Nowaczyk (Lund, Warsaw)
  The Laplace operator on metric graphs with different boundary conditions at the vertices
- R. Romanov (St. Petersburg)
  On the selfadjoint subspace of one-speed Boltzmann operator
- Yu. Safarov (London)
  Estimating the counting function with the use of coherent states
- A. A. Shkalikov (Moscow)
  On the maps associated with the inverse Sturm-Liouville problems in the scale of Sobolev spaces. Uniform stability
- S. Simonov (St. Petersburg)
  Weyl-Titchmarsh type formula for Hermite operator
- M. J. Zygmunt (Kraków)
  Tridiagonal block matrices and canonical moments

Spectral concentration and resonances for one-dimensional Schrödinger operators with integrable potentials

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1 Introduction

We consider the time independent Schrödinger operator H associated with the singular Sturm-Liouville boundary value problem

$$Lu := -u'' + q(x)u = \lambda u, \quad x \geq 0, \tag{1}$$

$$u(0) = 0, \tag{2}$$

where $q \in L_1([0,\infty))$ is a real valued potential function, $\lambda \in \mathbf{C}$ denotes the spectral parameter, and the domain of H, $\mathcal{D}(H)$, is defined by

$$\mathcal{D}(H) := \{f \in \mathcal{H} : Lf \in \mathcal{H}; f, f' \text{ locally a.c.}; f(0) = 0\},$$

with $\mathcal{H} := L_2([0,\infty))$. In this case it is well known that H is self-adjoint, so that its spectrum, $\sigma(H)$, is real; moreover, the spectrum is purely absolutely continuous on $(0,\infty)$, and the negative part, if any, consists of isolated eigenvalues, possibly accumulating at $\lambda = 0$. The point 0 may also be an eigenvalue, but only if $xq(x) \in L_1([0,\infty))$.

1.1 The spectral function

It is convenient in the context to use the spectral function, $\rho(\lambda) : \Re \to \Re$, rather than the resolvent operator, $(H - \lambda I)^{-1}$. The spectral function satisfies $\rho(0) = 0$, is non-decreasing on $\Re$, and is related to the spectrum by

$$\begin{aligned}\sigma(H) \quad = \quad & \Re \setminus \{\lambda \in \Re : \text{there exists a neighbourhood } N(\lambda) \text{ of } \lambda \\ & \text{such that } \rho(\lambda) \text{ is constant on } N(\lambda)\}\end{aligned}$$

More informally, $\sigma(H)$ may be regarded as the closure of the set of points of increase of $\rho(\lambda)$. Other notable features of the spectral function which hold for integrable potentials include:

- $\rho(\lambda)$ has jump discontinuities at the eigenvalues of H, but is otherwise constant for $\lambda \in (-\infty, 0]$,
- for $\lambda > 0$, $\rho(\lambda)$ is purely absolutely continuous, with spectral density, $\rho'(\lambda)$, satisfying $\rho'(\lambda) > 0$,
- if $x^n q(x) \in L_1([0,\infty))$, then $\rho^{(n+1)}(\lambda)$ exists and is continuous for $\lambda > 0$, $n = 0, 1, 2, \cdots$, where $\rho^{(n+1)}(\lambda)$ denotes the nth derivative of the spectral density.

1.2 Spectral concentration

In the context of integrable potentials, the concept of spectral concentration refers to a local intensification of the continuous spectrum. More precisely, we have

Definition A point $\lambda_c > 0$ is said to be a *point of spectral concentration* of H if $\rho'(\lambda)$ has a local maximum at $\lambda = \lambda_c$.

The following immediate consequence of this definition was noted in [1] by Eastham: if $\Lambda_0 \geq 0$ is such that $\rho''(\lambda)$ exists and has one sign for $\lambda > \Lambda_0$, then Λ_0 is an upper bound for points of spectral concentration on $\Re$. This fact has been used in various schemes for determining upper bounds for points of spectral concentration (see e.g. [1], [4]).

1.3 Resonances

There is a long standing conjecture that the phenomenon of spectral concentration as defined above is closely linked to resonances on the so-called unphysical sheet. To demonstrate the nature of this conjecture, we first introduce the Jost function for integrable potentials. Suppose that $\lambda \in \mathbf{C} \setminus \{[\mathbf{0}, \infty)\}$, and let $z = \sqrt{\lambda}$ be chosen so that $\Im z \geq 0$, $\Im z \neq 0$. Then the Jost solution, $\chi(x, z)$, of (1) is the unique solution satisfying

$$\chi(x, z) \sim \mathrm{e}^{\mathrm{i}zx}, \qquad \chi'(x, z) \sim \mathrm{i}z\mathrm{e}^{\mathrm{i}zx}$$

as $x \to \infty$, where $'$ denotes differentiation with respect to x. Evidently, $\chi(x, z)$ and $\chi'(x, z)$ are in $L_2([0, \infty); dx)$ for $\Im z > 0$; moreover, χ and χ' are analytic in x and z for $\Im z > 0$, and continuous in x and z for $\Im z \geq 0$ [3]. The Jost function, $\chi(z)$, is defined in terms of the Jost solution by $\chi(z) := \chi(0, z)$, and its analyticity properties in z are inherited from the Jost solution in the obvious way. Zeros of $\chi(z)$ on $\mathbf{C}^+$ can only occur at isolated points on the positive imaginary z-axis, which correspond to negative eigenvalues of H on the λ-plane.

For some integrable potentials the Jost function can be analytically continued into part or all of the negative half z-plane, otherwise known as the unphysical sheet. Zeros of $\chi(z)$ in such regions of $\mathbf{C}^-$ are known as resonances, and resonances on the negative imaginary z-axis are referred to as anti-bound states.

2. Spectral Concentration and Resonances

Since the Jost solution is in $L_2([0, \infty); dx)$ for $\Im z > 0$, it is a λ-multiple of the well known Weyl solution of (1). It follows that the spectral density for $\lambda > 0$ may be represented in terms of the Jost function by

$$\rho'(\lambda) = \frac{\sqrt{\lambda}}{\pi \mid \chi(\sqrt{\lambda}) \mid^2}, \qquad \lambda = z^2 > 0. \tag{3}$$

This equation is known as the Kodaira formula for the spectral density, and is valid whenever $q \in L_1([0, \infty))$.

If a resonance z_0 with $\Re z_0 \neq 0$ lies close to the real z-axis, the Kodaira formula (3) suggests that there may be a corresponding increase in $\rho'(\lambda)$ in the vicinity of

$(\Re z)^2$. This observation has led to many conjectures about the relationship between resonances and spectral concentration in the case of integrable potentials. The Kodaira formula also suggests that eigenvalues of H, which are zeros of the Jost function on the positive imaginary z-axis and also known as bound states, may influence the behaviour of $\rho'(\lambda)$ in a similar way to the anti-bound states.

2.1 Examples

We illustrate these ideas by some explicit examples.

Example 1 Let

$$q(x) = \frac{2a^2}{(1+ax)^2}, \qquad a > 0,\ x \geq 0,$$

where a is a fixed constant. It is easy to check that the Jost solution is given by

$$\chi(x,z) = \mathrm{e}^{\mathrm{i}zx}\left(1 - \frac{a}{\mathrm{i}z(1+ax)}\right),$$

so that the Jost function satisfies

$$\chi(z) = 1 + \frac{\mathrm{i}a}{z}, \qquad z \neq 0. \tag{4}$$

We note that the Jost function is analytic in z on $\mathbf{C}\setminus\{0\}$, and that the only resonance is an anti-bound state at $z = -\mathrm{i}a$. By (3) and (4), we have for $\lambda > 0$

$$\rho'(\lambda) = \frac{\lambda\sqrt{\lambda}}{\pi(\lambda + a^2)} > 0,$$

from which

$$\rho''(\lambda) = \frac{\sqrt{\lambda}(\lambda + 3a^2)}{2\pi(\lambda + a^2)^2} > 0, \tag{5}$$

so that by our remarks above, there are no points of spectral concentration of H for $\lambda > 0$.

This example shows that a resonance need not be associated with a point of spectral concentration, even if the resonance is arbitrarily close to the real axis.

Example 2 Let

$$q(x) = ce^{-ax}, \qquad a > 0,\ c < 0,$$

where a and c are constants. Then it may be shown that the Jost solution satisfies

$$\chi(x,z) = e^{\mathrm{i}zx}\left\{1 + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{ce^{-ax}}{a^2}\right)^n\left(\frac{1}{(1-\frac{2\mathrm{i}z}{a})}\cdots\frac{1}{(n-\frac{2\mathrm{i}z}{a})}\right)\right\},$$

so that $\chi(x,z)$ and hence the Jost function $\chi(z)$ is analytic in z on

$$\mathbf{C}\setminus\left\{-\frac{\mathrm{i}na}{2} : n = 1, 2, 3, \cdots\right\}.$$

Setting $z = \mathrm{i}t$, $t \in \Re$, we see that if $t > -\frac{a}{2}$, then $\chi(\mathrm{i}t)$ is real with

$$\chi(\mathrm{i}t) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{c}{a^2}\right)^n \left(\frac{1}{(1+\frac{2t}{a})} \cdots \frac{1}{(n+\frac{2t}{a})}\right). \tag{6}$$

It follows that $\chi(\mathrm{i}t)$ is strictly increasing from $-\infty$ to 1 as t increases on $(-\frac{a}{2}, \infty)$, so that $\chi(\mathrm{i}t)$ has precisely one zero t_0 in the t-interval $(-\frac{a}{2}, \infty)$. The point $\mathrm{i}t_0$ is an antibound state or $-t_0^2$ is an eigenvalue of H according as $t_0 < 0$ or $t_0 > 0$ respectively, the sign of t_0 depending on the choice of a and c.

By truncating the series in (6) and obtaining bounds for the remainder, it is possible to estimate the zeros of $\chi(\mathrm{i}t)$ for $t > -\frac{a}{2}$ to an arbitrary degree of accuracy. In a similar way, the Kodaira formula (3) may be used to investigate the behaviour of $\rho'(\lambda)$ for small values of λ on $\Re^+$. Numerical results suggest that, depending on the choice of the constants a and c, points of spectral concentration may be, but need not be associated with zeros of $\chi(z)$ on $\Re z = 0$, $-\frac{a}{2} < \Im z < \infty$, regardless of whether $t_0 < 0$ or $t_0 > 0$; that is to say, there may or may not be a point of spectral concentration associated with the eigenvalue or anti-bound state.

Further consideration of this example and other similar cases may be found in [2], [5], and the references contained therein.

2.2 Discussion

There has been a substantial literature on both spectral concentration and resonances since the early work of Titchmarsh, who provided a rigorous mathematical analysis of several important cases in which both phenomena were exhibited [8]. Interest in the two phenomena has often been motivated by such issues as impedance theory, resonance scattering and spectral stability, and both are associated with scattering states which remain localised for a long time [6], [7].

The situation when $q \in L_1([0, \infty))$ is of special interest because in this case the Jost function enables the analysis to be carried out in a particularly convenient and unified way. If q is real valued, then both resonances and and eigenvalues are isolated zeros in regions of analyticity of the Jost function (with the possible exception of the point $z = 0$, which is a spectral singularity if $\lambda = 0$ is an eigenvalue). This feature, together with the Kodaira formula (3), can greatly facilitate the process of estimating the location of resonances, eigenvalues and points of spectral concentration, and thus contribute more fully to understanding of the relationships between them.

