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IM PAN Preprint 720 (2010)

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Presented by Piotr M. Hajac

Published as manuscript

Received 05 May 2010

Mathematische Zeitschrift manuscript No.
(will be inserted by the editor)

Distributive lattices and cohomology

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Received: date / Accepted: date

Abstract A resolution of the intersection of a finite number of subgroups of an abelian group by means of their sums is constructed, provided the lattice generated by these subgroups is distributive. This is used for detecting singularities of modules over Dedekind rings. A generalized Chinese remainder theorem is derived as a consequence of the above resolution. The Gelfand-Naimark duality between finite closed coverings of compact Hausdorff spaces and the generalized Chinese remainder theorem is clarified.

Keywords Distributive lattice · Cohomology · Chinese Remainder Theorem

Mathematics Subject Classification (2000) 06D99, 46L52, 13D07, 13F05, 16E60.

1 Introduction

The Gelfand-Naimark duality identifies lattices of closed subsets in compact Hausdorff spaces with lattices opposite to surjective systems of quotients of unital commutative C*-algebras. Therefore, given a finite collection $I_0, \dots, I_n$ of closed *-ideals in a C*-algebra $A = C(X)$ of continuous functions on a compact Hausdorff space X, it identifies coequalizers in the category of compact Hausdorff spaces ($V(I) \subset X$ is the zero locus of the ideal $I \subset A = C(X)$)

$$\bigcup_{\alpha=0}^{n} V(I_\alpha) \leftarrow \coprod_{\alpha=0}^{n} V(I_\alpha) \leftleftarrows \coprod_{\alpha,\beta=0}^{n} V(I_\alpha) \cap V(I_\beta) \tag{1}$$

with equalizers in the category of unital commutative C*-algebras

$$A/\bigcap_{\alpha=0}^{n} I_\alpha \to \prod_{\alpha=0}^{n} A/I_\alpha \rightrightarrows \prod_{\alpha,\beta=0}^{n} A/I_\alpha + I_\beta. \tag{2}$$

The author was partially supported by KBN grants N201 1770 33 and 115/E-343/SPB/6.PR UE/DIE 50/2005-2008.

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In particular, finite families of closed *-ideals intersecting to zero correspond to finite families of closed subsets covering X. In general, lattices of closed *-ideals in commutative unital C*-algebras are always distributive, since they are isomorphic by the Gelfand-Naimark duality to lattices opposite to sublattices of subsets. Therefore one can think about finite families of closed subsets in a compact Hausdorff space as of finite subsets in a distributive lattice of ideals. By Hilbert's Nullstellensatz one can replace a compact Hausdorff space and its closed subsets by an affine algebraic set X over an algebraically closed field and its algebraic subsets on one hand, and closed ideals in a C*-algebra by radical ideals in the algebra $\mathcal{O}[X]$ of polynomial functions on X, on the other hand. One can take also a finite set of monomial ideals in a ring of polynomials over a field [5] as well as a finite set of congruences in the ring of integers and the family of corresponding ideals. In all above cases the fact that the diagram (2) is an equalizer is a consequence of distributivity of a corresponding lattice of ideals, and in view of the last example can be regarded as a generalized Chinese remainder theorem. More examples can be obtained from the fact that every algebra of finite representation type has distributive lattice of ideals [12] and the property of having distributive lattice of ideals is Morita invariant and open under deformations of finite dimensional algebras [14].

The aim of the present paper is to show that the generalized Chinese theorem is a consequence of vanishing of first cohomology of a canonical complex associated with a finite number of members $I_0, \ldots, I_n$ of a distributive lattice L of subgroups of an abelian group A. The respective vanishing theorem (Theorem 1) depends only on that lattice. Since it is independent of the ambient abelian group A, Theorem 1 is prior to the generalized Chinese remainder theorem. It is also more general, since it claims vanishing of the whole higher cohomology. This can be used for computing some higher Ext's detecting singularity of modules over Dedekind rings (Corollary 1).