There are two longstanding conjectures about spectral concentration and resonances which have generated a high level of interest: firstly, whether every resonance in the lower half z-plane induces a point of spectral concentration of H, and secondly, whether every local maximum of $\rho'(\lambda)$ on $(0, \infty)$ is induced by some resonance. The first conjecture is refuted by Example 1 above (see also [1]), although the influence of the resonance is still detectable because differentiation of (5) shows that $\rho'(\lambda)$ has a point of inflection at $\lambda = (2\sqrt{3} - 3)a^2$. In the second case, there is some evidence to suggest that points of spectral concentration may also be induced by eigenvalues [9].

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Inverse operator of the generator of the bounded C_0-semigroup

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1. Introduction. Let X be a Banach space, let $\mathcal{E} = \mathcal{E}(X)$ be the set of densely defined closed linear operators on X and let $\mathcal{L} = \mathcal{L}(X)$ be the algebra of bounded linear operators on X. Denote by $\mathcal{G} = \mathcal{G}(X)$ the set of generators of uniformly bounded C_0-semigroups and by $\mathcal{G}_{exp} = \mathcal{G}_{exp}(X)$ the set of generators of exponentially stable C_0-semigroup acting on X.

If the operator $A \in \mathcal{G}_{exp}(X)$, then the inverse operator $A^{-1} \in \mathcal{L}$, so that A^{-1} generates a C_0-semigroup $(e^{tA^{-1}})_{t\geq 0}$ given by

$$e^{tA^{-1}} = \sum_{m=0}^{\infty} \frac{t^m A^{-m}}{m!}, \quad t \geq 0. \tag{1}$$

It can also be shown that in this case (see [1], [2]) the semigroup $(e^{tA^{-1}})_{t\geq 0}$ has the following integral representation:

$$e^{tA^{-1}}x = x - \sqrt{t}\int_0^{\infty} \frac{J_1(2\sqrt{ts})}{\sqrt{s}} e^{sA}x\, ds, \quad t > 0, \quad x \in X, \tag{2}$$

where $J_1(\cdot)$ is the Bessel function of the first kind and the first order.

Let $X = C_0[0,1]$ be the Banach space of functions $f(s)$ continuous on the closed interval $[0,1]$ and vanishing at $s = 1$. In [1], by means of the formula (2), it was proved that there exists a nilpotent semigroup $(e^{tA})_{t\geq 0}$ such that the semigroup $(e^{tA^{-1}})_{t\geq 0}$ is not uniformly bounded (the norm of $e^{tA^{-1}}$ grows at infinity as $t^{1/4}$). Using the nilpotent semigoup $(e^{tA})_{t\geq 0}$ and a certain operator-theoretical construction, [1] presents an example of a uniformly bounded C_0-semigroup $(e^{tA})_{t\geq 0}$ on the space $l_2(\mathbb{N}, C_0[0,1])$, such that the operator $A^{-1} \in \mathcal{E}$ is not a generator of a C_0-semigroup.

In [2] it was shown that in any Banach space $X = l^p$, $p \in (1,2)\cup(2,\infty)$, there exists $A \in \mathcal{G}(X)$ such that $A^{-1} \in \mathcal{E}$, but nevertheless A^{-1} does not generate a C_0-semigroup on X.

Improving the results in [1] and in a sense in [2], we show in this paper that if $X = L_p(0,1)$, $p \in (1,2) \cup (2,\infty)$, then there exists a nilpotent C_0-semigroup $(e^{tA})_{t\geq 0}$ on X such that the semigroup $(e^{tA^{-1}})_{t\geq 0}$ is not uniformly bounded.

We point out that the need in the analysis of the inverse operators of the generators of C_0-semigroups arises in infinite-dimensional control theory and numerical analysis. See, for instance, [1] and the bibliography therein.

2. Inverse operators. If $A \in \mathcal{G}_{exp}$ then by (2) and a well-known estimate for Bessel function [3, Ch. 7], we obtain

$$\|e^{tA^{-1}}\| \leq 1 + \sqrt{t}\int_0^1 \frac{|J_1(2\sqrt{ts})|}{\sqrt{s}} ds = 1 + \int_0^{2\sqrt{t}} |J_1(s)| ds \leq 1 + ct^{1/4}, \quad t > 0. \tag{3}$$

In the space $L_p = L_p(0,1)$, $p \in [1,\infty)$, with the standard norm $\|\cdot\|_{L_p}$ we consider the differential operator

$$A = -\mathcal{D}, \quad (\mathcal{D}f)(y) = f'(y), \quad \text{with the domain } D(\mathcal{D}) = \{f \in W_p^1(0,1) : f(0) = 0\},$$

where W_p^1 is the Sobolev space. The operator A generates a nilpotent C_0-semigroup $(e^{tA})_{t\geq 0}$:

$$(e^{tA}f)(y) = f(y-t), \quad 0 < y - t \leq 1, \quad (e^{tA}f)(y) = 0, \quad 0 \leq y < t,$$

and $-A^{-1}$ is the classical Volterra integral operator:

$$(A^{-1}f)(y) = -\int_0^y f(s)ds.$$

The identity (2) applied to the C_0-semigroup $e^{tA^{-1}}$ yields

$$(e^{tA^{-1}}f)(y) = f(y) - (S(t)f)(y), \;\; t > 0, \tag{4}$$

where

$$(S(t)f)(y) = \sqrt{t}\int_0^y \frac{J_1(2\sqrt{ts})}{\sqrt{s}} f(y-s)ds =$$

$$= \sqrt{t}\int_0^y \frac{J_1(2\sqrt{t(y-s)})}{\sqrt{y-s}} f(s)ds. \tag{5}$$

Theorem. *For each* $p \in [1,\infty)$

$$\overline{\lim_{t>0}} \left(t^{-|1/4-1/(2p)|} \|e^{tA^{-1}}\|_{L_p} \right) > 0, \;\; p \in [1,\infty) \tag{6}$$

Proof. From (5) it follows that

$$\|S(t)f\|_{L_p(0,1)} \geq \|S(t)f\|_{L_p(1/t,1)} \geq \|S_0(t)f\|_{L_p(1/t,1)} - \|S_1(t)f\|_{L_p(1/t,1)}, \;\; t > 1,$$

where the operator functions $S_j(t)$, $j = 0, 1$ are given by

$$(S_0(t)f)(y) = \sqrt{t}\int_0^{y-1/t} \frac{J_1(2\sqrt{t(y-s)}}{\sqrt{y-s}} f(s)ds,$$

$$(S_1(t)f)(y) = \sqrt{t}\int_{y-1/t}^y \frac{J_1(2\sqrt{t(y-s)}}{\sqrt{y-s}} f(s)ds.$$

By Minkowski's inequality, we obtain

$$\|S_1(t)f\|_{L_p(1/t,1)} \leq \sqrt{t}\int_0^{1/t} \frac{|J_1(2\sqrt{ts})|}{\sqrt{s}} \left(\int_{1/t}^1 |f(y-s)|^p dy \right)^{1/p} ds \leq$$

$$\leq \sqrt{t}\int_0^{1/t} \frac{|J_1(2\sqrt{ts})|}{\sqrt{s}} ds \|f\|_{L_p(0,1)} = \int_0^2 |J_1(s)| ds \|f\|_{L_p} = c\|f\|_{L_p},$$

where the constant $c > 0$ does not depend on $t > 1$. So, to prove (6) it suffices to show that

$$\overline{\lim_{t>0}} \left\{ t^{-|1/4-1/(2p)|} \left(\sup_{f \in L_p(0,1)} \frac{\|S_0(t)f\|_{L_p(1/t,1)}}{\|f\|_{L_p(0,1-1/t)}} \right) \right\} > 0. \tag{7}$$

Further, we will use the following asymptotic formula for the Bessel function [3, § 7.4]:

$$J_1(2\sqrt{t(y-s)}) = -\frac{\cos(2\sqrt{t(y-s)} + \pi/4)}{\sqrt{\pi}t^{1/4}(y-s)^{1/4}} + \mathrm{O}(t^{-3/4}(y-s)^{-3/4}), \;\; t(y-s) \to \infty.$$

Observe that $S_0(t)f$ can be decomposed as

$$(S_0(t)f)(y) = -(S_{0,1}f)(y) + (S_{0,2}(t)f)(y), \;\; y \in (1/t, 1), \tag{8}$$

where

$$(S_{0,1}f)(y) = \frac{t^{1/4}}{\sqrt{\pi}} \int_0^{y-1/t} \frac{\cos(2\sqrt{t(y-s)}+\pi/4)}{(y-s)^{3/4}} f(s)ds,$$

and

$$|(S_{0,2}(t)f)(y)| \le ct^{-1/4} \int_0^{y-1/t} \frac{|f(s)|}{(y-s)^{5/4}} ds = ct^{-1/4} \int_{1/t}^{y} \frac{|f(y-s)|}{s^{5/4}} ds.$$

Then, using Minkowski's inequality, we get the estimate

$$\|S_{0,2}(t)f\|_{L_p(1-1/t,1)} \le ct^{-1/4} \int_{1/t}^{1} \frac{1}{s^{5/4}} \left(\int_{1-1/t}^{1} \chi(y-s)|f(y-s)|^p dy \right\}^{1/p} ds \le$$

$$\le ct^{-1/4} \int_{1/t}^{1} \frac{1}{s^{5/4}} ds \|f\|_{L_p(0,1-1/t)} \le 4c\|f\|_{L_p(0,1-1/t)}.$$

Furthermore

$$\|S_{0,1}(t)f\|_{L_p(1/t,1)} = \frac{t^{1/4}}{\sqrt{\pi}} \|\tilde{S}(t)f\|_{L_p(0,1-1/t)}, \tag{9}$$

where the operator function $\tilde{S}(t)$ is defined as

$$(\tilde{S}(t)f)(y) = \int_0^y \frac{\cos(2\sqrt{t(y-s+1/t)}+\pi/4)}{(y-s+1/t)^{3/4}} f(s)ds, \quad y \in (0, 1-1/t).$$

Thus, by (8) and (9) we conclude that for the proof of the inequality (7), and then for the proof of the theorem, it is enough to prove that

$$\overline{\lim_{t>0}} \left\{ t^{-|1/4-1/(2p)|} t^{1/4} \left(\sup_{f \in L_p(0,1-1/t)} \frac{\|\tilde{S}(t)f\|_{L_p(0,1-1/t)}}{\|f\|_{L_p(0,1-1/t)}} \right) \right\} > 0. \tag{10}$$

Let $t_n = \pi^2 n^2$, where $n = 2, 4, \ldots$. Then for $k = 1, 2, \ldots, n$ we have

$$\cos(2\sqrt{t_n z}+\pi/4) \ge \frac{\sqrt{2}}{2}, \quad z \in I_k := \left[\frac{(k-1/4)^2}{n^2}, \frac{k^2}{n^2} \right] \subset [0,1]. \tag{11}$$

Note that

$$y - s + 1/t_n \in I_k, \quad s \in N, \quad y \in M_k, \tag{12}$$

where

$$N = [0, 1/(4n)], \quad M_k = \left[\frac{(k-1/4)^2}{n^2} - \frac{1}{t_n} + \frac{1}{4n}, \frac{k^2}{n^2} - \frac{1}{t_n} \right], \quad k = \frac{n}{2}+1, \ldots, n.$$

The lengths $|M_k|$ of the segments M_k satisfy

$$|M_k| = \frac{k^2}{n^2} - \frac{(k-1/4)^2}{n^2} - \frac{1}{4n} = \frac{k-n/2-1/8}{2n^2},$$

so that

$$d_n := \sum_{k=n/2+1}^{n} |M_k| = \frac{1}{2n^2} \sum_{k=n/2+1}^{n} (k-n/2-1/8) = \frac{4n-9}{32n} \ge \frac{1}{9}, \quad n > 20.$$

Let now $n > 20$ be an even integer, let g be the characteristic function of $M = \cup M_k$, $k = n/2+1,\dots,n$, and let f be the characteristic function of the segment N. Set $\beta_n = 1 - 1/t_n$. Then

$$\|f\|_{L_p(0,\beta_n)} = \left(\frac{1}{4n}\right)^{1/p} \le \frac{1}{n^{1/p}}, \quad \|g\|_{L_q(0,1-1/t_n)} \le 1,$$

$$\left|\int_0^{\beta_n} (\tilde{S}f)(y)g(y)dy\right| \le \|\tilde{S}f\|_{L_p(0,\beta_n)}\|g\|_{L_q(0,\beta_n)} \le \|\tilde{S}f\|_{L_p(0,\beta_n)}, \quad \frac{1}{p}+\frac{1}{q} = 1.$$

On the other hand, using properties of the segments M_k (see (11), (12)), we obtain

$$\left|\int_0^{\beta_n} (\tilde{S}_0 f)(y)g(y)dy\right| = \left|\int_{y\in M} (\tilde{S}_0 f)(y)dy\right| =$$

$$= \int_{y\in M}\int_0^{1/4n} \frac{\cos(2\sqrt{t_n(y-s+1/t_n} + \pi/4)}{(y-s+1/t_n)^{3/4}} dsdy \ge$$

$$\ge \frac{1}{\sqrt{2}}\int_{y\in M}\int_0^{1/4n} \frac{ds\,dy}{(y-s+1/t_n)^{3/4}} \ge \frac{1}{\sqrt{2}}\int_{y\in M}\int_0^{1/4n} dsdy = \frac{1}{\sqrt{2}}\frac{d_n}{4n} \ge \frac{1}{72n}.$$

Thus

$$t_n^{1/4}\frac{\|\tilde{S}f\|_{L_p(0,\beta_n)}}{\|f\|_{L_p(0,\beta_n)}} \ge \sqrt{\pi}\frac{n^{1/p}}{72n^{1/2}} = ct_n^{(1/p-1/2)/2}, \quad c > 0.$$

If $p \in [1,2]$, then the last inequality implies (10), and then (6) follows.

If $p \in (2,\infty)$ then by observing that

$$\|S(t)\|_{L_p} = \|S(t)\|_{L_q} \quad 1/p + 1/q = 1, \tag{13}$$

and using (6) for $p \in [1,2]$, we get the assertion of our theorem for $p \in (2,\infty)$ too. The proof is complete. □

As pointed out in Introduction, the above theorem provides the example of a nilpotent C_0-semigroup $(e^{tA})_{t\ge 0}$ in the Banach space $L_p(0,1)$, $p \in (1,\infty)$, $p \ne 2$, such that the C_0-semigroup $(e^{tA^{-1}})_{t\ge 0}$ is not uniformly bounded (the norm of $e^{tA^{-1}}$ grows at infinity as t^{α_p}, $\alpha_p = |1/4 - 1/(2p)|$). Thus, in the case $p = 1$ the theorem shows the sharpness of (3) for $A \in \mathcal{G}_{exp}(L_1(0,1))$. Recall that by [5, Theorem 2.2] one has $\|(I + A^{-1})^n\|_{L_p} \approx n^{\alpha_p}$, $n \to \infty$.

Remark. *The estimate* (6) *is sharp for every* $p > 1$. *Indeed,* $(e^{tA})_{t\ge 0}$ *is a contractive semigroup in the Hilbert space* $L_2(0,1)$ *and then* $(e^{tA^{-1}})_{t\ge 0}$ *is a contractive semigroup in* $L_2(0,1)$ *too. If* $1/p + 1/q = 1$, $p \in (1,2)$ *and* $t \ge 1$, *then by the Riesz-Thorin interpolation theorem* [*4, p.* 97], (3) *and* (13) *we obtain the estimate*

$$\|e^{tA^{-1}}\|_{L_q} \le \|e^{tA^{-1}}\|_{L_p} \le \|e^{tA^{-1}}\|_{L_2}^{2-2/p}\|e^{tA^{-1}}\|_{L_1}^{2/p-1} \le ct^{1/(2p)-1/4}.$$

Since $1/(2p) - 1/4 = |1/(2q) - 1/4|$, $p \in (1,2)$, *we then have* $\|e^{tA^{-1}}\|_{L_p} \le ct^{|1/4-1/(2p)|}$, $t \ge 1$, *for each* $p \in (1,\infty)$.

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Annihilation phenomenon in self-adjoint operators: a criterion of pure singularity of the spectrum

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We show that a natural generalization of the Cayley identity from the case of matrices to the case of general self-adjoint (possibly, unbounded) operators in Hilbert spaces is readily available. Moreover, the named generalization constitutes a rather transparent new criterion of the absence of absolutely continuous spectral subspace.

Theorem 1. *Let A be a (possibly, unbounded) self-adjoint operator in the Hilbert space H. Then the following two statements are equivalent.*

(i) The spectrum of A is purely singular;

(ii) There exists an outer bounded in the upper half-plane scalar function $\gamma_A(\lambda)$, weakly annihilating the operator A, i.e.,

$$w - \lim_{\varepsilon \downarrow 0} \gamma_A(A + i\varepsilon) = 0.$$

Moreover, the function γ_A can be chosen as follows:

$$\gamma_A(\lambda) = \det(I + i\sqrt{V}(A - iV - \lambda)^{-1}\sqrt{V})$$

for any trace class (or relatively trace class) non-negative operator V in H such that $\bigvee_{Im\ \lambda \neq 0}(A - \lambda)^{-1}VH = H$.

Remark 1. It is easy to see that $\gamma_A(\lambda)$ in fact coincides with the perturbation determinant $D_{A/A-iV}(\lambda)$ of the pair A, $A - iV$

Remark 2. Suppose that the operator A is a self-adjoint operator with simple spectrum. Then the trace class operator V of our Theorem can clearly be chosen as a rank one operator in Hilbert space H. In this situation, the statement of Theorem 1 can be modified in the part concerning the choice of the annihilator in the following way: the annihilator can be chosen as

$$\gamma_A(\lambda) := \frac{1}{1 - i(D(\lambda) - 1)},$$

where $D(\lambda) := 1 + \langle (A-\lambda)^{-1}\phi, \phi\rangle$ is the perturbation determinant of the pair A, $A + \langle \cdot, \phi\rangle\phi$ and ϕ is the generating vector for the operator A.

Our results effectively show [1, 2, 3], that in terms of weak outer annihilation the singular spectral subspace N_i^0 of a nonself-adjoint operator behaves in exactly the same way as the singular spectral subspace of a self-adjoint operator. Moreover, due to this result it would seem reasonable to include the singular component of the self-adjoint part of the operator L (in general case, when L is not necessarily completely nonself-adjoint) into the singular subspace N_i^0. It is also worth mentioning that not only the proof of this theorem exploits essentially nonself-adjoint (in particular, functional model related) techniques, but even certain crucial objects of the nonself-adjoint spectral theory appear already in its statement.

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Schrödinger operators on graphs and geometry

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The main aim of this lecture is to study the relation between the spectrum of a Schrödinger operator on a metric graph and geometric properties of the graph. This question has already been studied for Laplace operators with standard boundary conditions at the vertices and it was proven that the spectrum of the Laplace operator determines the total length, the number of connected components and the Euler characteristics of the underlying graph (see [3, 4]). Originally to establish the relation between the spectrum and the Euler characteristics one used so-called trace formula connecting the spectrum of the Laplacian with the set of periodic orbits on the graph [1, 2, 6]

$$\begin{aligned} u(k) &\equiv 2m_s(0)\delta(k) + \sum_{k_n \neq 0} \left(\delta(k-k_n) + \delta(k+k_n)\right) \\ &= \chi\delta(k) + \frac{\mathcal{L}}{\pi} + \frac{1}{\pi}\sum_{p\in\mathcal{P}} l(\mathrm{prim}\,(p))S(p)\cos kl(p), \end{aligned} \tag{1}$$

where

- k_n^2 are the eigenvalues of the Laplacian,
- $m_s(0)$ is the multiplicity of the eigenvalue zero*;
- p is a closed path on Γ;
- $l(p)$ is the length of the closed path p;
- $\mathrm{prim}\,(p)$ is one of the primitive paths for p;
- $S(p)$ is the product of all vertex scattering coefficients along the path p.

Based on this relation we prove the following formula for the Euler characteristic

$$\begin{aligned} \chi &= 2m_s(0) + 2\lim_{t\to\infty}\sum_{k_n\neq 0}\cos k_n/t \left(\frac{\sin k_n/2t}{k_n/2t}\right)^2 \\ &= 2m_s(0) - 2\lim_{t\to\infty}\sum_{k_n\neq 0}\frac{1-2\cos k_n/t+\cos 2k_n/t}{(k_n/t)^2}. \end{aligned} \tag{2}$$

This formula allows one to calculate the Euler characteristic of the metric graph using not only the spectrum of the Laplace operator, but the spectrum of any Schrödinger operator with essentially bounded potential.

Theorem 1. *Let Γ be a finite compact metric graph and $L(\Gamma)$ - the corresponding Laplace operator (with standard boundary conditions). Let $q \in L_\infty(\Gamma)$ be a real valued potential and $S = L(\Gamma) + Q$ - the corresponding Schrödinger operator, where Q is the operator of multiplication by q. Then the Euler characteristic $\chi(\Gamma)$ of the graph Γ is uniquely determined by the spectrum $\lambda_n(S)$ of the operator S and can be calculated using the limit*

$$\chi(\Gamma) = 2\lim_{t\to\infty}\sum_{n=0}^{\infty}\cos\sqrt{\lambda_n(S)}/t\left(\frac{\sin\sqrt{\lambda_n(S)}/2t}{\sqrt{\lambda_n(S)}/2t}\right)^2, \tag{3}$$

*It is equal to the number C of connected components in accordance with Theorem 1 from [4].

where we use the following natural convention

$$\lambda_m = 0 \Rightarrow \frac{\sin \sqrt{\lambda_m(S)}/2t}{\sqrt{\lambda_m(S)}/2t} = 1. \tag{4}$$

The proof is based on the following asymptotic formula connecting the spectra of Laplace and Schrödinger operators

$$k_n(S) = k_n(L) + O(1/n), \quad \text{as} \quad n \to \infty, \tag{5}$$

and the following two estimates

Estimate 1 (suitable for small values of n)

$$|a_n(t) - a_n^0(t)| \le c\frac{(n+B)^2}{t^2}. \tag{6}$$

Estimate 2 (suitable for large values of n)

$$\left|a_n(t) - a_n^0(t)\right| \le d\frac{t}{(n-B)^3}, \; n > B, \tag{7}$$

where c and d are certain positive constant $d > 0$.

Most of the results discussed here are published in [5].

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A singular Sturm-Liouville operator: Boundary conditions in the limit point case

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Our main object is the singular ordinary differential expression

$$\ell(y) := -y''(x) + \left(\frac{q_0 + q_1 x}{x^2}\right) y(x), \qquad x \in (0, \infty), \tag{1}$$

with $q_0 \leq -\frac{1}{4}$, $q_1 \in \mathbb{R}$, which is known as the "Hydrogen atom differential expression" (see [3], Section 39), since it appears after separation of variables in two- and three-dimensional Schrödinger equations with Coulomb potential. Even if the corresponding differential equation

$$-y''(x) + \left(\frac{q_0 + q_1 x}{x^2}\right) y(x) = \lambda y(x), \qquad \lambda \in \mathbb{C}^+, \tag{2}$$

is extremely well studied (and its solutions can be expressed in terms of Whittaker functions) it has recently again become subject of interest in several publications.