2 Distributive lattices and homological algebra

In this section we consider lattices of subgroups of a given abelian group with the intersection and the sum as the meet and the join operations, respectively. As for a general lattice the distributivity condition can be written in two equivalent forms:

- The sum distributes over the intersection

$$P_0 \cap (P_1 + P_2) = (P_0 \cap P_1) + (P_0 \cap P_2). \tag{3}$$

- The intersection distributes over the sum

$$I_0 + I_1 \cap I_2 = (I_0 + I_1) \cap (I_0 + I_2). \tag{4}$$

The aim of this section is to explain *homological* nature of the first and *cohomological* nature of the second form of distributivity. Homological characterization of distributivity was used by Zharinov in [18] to generalize famous edge-of-the-wedge theorem of Bogolyubov. In the present paper we prove a cohomological characterization of distributivity and derive from it a generalized Chinese remainder theorem. Note that in general the category of modules is not selfdual, so our cohomological characterization requires a proof independent of the homological proof of Zharinov. In this section we show also that both characterizations have consequences for arithmetic.

2.1 *Homology*

Let $P_0, \dots, P_n$ be a finite family of members of some fixed lattice L of subgroups in an abelian group A. We define a group of q-chains $\mathrm{C}_q(P_0, \dots, P_n)$ as a quotient of the direct sum

$$\bigoplus_{0 \le \alpha_0, \dots, \alpha_q \le n} P_{\alpha_0} \cap \ldots \cap P_{\alpha_q} \tag{5}$$

by a subgroup generated by elements

$$p_{\alpha_0,\dots,\alpha',\dots,\alpha'',\dots,\alpha_q} + p_{\alpha_0,\dots,\alpha'',\dots,\alpha',\dots,\alpha_q}, \qquad p_{\alpha_0,\dots,\alpha,\dots,\alpha,\dots,\alpha_q}, \tag{6}$$

and boundary operators (in terms of representatives of elements of quotient groups)

$$\partial : \mathrm{C}_q(P_0, \dots, P_n) \to \mathrm{C}_{q-1}(P_0, \dots, P_n), \tag{7}$$

$$(\partial p)_{\alpha_0 \dots \alpha_{q-1}} = \sum_{\alpha_q} p_{\alpha_0 \dots \alpha_{q-1} \alpha_q}. \tag{8}$$

By a standard argument from homological algebra $\partial \circ \partial = 0$. We denote by $\mathrm{H}_\bullet(P_0, \dots, P_n)$ the homology of the complex $(\mathrm{C}_\bullet(P_0, \dots, P_n), \partial)$.

Theorem 1 (Zharinov [18]) *1)* $\mathrm{H}_0(P_0, \dots, P_n) = P_0 + \cdots + P_n$*,*
2) If the lattice L is distributive then $\mathrm{H}_q(P_0, \dots, P_n) = 0$ *for* $q > 0$*,*
3) If $\mathrm{H}_1(P_0, P_1, P_2) = 0$ *for all* $P_0, P_1, P_2 \in L$ *then the lattice L is distributive.*

The following corollary provides a *homological characterization* of distributivity of a lattice L.

Corollary 1 *The following conditions are equivalent.*
1) L is distributive,
2) For all $P_0, \dots, P_n \in L$ the canonical morphisms of complexes

$$\mathrm{C}_\bullet(P_0, \dots, P_n) \to P_0 + \ldots + P_n \tag{9}$$

are quasiisomorphisms.

In particular, since $\mathrm{C}_q(P_0, \dots, P_n)$ can be identified with the direct sum

$$\mathrm{C}_q(P_0, \dots, P_n) = \bigoplus_{0 \le \alpha_0 < \ldots < \alpha_q \le n} P_{\alpha_0} \cap \ldots \cap P_{\alpha_q}, \tag{10}$$

the above corollary provides a *homological* resolution of the sum of subgroups $P_0 + \ldots + P_n$ by means of their intersections $P_{\alpha_0} \cap \ldots \cap P_{\alpha_q}$, $0 \le \alpha_0 < \ldots < \alpha_q \le n$, provided the lattice L is distributive.