Different authors, see [4, 5], have studied singular differential equations from the point of view of Weyl theory. In the above example both endpoints are in limit point case, which according to the classical theory (by introducing a regular reference point) would give rise to a 2×2-matrix Titchmarsh-Weyl-coefficient. In their articles the above named authors modified the well-known Titchmarsh-Weyl-approach by considering certain singular and regular solutions of (2) and obtain so a generalized Titchmarsh-Weyl-coefficient $m(\lambda)$, which is a scalar function but no longer belongs to the Nevanlinna class (mapping the upper half plane into itself).

We refine their construction by investigating these singular solutions further, which leads us to the interpretation of a certain singular solution $g(x, \lambda)$ as an element from the space $\mathcal{H}_{-n+2}(L_0)$ in the scale of Hilbert spaces associated to the operator L_0, the (unique) self-adjoint realization of (1) in $L^2(\mathbb{R}^+)$ with index $-n+2 = -\left[\sqrt{\frac{1}{4} + q_0}\right]$. This rather technical observation gives us the possibility to define

$$\varphi := (L_0 - \lambda_0) g(\,\cdot\,, \lambda_0) \in \mathcal{H}_{-n}(L_0)$$

as a distribution with support at the origin. Supersingular perturbations of the type

$$L_0 + t\langle \varphi, \,\cdot\, \rangle \varphi$$

are well studied in a series of papers [6, 1, 7]. We use these kind of models in order to construct a Hilbert space of functions, $\mathbb{H}$, which includes certain non-square integrable functions. Therein a family of self-adjoint operators, $\mathbb{L}_\theta$, with $\theta \in \mathbb{R}$ is defined,

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which act as the differential expression (1) and is characterized by certain boundary conditions.

This family is also described by the Krein-type formula

$$\rho(\mathbb{L}_\theta - \lambda)^{-1}|_{\mathcal{H}_{n-2}(L_0)} = (L_0 - \lambda)^{-1} - \frac{1}{b(\lambda)(Q(\lambda) + \cot\theta)} \langle g(\,\cdot\,, \overline{\lambda}), \,\cdot\,\rangle g(\,\cdot\,, \lambda),$$

where b is a certain polynomial that reflects the regularization process and Q is an explicitly computable Q-function. The main result is the following connection between the two generalized Nevanlinna functions $Q(\lambda)$ and $m(\lambda)$.

Theorem 1. *Let the generalized Titchmarsh-Weyl-coefficient $m(\lambda)$ and the Q-function $Q(\lambda)$ be given as before. Then there exists a polynomial $p(\lambda)$ (of low degree) such that*

$$Q(\lambda) - m(\lambda) = p(\lambda) \qquad \textit{with } \deg p \leq n - 2 = \left[\sqrt{\tfrac{1}{4} + q_0}\right].$$

This result gives an operator theoretical interpretation of the generalized Titchmarsh Weyl coefficient m, which also reveals the analogy to the classical situation.

We would like to mention that already earlier, [2], in the so-called Bessel case, that is, $q_1 = 0$, a model of $L_0 + t\langle\varphi, \cdot\rangle\varphi$ was given. However, there the space and the operators are fairly abstract and, in particular, an abstract Pontryagin space rather than a Hilbert space of functions is used. Moreover, that approach is confined to the particular operator under consideration.

Our proofs make only use of the local behaviour of the solutions of (2) and can hence be generalized to all equations with only the same asymptotic properties. Finally we also use the explicitly known solutions of (2) in order to give new expansions in terms of scattered waves which are not square integrable at the origin. the corresponding scattering matrix is not trivial in the p-channel.

This is based on the joint work with P. Kurasov, [8].

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Mixed spectrum of periodically modulated Jacobi matrices

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We are interested in spectral properties of the Jacobi operator $\mathcal{J}$ acting in $l^2(\mathbb{N};\mathbb{C})$ like

$$(\mathcal{J}u)(n) = \lambda_{n-1}u(n-1) + q_n u(n) + \lambda_n u(n+1), \qquad n > 1, \tag{1}$$

$$(\mathcal{J}u)(1) = q_1 u(1) + \lambda_1 u(2), \tag{2}$$

$$D(\mathcal{J}) = \left\{ u \in l^2(\mathbb{N};\mathbb{C}) : \mathcal{J}u \in l^2(\mathbb{N};\mathbb{C}) \right\},$$

where

$$\lambda_n := n^\alpha + c_n, \qquad q_n := -2n^\alpha + b_n, \qquad n \in \mathbb{N}, \tag{3}$$

α is a real number from $(0,1)$ and (c_n), (b_n) are some real periodic sequences with period $N = 2$. This operator is in the double root case. To describe its spectrum we use asymptotic formulas of solutions of the generalized eigenequation

$$\lambda_{n-1}u(n-1) + q_n u(n) + \lambda_n u(n+1) = \lambda u(n), \qquad \lambda \in \mathbb{R},\ n > 1, \tag{4}$$

and the subordination theory [5]. The asymptotic behavior of solutions of (4) is obtained, in the non-oscillatory case, by W.Kelley's approach presented in [4](see also [1], [7], [6]) or, in the oscillatory case, using an ansatz like in [3] (see also [1] and [6]). We do not apply the methods mentioned above directly to the equation (4). It is impossible because the sequences (b_n) and (c_n) are periodic. First we split (4) into N equations (compare [9] or [7]). Then we find a basis of solutions of each of the equations. In the last step we combine all the base solutions from the previous step to obtain the asymptotics of a basis of solutions of our primary system (4),

$$u_\pm^{>}(k) \sim k^{-\alpha/4} \exp\left(\pm A k^{1-\alpha/2}\right), \quad \text{for} \quad \lambda > \frac{1}{2}(b_1 + b_2) + (c_1 + c_2),$$

and

$$u_{\pm}^{<}(k) \sim k^{-\alpha/4} \exp\left(\pm iFk^{1-\alpha/2}\right), \quad \text{for} \quad \lambda < \frac{1}{2}(b_1+b_2)+(c_1+c_2).$$

Here A and F are some positive constants depending on the spectral parameter λ, the power α and c_1, c_2, b_1, b_2 - the generators of the perturbations (c_n) and (b_n). Using the above formulas we are able to prove the following.

Theorem 1. *Let the operator $\mathcal{J}$ be defined by (1), (2) and (3).
Then we have $\left(-\infty, \frac{1}{2}(b_1+b_2)+c_1+c_2\right) \subset \sigma_{ac}(\mathcal{J})$ and in the interval $\left(\frac{1}{2}(b_1+b_2)+c_1+c_2, +\infty\right)$ we may have some discrete eigenvalues.*

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The Laplace operator on metric graphs with different boundary conditions at the vertices

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The Laplace operator on metric graphs with different boundary conditions at the vertices is investigated.

In the first part of a talk we sketch a proof of the trace formula, in order to show that it is valid for all boudary conditions leading to vertex scattering matrix independent of energy parameter k.

In the second part we review different parameterizations of matching conditions given by V. Kostrykin and R. Schrader, M. Harmer, P. Kuchment and also by ourselves. It is proven that vertex boundary conditions can be successfully parameterized by the vertex scattering matrix at the energy equal to 1. This parametrization is a slight modification of that given by M. Harmer but with clear physical meaning.

Moreover the set of matching conditions leading to energy independent vertex scattering matrices is characterised and relations with known parameterizations are established. We propose to call such matching conditions *non-resonant*. The connectivity of a metric graph is reflected by properties of scattering matrix, therefore we analyse families of scattering matrices leading to properly connecting matching conditions. We also enquire the behaviour of scattering matrices when the energy parameter tends to infinity and define asymptotical proper connectivity of boundary conditions.

On the selfadjoint subspace of one-speed Boltzmann operator

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We have studied the problem of complete non-selfajointness for Boltzmann (transport) operator corresponding to a slab of multiplicative material. By definition, a closed operator in a Hilbert space is *completely non-selfadjoint* if it has not got reducing subspaces on which it induces a selfadjoint operator. If it has, the maximal of them is called a *selfadjoint subspace*.

The transport operator L acts in the Hilbert space $L^2(\mathbb{R} \times [-1,1])$ of functions of the variables $x \in \mathbb{R}$, $\mu \in [-1,1]$ by the formula:

$$L = i\mu \frac{\partial}{\partial x} + ic(x)K, \tag{1}$$

where $c \in L^\infty(\mathbb{R})$ is a real-valued function, and K is an integral operator in the μ-variable of the form

$$K = \sum_{\ell=1}^{n} \varphi_\ell(\mu) \int_{-1}^{1} \cdot \overline{\varphi_\ell(\mu')} \, \mathrm{d}\mu', \ n < \infty.$$

The operator (1) generates the evolution of particle densities in a medium with a local multiplication coefficient $c(x)$. The collision integral K describes the angle distribution of the secondary particles.

Our main result is the following

Theorem. Assume there exist $a, \varepsilon > 0$ such that $c(x) = 0 \ \ |x - x_0 - aj| < \varepsilon$, $j \in \mathbb{Z}$, for some $x_0 \in \mathbb{R}$, and that all the functions $\varphi_\ell(\mu)$, $1 \le \ell \le n$, are polynomials. Then the selfadjoint subspace of the operator L is non-trivial, and moreover, the restriction of L to it has Lebesgue spectrum of countable multiplicity on the interval $[-\pi/a, \pi/a]$.

If c is compactly supported, a stronger assertion holds.

Proposition. If additionally the function c vanishes outside of an interval, then the restriction of the operator L to its selfadjoint subspace has Lebesgue spectrum of countable multiplicity on the whole real axis.

If the functions φ_ℓ cease to be polynomials, the operator L may be completely non-selfadjoint for arbitrary finite c.

Example. Let $n = 2$, and the functions $\varphi_{1,2} \in L^\infty(-1,1)$ are supported on $[0,1]$ and $[-1,0]$ respectively. Then the Boltzmann operator (1) is compltely non-selfadjoint if the function c does not vanish identically.

Estimating the counting function with the use of coherent states

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0. Introduction

The aim of this note is to discuss possible extensions and applications of the coherent sates technique originated in F. Berezin's paper [B]. This technique allows one to obtain estimates for eigenvalues of a self-adjoint operator under minimal assumptions about its properties. Nevertheless, it seems to be relatively little known (or well forgotten). A possible explanation is that F. Berezin did not bother to state his results in a sufficiently general and clear form, as he did not seem to be interested in obtaining estimates for eigenvalues.

In Section 1 we introduce basic notation and definitions. Section 2 contains new abstract results on coherent states which can be applied to various differential operators. It should be considered as an attempt to fill in the gaps left in Berezin's paper. Finally, in the last two sections we show how these results can be applied to obtain estimates and discuss some known inequalities and possible developments.

1. Coherent states

Let H be a Hilbert space, and let Ξ be a metric space provided with a Borel measure $\mathrm{d}\omega$.

Definition 1. If $U : H \mapsto L^2(\Xi, d\omega)$ is an isometric operator such that

1. for almost all $\omega \in \Xi$, there exists $F_\omega \in H$ such that $Uu(\omega) = (u, F_\omega)_H$, $\forall u \in H$,
2. F_ω depends continuously on ω, that is, $\|F_\omega - F_{\omega'}\|_H \to 0$ as $\omega' \to \omega$ in Ξ,

then the elements F_ω of the Hilbert space H are called *coherent states.*

Further on we shall always be assuming that U is an isometry satisfying the conditions of Definition 1 and that F_ω are coherent states. Since U is an isometry, we have $\int_\Xi |Uu(\omega)|^2 d\omega = \|Uu\|^2_{L_2(\Xi,d\omega)} = \|u\|^2_H$.