In [1] authors introduce a class of so called G*GCD rings, defined as such for which $\gcd(P_1, P_2)$ exists for all finitely generated projective ideals P_1, P_2. This class includes GGCD rings, semihereditary rings, f.f. rings (and hence flat rings), von Neumann regular rings, arithmetical rings, Prüfer domains and GGCD domains. For every such a ring the class of finitely generated projective ideals is closed under intersection [1]. Therefore, for an arithmetical ring R every sum $P_0 + \ldots + P_n$ of finitely generated projective ideals in A admits a canonical resolution (9) by finitely generated projective modules, which implies the following corollary.

Corollary 2 *Let $P_0, \dots, P_n$ be finitely generated projective ideals in an arithmetical ring R. Then*

$$\mathrm{Ext}^q_R(P_0 + \ldots + P_n, -) = \mathrm{H}^q(\mathrm{Hom}_R(\mathrm{C}_\bullet(P_0, \ldots, P_n), -)), \tag{11}$$

$$\mathrm{Tor}^R_q(P_0 + \ldots + P_n, -) = \mathrm{H}_q(\mathrm{C}_\bullet(P_0, \ldots, P_n) \otimes_R -). \tag{12}$$

2.2 *Cohomology*

Let $I_0, \dots, I_n$ be a finite family of members of some fixed lattice L of subgroups in an abelian group A. We define a group of q-cochains $\mathrm{C}^q(I_0, \dots, I_n)$ as a subgroup of the product

$$\prod_{0 \le \alpha_0, \ldots, \alpha_q \le n} I_{\alpha_0} + \ldots + I_{\alpha_q} \tag{13}$$

consisting of sequences $(i_{\alpha_0 \ldots \alpha_q})$ which are completely alternating with respect to indices $\alpha_0, \dots, \alpha_q$, i.e.

$$i_{\alpha_0, \ldots, \alpha', \ldots, \alpha'', \ldots, \alpha_q} + i_{\alpha_0, \ldots, \alpha'', \ldots, \alpha', \ldots, \alpha_q} = 0, \quad i_{\alpha_0, \ldots, \alpha, \ldots, \alpha, \ldots, \alpha_q} = 0, \tag{14}$$

and coboundary operators

$$d : \mathrm{C}^q(I_0, \ldots, I_n) \to \mathrm{C}^{q+1}(I_0, \ldots, I_n), \tag{15}$$

$$(di)_{\alpha_0 \ldots \alpha_{q+1}} = \sum_{p=0}^{q+1} (-1)^p i_{\alpha_0 \ldots \widehat{\alpha_p} \ldots \alpha_{q+1}}. \tag{16}$$

By a standard argument from homological algebra $d \circ d = 0$. We denote by $\mathrm{H}^\bullet(I_0, \dots, I_n)$ the cohomology of the complex $(\mathrm{C}^\bullet(I_0, \dots, I_n), d)$.

Theorem 2 *1) $\mathrm{H}^0(I_0, \dots, I_n) = I_0 \cap \ldots \cap I_n$,*
2) If the lattice L is distributive then $\mathrm{H}^q(I_0, \dots, I_n) = 0$ for $q > 0$,
3) If $\mathrm{H}^1(I_0, I_1, I_2) = 0$ for all $I_0, I_1, I_2 \in L$ then the lattice L is distributive.

Proof. Let us note first that $\mathrm{C}^q(I_0, \dots, I_n)$ can be identified with the product

$$\mathrm{C}^q(I_0, \ldots, I_n) = \prod_{0 \le \alpha_0 < \ldots < \alpha_q \le n} I_{\alpha_0} + \cdots + I_{\alpha_q}. \tag{17}$$

1) Since the difference $i_\beta - i_\alpha$ is alternating with respect to the indices α, β we have

$$\mathrm{H}^0(I_0, \ldots, I_n) = \ker(\prod_{0 \le \alpha \le n} I_\alpha \to \prod_{0 \le \alpha < \beta \le n} I_\alpha + I_\beta), (i_\alpha) \mapsto (i_\beta - i_\alpha), \tag{18}$$

which consists of constant sequences $(i_\alpha = i \ \mid i \in I_0 \cap \ldots \cap I_n)$.