Example 1. Let Ω be a domain in the Euclidean space, $H = L_2(\Omega)$ and $\Xi = \mathbb{R}^n$. Define $F_\omega(x) = (2\pi)^{-n/2} e^{ix\cdot\omega}$ where $x \in \Omega$ and $\omega \in \mathbb{R}^n$. Then F_ω are coherent states in H. The corresponding isometry U coincides with the Fourier transform in $\mathbb{R}^n$ restricted to $L_2(\Omega)$.

Lemma 1. *If Q is a closed semibounded quadratic form on H then the function $Q[F_\omega]$ is measurable.*

Proof. A closed semibounded quadratic form on a Hilbert space is lower semicontinuous. Since F_ω continuously depends on ω, it follows that $Q_B[F_\omega]$ is lower semicontinuous and, consequently, is measurable. □

Lemma 2. *If B is a nonnegative operator of trace class then* $\operatorname{Tr} B = \int_\Xi (BF_\omega, F_\omega)_H \, \mathrm{d}\omega$.

Proof. Let $\{u_j\}$ be an orthonormal basis generated by the eigenvectors u_j of the operator B. Then

$$\begin{aligned}(BF_\omega, F_\omega)_H &= \Big(\sum_j Bu_j(F_\omega, u_j)_H, \sum_k u_k(F_\omega, u_k)_H\Big)_H \\ &= \sum_j |(F_\omega, u_j)_H|^2\,(Bu_j, u_j)_H = \sum_j |Uu_j(\omega)|^2\,(Bu_j, u_j)_H\,, \qquad \forall \omega \in \Xi\,,\end{aligned}$$

so that $\int_\Xi (BF_\omega, F_\omega)_H\,\mathrm{d}\omega = \int_\Xi \sum_j |Uu_j(\omega)|^2\,\mathrm{d}\omega\,(Bu_j, u_j)_H = \sum_j (Bu_j, u_j)_H = \operatorname{Tr} B.$ □

2. U-symbols of operators and forms

Given a semibounded self-adjoint operator B in H, we shall denote by Q_B the corresponding quadratic form with domain $\mathcal{D}(|B|^{1/2})$.

Lower bounds for the counting function of B can be obtained only under the assumption that $F_\omega \in \mathcal{D}(Q_B)$. However, in order to obtain upper bounds one has to represent the operator via the coherent states (or the isometry U). It is not always easy, as the domain of operator may be rather complex. In this section we shall discuss possible ways of finding such a representation, using the closures of quadratic forms and operators.

Definition 2. Let Q be a quadratic form on H. We shall say that a measurable real-valued function $\mathbf{b}$ on Ξ is a U-symbol of the form Q if $\mathbf{b}|Uu|^2 \in L^1(\Xi, \mathrm{d}\omega)$ and $Q[u] = \int \mathbf{b}|Uu|^2\,\mathrm{d}\omega$ for all $u \in D(Q)$.

Definition 3. Given a measurable function $\mathbf{b}$ on Ξ, we shall denote by $Q_{\mathbf{b}}$ the quadratic form $Q_{\mathbf{b}}[u] = \int_\Xi \mathbf{b}|Uu|^2\,\mathrm{d}\omega$ with domain $D(Q_{\mathbf{b}}) := \{u \in H : \mathbf{b}|Uu|^2 \in L^1(\Xi, \mathrm{d}\omega)\}$.

Clearly, $\mathbf{b}$ is a U-symbol of $Q_{\mathbf{b}}$. We shall normally be assuming that $\mathbf{b}$ satisfies

Condition 1. The function $\mathbf{b}$ is real valued and semibounded from below.

If Condition 1 is fulfilled then any quadratic form Q with a U-symbol $\mathbf{b}$ is also semibounded from below. Moreover, if $\mathbf{b}$ is bounded then Q is also bounded and

$$\operatorname{ess\,inf} \mathbf{b}(\omega)\|u\|_H^2 \;\le\; Q[u] \;\le\; \operatorname{ess\,sup} \mathbf{b}(\omega)\|u\|_H^2.$$

Lemma 3. *If Condition* 1 *is fulfilled then the form* $Q_{\mathbf{b}}$ *is closed.*

Proof. Since $\mathbf{b}$ is bounded from below, there exists a real k such that

$$\mathbf{b}(\omega) \ge k \quad \forall \omega \in \Xi.$$

Without loss of generality, we assume that $k = 1$ (otherwise we replace $\mathbf{b}$ with $\mathbf{b}-k+1$) and denote

$$\|u\|_R^2 := \int_{\mathbf{b}<R} \mathbf{b}|Uu|^2\,\mathrm{d}\omega.$$

If $u_n \in D(Q_{\mathbf{b}})$, $\|u_n - u\|_H \to 0$ and $Q_{\mathbf{b}}[u_n - u_m] \to 0$ as $n, m \to \infty$, then

$$\begin{aligned} \|u - u_n\|_R^2 &\leq \|u - u_m\|_R^2 + \|u_m - u_n\|_R^2 \\ &\leq R\|u - u_m\|^2 + Q_{\mathbf{b}}[u_m - u_n] \qquad \forall R, n, m. \end{aligned}$$

Since m can be chosen arbitrarily large, this implies that $\|u - u_n\|_R^2$ converges to zero as $n \to \infty$ uniformly with respect to R. Passing to the limit as $R \to \infty$, we see that $\mathbf{b}|Uu|^2 \in L^1(\Xi, \mathrm{d}\omega)$ and $Q_{\mathbf{b}}[u - u_n] \to 0$ as $n \to \infty$. □

Lemma 4. *Let Q_0 be a quadratic form with U-symbol $\mathbf{b}$ satisfying Condition 1, and let Q be its closure. Then $D(Q) \subset D(Q_{\mathbf{b}})$.*

Proof. For every $u \in D(Q)$, there exists a sequence $\{u_n\} \subset D(Q_0)$ such that $\|u - u_n\|_H \to 0$ and $Q_0[u_n] \to Q[u]$ as $n \to \infty$. We have

$$Q_0[u_n] \geq \|u_n\|_R^2 \geq \|u\|_R^2 - \|u - u_n\|_R^2 \geq \|u\|_R^2 - R\|u - u_n\|_H^2,$$

where $\|.\|_R$ is the same as in the proof of Lemma 3. Letting $n \to \infty$, we see that $\|u\|_R^2 \leq Q[u]$ for all $R > 0$. This implies that $\mathbf{b}|Uu|^2 \in L^1(\Xi, \mathrm{d}\omega)$. □

Lemmas 3 and 4 immediately imply the following corollaries.

Corollary 1. *Let Q_0 be a quadratic form with a U-symbol $\mathbf{b}$ satisfying Condition 1, and Q be its closure. Then Q also has the U-symbol $\mathbf{b}$. If, in addition, the form Q_0 is bounded then $\mathbf{b}|Uu|^2 \in L^1(\Xi, \mathrm{d}\omega)$ for all $u \in H$.*

Corollary 2. *If there is a constant $C > 0$ such that $\int_\Xi \mathbf{b}|Uu|^2 \,\mathrm{d}\omega \leq C\,\|u\|_H^2$ on some dense subset of H then the same inequality holds for all $u \in H$.*

Definition 4. Let B be a linear densely defined operator in H. We shall say that a function $\mathbf{b}$ on Ξ is a U-symbol of the operator B if $\mathbf{b}$ is a U-symbol of the quadratic form (Bu, u) with domain $D(B)$.

Remark 1. Clearly, $\mathbf{b}$ is a U-symbol of B if and only if $Bu = U^*\mathbf{b}\,Uu$ for all $u \in D(B)$.

In the next two theorems we shall assume that

Condition 2. f is a nonnegative real-valued convex function such that the quadratic form $Q_{f(\mathbf{b})}$ is defined on a dense subset of H and generates a trace class operator B_f.

Theorem 1. *Assume that a symmetric linear operator has a U-symbol $\mathbf{b}$ satisfying Condition 1. Let B be the self-adjoint operator generated by the closure of the form $(B_0u, u)_H$, and λ_j be its eigenvalues. If Condition 2 is fulfilled then*

$$\sum_{j=1}^{\infty} f(\lambda_j) \leq \operatorname{Tr} B_f. \tag{1}$$

Proof. Let $\{u_j\}$ be a basis in H such that $Q_B[u_j] = \lambda_j$. In view of Corollary 1 the function $\mathbf{b}$ is a U-symbol of the form Q_B. By Jensen's inequality,

$$f(\lambda_j) = f(Q_B[u_j]) = f\left(\int \mathbf{b}|Uu_j|^2 \mathrm{d}\omega\right) \leq \int f(\mathbf{b})|Uu_j|^2 \mathrm{d}\omega = (B_f u_j, u_j)_H, \quad \forall j = 1, 2, \ldots$$

Summing up over j, we obtain $\sum_{j=1}^{\infty} f(\lambda_j) \leq \sum_{j=1}^{\infty} (B_f u_j, u_j)_H \leq \operatorname{Tr} B_f$. □

Theorem 2. *Let B be the closure of B_0. If the operator B_0 has a U-symbol $\mathbf{b}$ and Condition 2 is fulfilled then* (1) *holds.*

Proof. Let u_j be the same basis as in Theorem 1. Since B is the closure of B_0, we can find $u_{j,\varepsilon} \in D(B_0)$ such that

$$|u_j - u_{j,\varepsilon}|_H \to 0 \text{ and } |Bu_j - Bu_{j,\varepsilon}|_H \to 0,$$

as $\varepsilon \to 0$. Under Condition 2, the form $Q_{f(B)}$ is bounded. Therefore

$$f(\lambda_j) = f((Bu_j, u_j)) = f((Bu_{j,\varepsilon}, u_{j,\varepsilon})) = \lim_{\varepsilon\to 0} f(\int \mathbf{b}|Uu_{j,\varepsilon}|^2 \,\mathrm{d}\omega)$$
$$\leq \lim_{\varepsilon\to 0} \int f(\mathbf{b})|Uu_{j,\varepsilon}|^2 \,\mathrm{d}\omega) = \lim_{\varepsilon\to 0} Q_{f(\mathbf{b})}[u_{j,\varepsilon}] = Q_{f(\mathbf{b})}[u_j] = (B_f u_j, u_j)\,, \qquad \forall j = 1, 2, \ldots$$

Therefore, summing up over j, we obtain (1). □

Thus we see that, in order to obtain the upper bound (1), it is sufficient to construct a coherent state representation not for the operator itself but

either for a quadratic form Q whose closure coincides with Q_B,

or for an operator B_0 whose closure coincides with B.

Example 2. The Dirichlet Laplacian on a domain Ω is the closure of the operator $-\Delta|_{C_0^\infty(\Omega)}$ with U-symbol $|\omega|^2$, provided that F_ω are chosen as in Example 1.

3. Estimates of the Counting function

Throughout this section we assume that B is a semibounded operator and denote by λ_j its eigenvalues counted with their multiplicities. Let $N(B, \lambda)$ be the number of its eigenvalues lying below λ. If $(-\infty, \lambda)$ contains the essential spectrum of B, we set $N(B, \lambda) := +\infty$.

3.1 Lower bounds

The following theorem was proved in [Sa2] and (in a slightly less general form) in [B].

Theorem 3. *Let f be a non-negative convex function such that $f(B)$ is an operator of trace class. If $F_\omega \in \mathcal{D}(Q_B)$ then*

$$\operatorname{Tr} f(B) \;\geq\; \int_\Xi f\left(\|F_\omega\|_H^{-2} Q_B[F_\omega]\right) \|F_\omega\|_H^2 \,\mathrm{d}\omega. \tag{2}$$

Proof. By the spectral theorem,

$$\begin{aligned} Q_B[F_\omega] &= \int_{\mathbb{R}} \mu \,\mathrm{d}(P(\mu)F_\omega, F_\omega)_H, \\ (f(B)F_\omega, F_\omega)_H &= \int_{\mathbb{R}} f(\mu) \,\mathrm{d}(P(\mu)F_\omega, F_\omega)_H. \end{aligned}$$

Therefore Jensen's inequality implies that

$$(f(B)F_\omega, F_\omega)_H \;\geq\; f(\|F_\omega\|_H^{-2} Q_B[F_\omega])\|F_\omega\|_H^2, \qquad \forall \omega \in \Xi.$$

Now, by Lemma 2, $\operatorname{Tr} f(B) = \int_\Xi (f(B)F_\omega, F_\omega)_H \, d\omega \geq \int_\Xi f(\|F_\omega\|_H^{-2} Q_B[F_\omega])\|F_\omega\|_H^2 \, d\omega$.
□

In particular, we see $f(B)$ does not belong to the trace class whenever the integral on the right hand side of (2) is infinite. Theorem 3 also implies the following corollary.

Corollary 3. *If the interval* $(-\infty, \lambda)$ *does not contain the essential spectrum of* B *and* $F_\omega \in \mathcal{D}(Q_B)$ *then*

$$(\lambda - \mu)\, N(B, \lambda) + \int_{-\infty}^{\mu} N(B, \tau)\, \mathrm{d}\tau \;\geq\; \int_\Xi (\lambda \|F_\omega\|_H^2 - Q_B[F_\omega])_+ \, \mathrm{d}\omega, \qquad \forall \mu \leq \lambda. \quad (3)$$

Proof. Since $N(B, \tau)$ is non-decreasing, applying the theorem to the convex function $f_\lambda(\tau) = (\lambda - \tau)_+$, we obtain

$$\begin{aligned} (\lambda - \mu) N(B, \lambda) + \int_{-\infty}^{\mu} N(B, \tau)\, d\tau &\geq \int_{-\infty}^{\lambda} N(B, \tau)\, d\tau \\ &= \operatorname{Tr} f_\lambda(B) \geq \int_\Xi (\lambda \|F_\omega\|_H^2 - Q_B[F_\omega])_+ \, \mathrm{d}\omega\,. \end{aligned}$$

□

Enumerating the eigenvalues lying below λ in the increasing order and taking $\lambda = \mu = \lambda_k$ in (3), we see that

$$\int_\Xi (\lambda_k \|F_\omega\|_H^2 - Q_B[F_\omega])_+ \, d\omega \;\leq\; \int_{-\infty}^{\lambda_k} N(B, \tau)\, d\tau \;=\; k\lambda_k - \sum_{j=1}^{k} \lambda_j\,. \qquad (4)$$

Substituting in (3) $\mu = \lambda_1$, where λ_1 is the minimal eigenvalue, we obtain

Corollary 4. *Under the conditions of Corollary* 3*,*

$$N(B, \lambda) \geq (\lambda - \lambda_1)^{-1} \int_\Xi (\lambda \|F_\omega\|_H^2 - Q_B[F_\omega])_+ \, d\omega, \quad \lambda > \lambda_1\,. \qquad (5)$$

Example 3. In [La], the inequality (5) with F_ω as in Example 1 was implicitly used to obtain the estimate

$$N(-\Delta_{\mathrm{N}}, \lambda) \;\geq\; \left(\frac{2}{n+2}\right) C_n\, |\Omega|\, \lambda^{n/2}\,, \qquad \forall \lambda > 0\,,$$

for the Neumann Laplacian on a domain Ω. Here C_n is the constant appearing in the Weyl asymptotic formula and $|\Omega|$ is the volume of Ω.

Our abstract theorems allow one to extend the above estimate to a general elliptic operator with the Neumann boundary condition. Choosing coherent states which vanish near the boundary, one can also obtain lower bounds for operators satisfying other boundary conditions (see, for example, [Sa2]).

3.2 Upper bounds

The following theorem clarifies the role of U-symbols.

Theorem 4. *Let the conditions of Theorem* 1 *or Theorem* 2 *be fulfilled. Then for any convex nonnegative function* f *we have*

$$\sum_{j=1}^{\infty} f(\lambda_j) \;\le\; \int_{\Xi} f\left(\mathbf{b}(\omega)\right) \|F_\omega\|_H^2 \,\mathrm{d}\omega\,. \tag{6}$$

Proof. Let u_j be orthonormal eigenvectors corresponding to the eigenvalues λ_j, and let $\{u_k'\}$ be an orthonormal basis in the orthogonal complement to the subspace spanned by u_j. The estimate (1) implies that

$$\begin{aligned}\sum_{j=1}^{\infty} f(\lambda_j) \le \operatorname{Tr} B_f &= \sum_{j=1}^{\infty}(B_f u_j, u_j) + \sum_{k=1}^{\infty}(B_f u_k', u_k')\\ &= \sum_{j=1}^{\infty}\int_{\Xi} f(\mathbf{b}(\omega))|U u_j(\omega)|^2\, d\omega + \sum_{k=1}^{\infty}\int_{\Xi} f(\mathbf{b}(\omega))|U u_k'(\omega)|^2\, d\omega\\ &= \int_{\Xi} f(\mathbf{b}(\omega))\left(\sum_{j=1}^{\infty}|(u_j, F_\omega)_H|^2 + \sum_{j=1}^{\infty}|(u_k', F_\omega)_H|^2\right) d\omega\end{aligned}$$

where, by Bessel's equality, the right hand side is equal to $\int_{\Xi} f(\mathbf{b}(\omega))\|F_\omega\|_H^2\, d\omega$. □

Applying Theorem 4 to $f(\tau) = (\lambda-\tau)_+$ and taking into account the monotonicity of $N(B,\lambda)$, we obtain

Corollary 5. *If the interval* $(-\infty,\lambda)$ *does not contain the essential spectrum of* B *then*

$$(\lambda-\mu)\,N(B,\mu) + \int_{-\infty}^{\mu} N(B,\tau)\,d\tau \le \int_{\Xi}(\lambda-\mathbf{b}(\omega))_+\|F_\omega\|^2\,d\omega, \qquad \forall \mu\le\lambda. \tag{7}$$

The easiest way to deduce an upper bound for $N(B,\mu)$ from (7) is to estimate

$$N(B,\lambda) \;\le\; \varepsilon^{-1}\int_{-\infty}^{\lambda+\varepsilon} N(B,\tau)\,d\tau \;\le\; \varepsilon^{-1}\int_{\Xi}(\lambda+\varepsilon-\mathbf{b}(\omega))_+\|F_\omega\|^2\,d\omega \tag{8}$$

and to optimize the choice of $\varepsilon > 0$.

Example 4. For the Dirichlet Laplacian on an n-dimensional domain Ω, the inequality (7) with $\mu=\lambda$ and F_ω defined in Example 1 takes the form

$$\int_0^{\lambda} N(-\Delta_{\mathrm{D}},\mu)\,\mathrm{d}\mu \;\le\; \left(\frac{2}{n+2}\right)\, C_n\,|\Omega|\,\lambda^{n/2+1} \tag{9}$$

where C_n is the Weyl constant and $|\Omega|$ is the volume of Ω. This inequality was proved by F. Berezin (see [B]). In [La], A. Laptev used (9) and (8) to give a new proof of the Li–Yao estimate

$$N_{-\Delta}(\lambda) \le (1+2/n)^{n/2} C_n\,|\Omega|\,\lambda^{n/2}$$

previously obtained in [LY] by a different method.

4. Concluding remarks

A more sophisticated way of getting two-side estimates for $N(B,\lambda)$, which does not seem to have been exploited in research papers but should give better results, is to use the estimate

$$\varepsilon^{-1}\left(\int_{-\infty}^{\lambda} N(B,\tau)\,d\tau - \int_{-\infty}^{\lambda-\varepsilon} N(B,\tau)\,d\tau\right) \;\le\; N(B,\lambda) \;\le\; \varepsilon^{-1}\left(\int_{-\infty}^{\lambda+\varepsilon} N(B,\tau)\,d\tau - \int_{-\infty}^{\lambda} N(B,\tau)\,d\tau\right) \quad (10)$$

instead of (8), and then to apply **both** inequalities (3) and (7) in the right and left hand sides. Remarkably, when using this approach, every improvement in the lower bound for the integral $\int_{-\infty}^{\lambda} N(B,\tau)\,d\tau$ leads to an improvement of the upper bound for $N(B,\lambda)$, and the other way round.

One can argue that, for a differential operator B with boundary conditions, it is not easy to find coherent states which lie in $\mathcal{D}(Q_B)$ (that is, satisfy the boundary condition) and, at the same time, diagonalize the quadratic form Q_B in the sense of Definition 2. However, one can use different coherent states in (3) and (7).

For a differential operator on a closed manifold, this problem does not arise. In particular, on a Riemannian manifold without boundary, coherent states can be constructed with the use of global phase functions introduced in [Sa1], which play the role of $e^{ix\cdot\omega}$ in Example 1. This has been done in [So], where the author used (10) to obtain explicit two-side estimates for the counting function of the Laplace–Beltrami operator. These estimates imply, in particular, the Weyl asymptotic formula with an order sharp reminder estimate under limited assumptions on the smoothness of the manifold.

A similar but slightly different approach was used in [ELSS] for the study of semi-classical asymptotics for the Dirac and Schrödinger operators on $\mathbb{R}^n$.

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On the maps associated with the inverse Sturm-Liouville problems in the scale of Sobolev spaces. Uniform stability

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We shall deal with the Sturm-Liouville operator

$$Ly = -y'' + q(x)\, y, \qquad x \in [0, \pi], \tag{1}$$

viewing in mind the classical inverse problems.† Our main goal is to prove the estimates which guarantee the uniform stability for solutions of the inverse problems. We shall solve the direct and inverse problems for potentials q belonging to Sobolev space $W_2^\theta[0, \pi]$ for fixed $\theta \geqslant -1$. However, the obtained results are new for the classical case when the potential q belongs to the space $L_2[0, \pi]$.

We shall use the language of nonlinear maps in Hilbert spaces to formulate the direct and inverse Sturm-Liouville problems and benefit much of using this language. Working with potentials $q \in W_2^\theta[0, 1]$ (with fixed $\theta \geqslant -1$) we have to define the spaces where the spectral data have to be placed in. The construction of these spaces is an important step to treat the problems in a new setting. We remark that in the classical case of L_2-potentials with zero mean value the language of nonlinear maps for investigating of some inverse problems was introduced by Pöschel and Trubowitz [11]. However, if we like to investigate the inverse problems for potentials in the whole scale of Sobolev spaces (including negative smooth indices) we must construct new spaces where the spectral data live in.

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Of course, the spectral data for different inverse problems have to be placed in different Hilbert spaces. It turns out, however, that all these spaces are finite dimensional extensions of usual weighted l_2 spaces.