2) For $q > 0$ induction on n. For $n = 0$ obvious. Inductive step: Consider $(I_0, \dots, I_n)$ for $n > 0$. Then every q-cochain $(i_{\alpha_0 \ldots \alpha_q}, i_{\alpha_0 \ldots \alpha_{q-1} n})$, for $q > 0$, can be identified with a sequence consisting of elements

$$i_{\alpha_0 \ldots \alpha_q} \in I_{\alpha_0} + \ldots + I_{\alpha_q}, \quad \text{for} \ \ 0 \le \alpha_0 < \ldots < \alpha_q \le n-1, \tag{19}$$

$$i_{\alpha_0 \ldots \alpha_{q-1} n} \in I_{\alpha_0} + \ldots + I_{\alpha_{q-1}} + I_n, \quad \text{for} \ \ 0 \le \alpha_0 < \ldots < \alpha_{q-1} \le n-1. \tag{20}$$

This is a cocycle iff

$$\sum_{p=0}^{q+1}(-1)^p i_{\alpha_0\ldots\widehat{\alpha_p}\ldots\alpha_{q+1}} = 0, \tag{21}$$

for all $0 \leq \alpha_0 < \ldots < \alpha_{q+1} \leq n-1$, and

$$\sum_{p=0}^{q}(-1)^p i_{\alpha_0\ldots\widehat{\alpha_p}\ldots\alpha_q n} + (-1)^{q+1} i_{\alpha_0\ldots\alpha_q} = 0. \tag{22}$$

for all $0 \leq \alpha_0 < \ldots < \alpha_q \leq n-1$.

By the inductive hypothesis $\mathrm{H}^q(I_0,\ldots,I_{n-1}) = 0$ for $q > 0$. Then (21) implies that for all $0 \leq \alpha_0 < \ldots < \alpha_{q-1} \leq n-1$ there exist $i_{\alpha_0\ldots\alpha_{q-1}} \in I_{\alpha_0} + \ldots + I_{\alpha_{q-1}}$, such that for all $0 \leq \alpha_0 < \ldots < \alpha_q \leq n-1$

$$i_{\alpha_0\ldots\alpha_q} = \sum_{p=0}^{q}(-1)^p i_{\alpha_0\ldots\widehat{\alpha_p}\ldots\alpha_q}, \tag{23}$$

hence by (22)

$$\sum_{p=0}^{q}(-1)^p i_{\alpha_0\ldots\widehat{\alpha_p}\ldots\alpha_q n} + \sum_{p=0}^{q}(-1)^{p+q+1} i_{\alpha_0\ldots\widehat{\alpha_p}\ldots\alpha_q} = 0, \tag{24}$$

which can be rewritten as

$$\sum_{p=0}^{q}(-1)^p (i_{\alpha_0\ldots\widehat{\alpha_p}\ldots\alpha_q n} - (-1)^q i_{\alpha_0\ldots\widehat{\alpha_p}\ldots\alpha_q}) = 0. \tag{25}$$

For $q = 1$ we know by already proven point 2) of the theorem that $\mathrm{H}^{q-1}(I_0 + I_n,\ldots,I_{n-1}+I_n) = (I_0+I_n)\cap\ldots\cap(I_{n-1}+I_n)$ which is equal to $I_0\cap\ldots\cap I_{n-1}+I_n$, since the lattice L is distributive. Therefore by (25) for $q=1$ there exist $i \in I_0 \cap \ldots \cap I_{n-1}$ and $i_n \in I_n$ such that for all $0 \leq \alpha \leq n-1$

$$i_{\alpha n} + i_\alpha = i + i_n, \tag{26}$$

hence

$$i_{\alpha n} = i_n - (i_\alpha - i). \tag{27}$$

Equations (23) for $q = 1$, which reads as

$$i_{\alpha\beta} = i_\beta - i_\alpha = (i_\beta - i) - (i_\alpha - i), \tag{28}$$

and (27) together mean that $i = (i_{\alpha\beta}, i_{\alpha n} \;\mid 0 \leq \alpha < \beta \leq n-1) \in \mathrm{C}^1(I_0,\ldots,I_n)$ is coboundary of $(i_\alpha - i, i_n \;\mid 0 \leq \alpha \leq n-1) \in \mathrm{C}^0(I_0,\ldots,I_n)$ which proves that $\mathrm{H}^1(I_0,\ldots,I_n) = 0$.