Now let us organize these ideas in a rigorous setting and formulate the main results. First we remind that the definition of the Sturm-Liouville operators with the classical potentials $q \in L_1[0,1]$ can be extended for distribution potentials q belonging to the Sobolev space $W_2^{-1}[0,\pi]$. Suppose that a complex valued function q belongs to Sobolev space $W_2^{\alpha}[0,\pi]$ with some $\alpha \geqslant -1$. Set $\sigma(x) = \int q(x)\,dx$, where the primitive in understood in the sense of distributions. Following the paper [12] (see also [13] for more details), we define the Dirichlet operator by the equality

$$L_D y = Ly = -(y^{[1]})' - \sigma(x)y^{[1]} - \sigma^2(x)y, \quad y^{[1]}(x) := y'(x) - \sigma(x)y(x), \tag{2}$$

on the domain

$$\mathcal{D}(L_D) = \{y,\, y^{[1]} \in W_1^1[0,\pi] \ \mid \ y(0) = y(\pi) = 0\}.$$

The Dirichlet-Neumann operator is defined similarly: $L_{DN}y = Ly$ on the domain

$$\mathcal{D}(L_{DN}) = \{y,\, y^{[1]} \in W_1^1[0,\pi] \ \mid \ y(0) = y^{[1]}(\pi) = 0\}.$$

For smooth functions σ the right hand-sides of (1) and (2) coincide and we get the classical Sturm-Liouville operators with Dirichlet and Dirichlet-Neumann boundary conditions (in the latter case we have to assume in addition $\int_0^\pi q(x)\,dx + \sigma(0) = \sigma(\pi) = 0$).

Denote by $s(x,\lambda)$ a unique solution of the equation $Ly - \lambda y = 0$ that satisfies the conditions $s(0,\lambda) = 0$ and $s^{[1]}(0,\lambda) = \sqrt{\lambda}$ (it is known [12] that such a solution does exists). Obviously, the zeros $\{\lambda_k\}_1^\infty$ and $\{\mu_k\}_1^\infty$ of the entire functions $s(1,\lambda)$ and $s^{[1]}(1,\lambda)$ coincide with the eigenvalues of the operator L_D and L_{DN}, respectively. We enumerate these sequences in such a way that the sequences $\{|\lambda_k|\}_1^\infty$ and $\{|\mu_k|\}_1^\infty$ are asymptotically increasing (in the case of a real potential q all the zeros of the functions $s(1,\lambda)$ and $s^{[1]}(1,\lambda)$ are simple and real; so, they can be enumerated in the increasing order for all indices $k \geqslant 1$). Introduce also the numbers

$$\alpha_k = \int_0^\pi s^2(x,\lambda_k)\,dx, \quad \beta_k = \int_0^\pi s^2(x,\mu_k)\,dx,$$

which in the case of real potentials are called *the norming constants* (we save this definition for complex potentials). The sequences

$$\{\lambda_k\}_1^\infty \cup \{\alpha_k\}_1^\infty \quad \text{and} \quad \{\mu_k\}_1^\infty \cup \{\beta_k\}_1^\infty$$

form the so-called spectral data of the operators L_D and L_{DN}, respectively. Further, investigating the operator L_D, it will be more convenient to work with the numbers

$$s_{2k} = \sqrt{\lambda_k} - k, \qquad s_{2k-1} = \alpha_k - \pi/2, \qquad k = 1, 2, \ldots.$$

We say that the sequence $\{s_k\}_1^\infty = \{s_k(D)\}_1^\infty$ defines *the regularized spectral data of the operator* L_D. Analogously, we can consider the numbers

$$s_{2k-1} = \sqrt{\mu_k} - (k - 1/2), \qquad s_{2k} = \beta_k - \pi/2, \qquad k = 1, 2, \dots.$$

Then the sequence $\{s_k\}_1^\infty = \{s_k(DN)\}_1^\infty$ defines *the regularized spectral data of the operator* L_{DN}. We imply that in all the above formulae the branches of the squire roots are fixed. We assume, for example, that the arguments of the numbers $\sqrt{\lambda_k}$ and $\sqrt{\mu_k}$ lie in the segment $(-\pi/2, \pi/2]$.

It was first discovered by Borg [1] that two sequences $\{\lambda_k\}$ and $\{\mu_k\}$ define a potential q in a unique way (at least if q is real). Equivalently, we can work with the numbers

$$s_{2k-1} = \sqrt{\mu_k} - (k - 1/2), \qquad s_{2k} = \sqrt{\lambda_k} - k, \qquad k = 1, 2, \dots.$$

Then the sequence $\{s_k\}_1^\infty = \{s_k(B)\}_1^\infty$ defines *the regularized spectral data of the Borg problem.*

Now we come to the following problems: to prove that the spectral data define a potential in a unique way. Next step is to give a complete characterization of the spectral data provided that the potential runs through a given space (in the classical case it is the space $L_2[0, \pi]$). These problems have been investigated starting from pioneering papers of Borg [1], Marchenko [9], Gelfand and Levitan [2]. For historical overview we refer the readers to the books of Marchenko [10], Levitan [8], Freiling and Yurko [3] and to a recent paper of Hryniv and Mykytyuk [6].

To cast a new light on these problems we shall introduce the special Hilbert spaces associated with the spectral data. Certainly these spaces will be different for different inverse spectral problems.

Denote by l_2^θ the usual weighted l_2 space consisting of the sequences of complex numbers $x = \{x_1, x_2, \dots\}$ such that

$$\|x\|_\theta := \sum_1^\infty |x_k|^2 \, k^{2\theta} < \infty.$$

Consider the sequences

$$e^{2s-1} = \{(2k)^{-(2s-1)}\}_{k=1}^\infty, \quad e^{2s} = \{(2k)^{-(2s)}\}_{k=1}^\infty, \qquad s = 1, 2, \dots.$$

Remark that the sequence e^p belongs to the spaces l_2^θ for $0 \leqslant \theta < p - 1/2$ and $e^p \notin l_2^\theta$ for $\theta \geqslant p - 1/2$. For a fixed $\theta \geqslant 0$ there is a unique integer m such that $m - 1/2 \leqslant \theta < m + 1/2$. For this number θ define l_D^θ as a finite-dimensional extension of the space l_2^θ, namely

$$l_D^\theta = l_2^\theta \oplus span\{e^k\}_{k=1}^m.$$

Thus, l_D^θ consists of elements $\hat{x} = x + \sum_{k=1}^m c_k e^k$, where $x \in l_2^\theta$ and $\{c_k\}_1^m$ are complex numbers. The inner product of elements $\hat{x},\ \hat{y} \in l_D^\theta$ is defined by the formula

$$(\hat{x}, \hat{y})_\theta = (x, y)_\theta + \sum_{k=1}^m c_k \overline{d_k},$$

where $(x, y)_\theta$ is the inner product in l_2^θ, and d_k are the coefficients of e^k in the decomposition of $\hat{y}$.

Similarly, we construct the spaces corresponding to the spectral data of the operator L_{DN} and to the spectral data of the Borg problem. Namely, introduce the sequences

$$\tilde{e}^{\,2s-1} = \{(2k+1)^{-(2s-1)}\}_{k=1}^{\infty}, \qquad \tilde{e}^{\,2s} = \{(2k-1)^{-(2s)}\}_{k=1}^{\infty}, \qquad s = 1, 2, \dots,$$
$$\hat{e}^{\,2s-1} = \{k^{-(2s-1)}\}_{k=1}^{\infty}, \qquad \tilde{e}^{\,2s} = \{(-1)^k\, k^{-(2s-1)}\}_{k=1}^{\infty}, \qquad s = 1, 2, \dots,$$

and define the spaces

$$l_{DN}^\theta = l_2^\theta \oplus span\{\tilde{e}^{\,k}\}_{k=1}^m, \qquad l_D N^\theta = l_2^\theta \oplus span\{\hat{e}^{\,k}\}_{k=1}^m.$$

We imply that the inner product in these spaces is defined in the same way as in l_D^θ.

First we note that $l_D^\theta = l_{DN}^\theta = l_B^\theta = l_2^\theta$ for $0 \leqslant \theta < 1/2$. This follows from the definition. Then we note also that all these spaces with $\theta > 0$ are continuously embeded in l_2 since all the sequences $e^{\,s}, \tilde{e}^{\,s}, \hat{e}^{\,s}$ belong to l_2.

Now let us define the operators

$$F_D\ \sigma = \{s_k(D)\}_1^\infty, \quad F_{DN}\ \sigma = \{s_k(DN)\}_1^\infty, \quad F_B\ \sigma = \{s_k(B)\}_1^\infty. \tag{3}$$

It follows from results of [13] and [4] that regularized spectral data in the right hand-sizes of (3) are the sequences from l_2 (for any primitive $\sigma = \int q(x)\,dx$). Hence, all the operators in (3) are well defined as operators from L_2 to l_2. We will see more below: the operators F_D, F_{DN} and F_B map the Sobolev space W_2^θ (for any fixed $\theta \geqslant 0$) into l_D^θ, l_{DN}^θ and l_B^θ, respectively.

It is proved in [15, 16] that the maps F_D, F_{DN} and F_B are Frechet differentiable at every point (function) σ, provided that it is real valued, in particular, at the point $\sigma = 0$. The Frechet derivatives at this point are linear operators T_D, T_{DN}, T_B which are defined by the formulae

$$\begin{cases} (T_D\ \sigma)_{2k-1} = -\frac{1}{\pi}\int\limits_0^\pi \sigma(t)\sin(2kt)\,dt, \quad k = 1, 2, \dots, \\ (T_D\ \sigma)_{2k} = -\int\limits_0^\pi (\pi - t)\sigma(t)\cos(2kt)\,dt, \quad k = 1, 2, \dots; \end{cases}$$

$$\begin{cases} (T_{DN}\ \sigma)_{2k-1} = -\frac{1}{\pi}\int\limits_0^\pi \sigma(t)\sin((2k-1)t)\,dt, \quad k = 1, 2, \dots, \\ (T_{DN}\ \sigma)_{2k} = -\int\limits_0^\pi (\pi - t)\sigma(t)\cos((2k-1)t)\,dt, \quad k = 1, 2, \dots; \end{cases}$$

$$(T_D\ \sigma)_k = -\frac{1}{\pi}\int\limits_0^\pi \sigma(t)\sin(kt)\,dt, \quad k = 1, 2, \dots.$$

The next proposition helps to understand the importance of the spaces l_D^θ, l_{DN}^θ and l_B^θ.

Proposition 1.*Any given $\theta \geqslant 0$ the operators T_D and T_{DN} map the spaces $W_2^\theta \ominus \{1\}$ onto l_D^θ and l_{DN}^θ isomorphically. The operator T_B maps the space W_2^θ onto l_B^θ isomorphically.*

The proof of this proposition is given in [14, 16].

The next statement is the most difficult and the most essential result in the theory.

Theorem 2. *Fix any $\theta \geqslant 0$. The operator F_D maps the space W_2^θ into l_D^θ and admit a representation in the form*

$$F_D(\sigma) = T_D\ \sigma + \Phi_D(\sigma),$$

where T_D is the linear operator defined in Proposition 1 and Φ_D maps the space W_2^θ into l_D^τ where

$$\tau = \begin{cases} 2\theta, & \text{if } 0 \leqslant \theta \leqslant 1, \\ \theta + 1, & \text{if } 1 \leqslant \theta < \infty. \end{cases}$$

Moreover, $\Phi_D : W_2^\theta \to l_D^\tau$ is bounded at any ball, i.e.

$$\|\Phi(\sigma)\|_\tau \leq C\|\sigma\|_\theta,$$

with a constant C depending only on R. The same results are valid for the operators F_{DN} and F_B. Namely,

$$F_{DN}(\sigma) = T_{DN}\ \sigma + \Phi_{DN}(\sigma), \quad F_B(\sigma) = T_B\ \sigma + \Phi_B(\sigma),$$

and the maps Φ_{DN} and Φ_B have the same properties as Φ_D.

We remark that the embedding $l^\eta \hookrightarrow l^\theta$ is compact, provided that $\eta > \theta$ (we omit here the indices D, DN or B). Hence, from the above theorem we obtain: The maps F_D, F_{DN} and F_B are weakly nonlinear, i.e. they are compact perturbations of linear maps.

There is no way to describe the image of the maps F_D, F_{DN} or F_B acting from W_2^θ. However, the image of all real functions on W_2^θ we can describe explicitly. Let us first work with the map F_D. Fix a number $h \geqslant 0$ and define $\lambda_k := (s_{2k} + k)^2$. Denote by $\Sigma_{r,h}^\theta$ the set of all sequences in the ball of radius r in the space l_D^θ such that all the numbers λ_k are real and

$$s_{2k-1} > h, \quad \lambda_{k+1} - \lambda_k > h \quad \text{for all } k \geqslant 1. \tag{4}$$

In the case $h = 0$ and $r = \infty$ we denote this set by Σ^θ. The next theorem gives the solution of the inverse problem.

Theorem 2. *Denote by $W_{\mathbb{R},0}^\theta$ the set of all real functions $\sigma \in W_2^\theta$ such that $\int_0^\pi \sigma(x)\,dx = 0$, and by B_R^θ the ball of radius R in the set $W_{\mathbb{R},0}^\theta$. Any given $\theta \geqslant 0$ the map $F_D : W_{\mathbb{R},0}^\theta \to \Sigma^\theta$ is bijective. This map is real analytic at any point $\sigma \in W_{\mathbb{R},0}^\theta$. In the case $\theta > 0$ the Frechet derivative $d_\sigma F_D$ of this map is uniformly bounded in any ball B_R^θ. The image $F_D(B_R^\theta)$ is contained in the set $\Sigma_{r,h}^\theta$ with some $r, h > 0$ depending only on R. Conversely, pre-image $F^{-1}(\Sigma_{r,h}^\theta)$ is contained in a ball B_R^θ with*

R depending on r and h. For all sequences $\{s_k\}, \{\tilde{s}_k\} \in \Sigma^\theta_{r,h}$ the following estimates are valid

$$C_1\|\sigma - \tilde{\sigma}\|_\theta \leqslant \|\{s_k\} - \{\tilde{s}_k\}\|_\theta \leqslant C_2\|\sigma - \tilde{\sigma}\|_\theta, \tag{5}$$

where $\sigma = F^{-1}(\{s_k\}), \tilde{\sigma} = F^{-1}(\{\tilde{s}_k\})$, and the constants C_1 and C_2 depend only on r and h.

We remark that the estimate (5) is new even in the classical case $\theta = 1$. It expresses the uniform stability for solutions of the inverse problem. The dependence of the constants C_1 and C_2 on r and h is essential.

The same results become valid after some obvious changes for the maps F_{DN} and F_B corresponding to the other inverse problems. In formulation of a similar result for the map F_{DN} we have to replace the space l^θ_D by l^θ_{DN} and in the definition of the set $\Sigma^\theta_{r,h}$ to replace the condition (4) by

$$s_{2k} > h, \quad \mu_{k+1} - \mu_k > h \quad \text{for all } k \geqslant 1,$$

where $\mu_k := (s_{2k-1} + k - 1/2)^2$.

To formulate an analog of Theorem 2 for the map F_B one has to make the following changes. First, the space l^θ_D has to be replaced by l^θ_B and the set $W^\theta_{\mathbb{R},0}$ by the set of all real functions in W^θ_2. The set $\Sigma^\theta_{r,h}$ has to be define in this case in the following way. It consists of all sequences $\{s_k\}^\infty_{k=1} \in l^\theta_B$ such they lie in the ball of radius r in l^θ_B, the numbers $\lambda_k := (s_{2k} + k)^2$ and $\mu_k := (s_{2k-1} + k - 1/2^2$ are real and

$$\lambda_k - \mu_k > h, \quad \mu_k - \lambda_{k-1} > h \quad \text{for all} \quad k \geqslant 1.$$

The proof of Theorem 2 is based essentially on Theorem 1.

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Weyl-Titchmarsh type formula for Hermite operator

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Abstract

Small perturbations of the Jacobi matrix with weights $a_n = \sqrt{n}$ and zero diagonal are considered. A formula relating the asymptotics of polynomials of the first kind to the spectral density is stated, which is analogous to the Weyl-Titchmarsh formula for the Schrödinger operator on the half-line with summable potential.

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Let q be a summable real-valued function on $\mathbb{R}_+$, $q \in Ł_1(\mathbb{R}_+)$. Consider the Schrödinger operator on $\mathbb{R}_+$

$$\mathcal{L} = -\frac{d^2}{dx^2} + q(x)$$

with the Dirichlet boundary condition. The purely absolutely continuous spectrum of this operator coincides with $\mathbb{R}_+$ [4]. Let $\varphi(x, \lambda)$ be a solution of the spectral equation for $\mathcal{L}$,

$$-u''(x, \lambda) + q(x)u(x, \lambda) = \lambda u(x, \lambda),$$

such that $\varphi(0, \lambda) \equiv 0$, $\varphi'(0, \lambda) \equiv 1$ (satisfying the initial conditions). The following classical result holds [4].

Proposition 1. *If $q \in L_1(\mathbb{R}_+)$, then for every $k > 0$ there exist $a(k)$ and $b(k)$ such that*

$$\varphi(x, k^2) = a(k)\cos(kx) + b(k)\sin(kx) + o(1) \text{ as } x \to +\infty,$$

and for a.a. $\lambda > 0$

$$\rho'(\lambda) = \frac{1}{\pi\sqrt{\lambda}(a^2(\sqrt{\lambda}) + b^2(\sqrt{\lambda}))}$$

(the Weyl-Titchmarsh formula).

In the present note the analogous result for the Hermite operator is stated, see [12] for the details. "Free" Hermite operator is a Jacobi matrix with weights $\{\sqrt{n}\}_{n=1}^{\infty}$ and zero diagonal,

$$\mathcal{J}_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & \sqrt{2} & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let us call Jacobi matrix

$$\mathcal{J} = \begin{pmatrix} b_1 & a_1 & 0 & \cdots \\ a_1 & b_2 & a_2 & \cdots \\ 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

the Hermite operator with small perturbation, if it is defined by the sequences of weights $\{a_n\}_{n=1}^{\infty}$ and diagonal $\{b_n\}_{n=1}^{\infty}$ such that (let $c_n := a_n - \sqrt{n}$)

$$c_n = o(\sqrt{n}) \text{ as } n \to \infty \text{ and } \sum_{n=1}^{\infty}\left(\frac{|c_n|}{n} + \frac{|c_{n+1} - c_n| + |b_n|}{\sqrt{n}}\right) < \infty. \tag{1}$$

The spectrum of $\mathcal{J}_0$ is purely absolutely continuous on the whole real line with the spectral density

$$\rho_0'(\lambda) = \frac{e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}}.$$

Spectral density of $\mathcal{J}$ can be studied using the asymptotic analysis of generalized eigenvectors related to $\mathcal{J}$, i.e., solutions of the spectral equation

$$a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} = \lambda u_n, \ n \geq 2. \tag{2}$$

The method is based upon the comparison of solutions of (2) to solutions of the spectral equation for the free Hermite operator,

$$\sqrt{n-1}u_{n-1} + \sqrt{n}u_{n+1} = \lambda u_n, \ n \geq 2. \tag{3}$$

This situation is analogous to the one described by the Weyl-Titchmarsh theory for the Schrödinger operator on the half-line with the summable potential. The following results hold [12].

Theorem 1. *For every complex* λ *equation* (3) *has two linearly independent solutions* $I_n^+(\lambda)$ *and* $I_n^-(\lambda)$*, which are entire functions of* λ *for every* n *and have the following asymptotics as* $n \to \infty$*:*

$$I_n^{\pm}(\lambda) = \frac{\pm i e^{\frac{\lambda^2}{4}}}{(8\pi)^{1/4}} \frac{(\mp i)^n e^{\pm i\lambda\sqrt{n}}}{n^{1/4}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).$$

These asymptotics are uniform with respect to λ *in every compact set in* $\mathbb{C}$*.*

Solutions $I_n^+(\lambda)$ and $I_n^-(\lambda)$ are the direct analogues to the solutions $\frac{e^{ikx}}{2ik}$ and $\frac{e^{-ikx}}{-2ik}$ of the spectral equation for "free" Schrödinger operator,

$$-u''(x,k^2) = k^2 u(x,k^2).$$

The main technical difficulty of our problem is non-triviality of solutions $I_n^{\pm}(\lambda)$ compared to $\frac{e^{\pm ikx}}{\pm 2ik}$. The model of the Hermite operator was studied in the paper of Brown-Naboko-Weikard [6], but solutions $I_n^{\pm}(\lambda)$ were not introduced there.

Theorem 2. *[12] Let* (1) *hold. Then*

- *For every* $\lambda \in \mathbb{R}$ *there exists* $F(\lambda)$ *(the Jost function) such that*

  $$P_n(\lambda) = F(\lambda)I_n^-(\lambda) + \overline{F(\lambda)}I_n^+(\lambda) + o(n^{-1/4}) \ \text{as } n \to \infty.$$

  Function F *is continuous and non-vanishing on* $\mathbb{R}$*.*

- *The spectrum of* $\mathcal{J}$ *is purely absolutely continuous, and for a.a.* $\lambda \in \mathbb{R}$

  $$\rho'(\lambda) = \frac{e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}|F(\lambda)|^2}$$

  (the Weyl-Titchmarsh type formula).

The previous theorem can be proven by another method, based on the Levinson-type analytical and smooth theorem, cf. [3] and papers of Bernzaid-Lutz [5], Janas-Moszyński [8] and Silva [10], [11].

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Tridiagonal block matrices and canonical moments

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In this talk we present the connection between matrix measures and random walks with a block tridiagonal transition matrix. We derive sufficient conditions such that the blocks of the n-step block tridiagonal transition matrix of the Markov chain can be represented as integrals with respect to a matrix valued spectral measure. Several stochastic properties of the processes are characterized by means of this matrix measure. In many cases this measure is supported in the interval [-1, 1]. The results are illustrated by several examples including random walks on a grid.

Consider a homogeneous Markov chain with state space $\mathcal{C}_d = \{(i,j) \in \mathbb{N}_0 \times \mathbb{N} \mid 1 \leq j \leq d\}$ and block tridiagonal transition matrix

$$P = \begin{pmatrix} B_0 & A_0 & & & 0 \\ {C_1}^T & B_1 & A_1 & & \\ & {C_1}^T & B_2 & A_2 & \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}$$

where $d \in \mathbb{N}$ is finite, and $A_0, A_1, \ldots, B_0, B_1, \ldots, C_1, C_2, \ldots$ are $d{\times}d$ matrices containing the probabilities of one-step transitions. This means that the probability of going in one step from state (i,j) to (i',j') is given by the element in the position (j,j') of the matrix $P_{i,i'}$, where the one-step block tridiagonal transition matrix is represented by $P = \left(P_{i,i'}\right)_{i,i'=0,1,\ldots}$.

Matrices P of the above form are closely related to a sequence of matrix polynomials recursively defined by

$$xQ_n(x) = A_n Q_{n+1}(x) + B_n Q_n(x) + {C_n}^T Q_{n-1}(x), \qquad n \in \mathbb{N},$$

where $Q_{-1}(x) = 0$ and $Q_0(x) = I$.