For $q > 1$ by the inductive hypothesis $\mathrm{H}^{q-1}(I_0+I_n,\ldots,I_{n-1}+I_n) = 0$, hence (25) implies that for all $0 \leq \alpha_0 < \ldots < \alpha_{q-2} \leq n-1$ there exist $i_{\alpha_0\ldots\alpha_{q-2}n} \in (I_{\alpha_0}+I_n) + \ldots + (I_{\alpha_{q-2}}+I_n) = I_{\alpha_0}+\ldots+I_{\alpha_{q-2}}+I_n$, such that for all $0 \leq \alpha_0 < \ldots < \alpha_{q-1} \leq n-1$

$$i_{\alpha_0\ldots\alpha_{q-1}n} - (-1)^q i_{\alpha_0\ldots\alpha_{q-1}} = \sum_{p=0}^{q-1}(-1)^p i_{\alpha_0\ldots\widehat{\alpha_p}\ldots\alpha_{q-1}n}, \tag{29}$$

which can be rewritten as

$$i_{\alpha_0\ldots\alpha_{q-1}n} = \sum_{p=0}^{q-1}(-1)^p i_{\alpha_0\ldots\widehat{\alpha_p}\ldots\alpha_{q-1}n} + (-1)^q i_{\alpha_0\ldots\alpha_{q-1}}. \tag{30}$$

Equations (23) and (30) together mean that $(i_{\alpha_0\ldots\alpha_q}, i_{\alpha_0\ldots\alpha_{q-1}n})$ is coboundary of $(i_{\alpha_0\ldots\alpha_{q-1}}, i_{\alpha_0\ldots\alpha_{q-2}n})$, hence $\mathrm{H}^q(I_0,\ldots,I_n)=0$ for $q>1$.

3) We have to prove that for all $I_0, I_1, I_2 \in L$

$$(I_0+I_1)\cap(I_0+I_2) = I_0 + I_1\cap I_2. \tag{31}$$

The inclusion $(I_0+I_1)\cap(I_0+I_2) \supset I_0+I_1\cap I_2$ is obvious. To prove the opposite inclusion take $i \in (I_0+I_1)\cap(I_0+I_2)$. It can be written in two ways as

$$i = i_{01} + i'_{12}, \quad \text{where} \quad i_{01}\in I_0 \subset I_0+I_1, \;\; i'_{12}\in I_1\subset I_1+I_2, \tag{32}$$

$$i = i_{02} + i''_{12}, \quad \text{where} \quad i_{02}\in I_0 \subset I_0+I_2, \;\; i''_{12}\in I_2\subset I_1+I_2. \tag{33}$$

Define $i_{12} := i'_{12} - i''_{12}$. Subtracting (33) from (32) we get the cocycle condition

$$i_{12} - i_{02} + i_{01} = 0. \tag{34}$$

Since $\mathrm{H}^1(I_0,I_1,I_2)=0$ (34) implies that there exist $i_\alpha\in I_\alpha$, $\alpha=0,1,2$, such that

$$i_{\alpha\beta} = i_\beta - i_\alpha, \tag{35}$$

in particular

$$i'_{12} - i''_{12} = i_{12} = i_2 - i_1, \tag{36}$$

hence

$$i_1 + i'_{12} = i_2 + i''_{12}. \tag{37}$$

Since $i_0\in I_0$ and by (32) $i_{01}\in I_0$ (35) implies that

$$i_1 = i_0 + i_{01} \in I_0. \tag{38}$$

By (32) (resp. (33)) the left (resp. right) hand side of (37) belongs to I_1 (resp. I_2), hence

$$i_1 + i'_{12} \in I_1\cap I_2. \tag{39}$$

Finally, by (32), (38) and (39)

$$i = i_{01} + i'_{12} = (i_{01} - i_1) + (i_1 + i'_{12}) \in I_0 + I_1\cap I_2. \square \tag{40}$$

The following corollary provides a *cohomological characterization* of distributivity of a lattice L.

Corollary 3 *The following conditions are equivalent.*

1) L is distributive,

2) For all $I_0,\ldots,I_n\in L$ the canonical morphisms of complexes

$$I_0\cap\ldots\cap I_n \to \mathrm{C}^\bullet(I_0,\ldots,I_n), \tag{41}$$

are quasiisomorphisms.

In particular, by the identification (17) the above corollary provides a *cohomological* resolution of the intersection of subgroups $I_0 \cap \ldots \cap I_n$ by means of their sums $I_{\alpha_0} + \cdots + I_{\alpha_q}$, $0 \leq \alpha_0 < \ldots < \alpha_q \leq n$, provided the lattice L is distributive.

This fact together with the fact that each finitely generated module over a Dedekind ring is a direct sum of distributive modules (i.e. modules whose lattice of submodules is distributive) [15] can be used for detecting singularities of modules over Dedekind rings. First of all, in a non-singular left module (i.e. left module without nonzero elements annihilated by all essential left ideals) the intersection of injective submodules is again injective [17]. Therefore, given injective submodules $I_0, \ldots, I_n$ in a left module A over a ring R, the functors $\mathrm{Ext}^q_R(-, I_0 \cap \ldots \cap I_n)$ for $q > 0$ detect singularity of A. These functors can be computed with use of our resolution whenever every sum of injective submodules of a left R-module A is injective. The latter property characterizes left Noetherian left hereditary rings [9], hence it holds for Dedekind rings. Therefore we can apply Theorem 2 to obtain the following corollary.

Corollary 4 *Let $I_0, \ldots, I_n$ be injective submodules in a distributive left R-module A over a left Noetherian and left hereditary ring R. Then*

$$\mathrm{Ext}^q_R(-, I_0 \cap \ldots \cap I_n) = \mathrm{H}^q(\mathrm{Hom}_R(-, \mathrm{C}^\bullet(I_0, \ldots, I_n))). \tag{42}$$

Therefore, if A is a finitely generated and non-singular module over a Dedekind ring R

$$\mathrm{H}^q(\mathrm{Hom}_R(-, \mathrm{C}^\bullet(I_0, \ldots, I_n))) = 0 \tag{43}$$

for $q > 0$.

3 Generalized Chinese Remainder Theorem

As next application we will prove the following generalized Chinese remainder theorem.

Corollary 5 *For any finite family $I_0, \ldots, I_n$ of members of some fixed distributive lattice L of subgroups in an abelian group A the canonical diagram*

$$A/\bigcap_{\alpha=0}^{n} I_\alpha \to \prod_{\alpha=0}^{n} A/I_\alpha \rightrightarrows \prod_{\alpha,\beta=0}^{n} A/I_\alpha + I_\beta. \tag{44}$$

is an equalizer.

Proof. Injectivity of the first arrow is obvious. Exactness of (44) in the middle term is equivalent to exactness in the middle term of the canonical complex

$$A \xrightarrow{\pi} \prod_{0 \leq \alpha \leq n} A/I_\alpha \xrightarrow{\delta} \prod_{0 \leq \alpha < \beta \leq n} A/I_\alpha + I_\beta, \tag{45}$$

where $\pi(a) = (a + I_\alpha \mid 0 \leq \alpha \leq n)$, $\delta(a_\alpha + I_\alpha \mid 0 \leq \alpha \leq n) = (a_\beta - a_\alpha + I_\alpha + I_\beta \mid 0 \leq \alpha < \beta \leq n)$. We have

$$\ker \delta = (a_\alpha + I_\alpha \mid a_\beta - a_\alpha \in I_\alpha + I_\beta), \tag{46}$$

hence $(i_{\alpha\beta} := a_\beta - a_\alpha \mid 0 \leq \alpha < \beta \leq n)$ is a cocycle in $\mathrm{C}^1(I_0, \ldots, I_n)$. By Theorem 1 there exist $i_\alpha \in I_\alpha$ such that $a_\beta - a_\alpha = i_\beta - i_\alpha$. Let $a := a_\alpha - i_\alpha = a_\beta - i_\beta$. Then $(a_\alpha + I_\alpha \mid 0 \leq \alpha \leq n) = \pi(a)$, which proves exactness of (45) in the middle term. □

Remark. It is well known that if all I_α's are pairwise coprime ideals in a unital associative ring A, i.e. $I_\alpha + I_\beta = A$ for $\alpha \neq \beta$, then the diagram (44) is an equalizer and $I_1 \cap \ldots \cap I_n = \sum_\sigma I_{\sigma(1)} \ldots I_{\sigma(n)}$, where σ's are sufficiently many permutations of $\{1, \ldots, n\}$. These facts are independent of distributivity of the lattice of ideals. Therefore Corollary 5 (essentialy present already in [10], next rediscovered and generalized many times, e.g. [4], [3], [16], [5]) should be understood as a generalization of the Chinese remainder theorem to the non-coprime case, for which distributivity of the lattice of ideals is a sufficient condition. In fact, the lattice of left ideals in a (unital associative) ring is distributive iff the above generalized Chinese remainder theorem holds for such ideals [3]. Therefore in the commutative case there is *"one necessary and sufficient condition that places the theorem in proper perspective. It states that the Chinese remainder theorem holds in a commutative ring if and only if the lattice of ideals of the ring is distributive"* [13]. The aim of this section was to show how lattice theory communicates with modular arithmetic through homology theory.

4 Noncommutative Topology

Finite families of closed subsets covering a topological space are important for the Mayer-Vietoris principle in sheaf cohomology with supports and topological K-theory. Since by the Gelfand-Naimark duality gluing of a compact Hausdorff space X from finite number of compact Hausdorff pieces is equivalent to a generalized Chinese remainder theorem (2) for closed *-ideals in a commutative unital C*-algebra $C(X)$, one is tempted to define a *"noncommutative closed covering of a noncommutative space dual to an associative C*-algebra A"* as a finite collection of closed *-ideals intersecting to zero and generating a distributive lattice [6], [8].

In [8] authors focus on the combinatorial side of such gluing in terms of the poset structure on X induced by such a covering. This poset structure has its own topology (Alexandrov topology), drastically different from the original compact Hausdorff one. After fixing an order of the finite closed covering, they represent the distributive lattice generated by these originally closed (now Alexandrov open) subsets as a homomorphic image of the free distributive lattice generated by the same finite set of generators. Next, authors pull-back quotient C*-algebras A/I to that free lattice and view the resulting surjective system of quotient algebras as *flabby sheaf* of C*-algebras on the Alexandrov topology corresponding to that free lattice. Finally, they formulate the Gelfand-Naimark duality between ordered coverings of compact Hausdorff spaces by N closed sets and flabby sheaves of commutative unital C*-algebras on the Alexandrov topology corresponding to the free distributive lattice with N generators.

The aim of the present section is to avoid the auxiliary Alexandrov topology. In fact, creating new topology by declaring old closed subsets to be new opens is not necessary. The reason is that there is no need to see the generalized Chinese remainder theorem as the sheaf condition. The following definitions introduce a notion, which replaces sheaves when finite closed coverings replace open coverings.

Definition. For any topological space X we define a category of functors $\mathcal{P}$ (we call them *patterns*) from the lattice of *closed subsets* of X to the category of sets (abelian groups, rings, algebras etc) satisfying the following *unique gluing property* with respect to finite closed coverings $(C_0, \ldots, C_n)$ of closed subsets $C = C_0 \cup \ldots \cup C_n \subset X$, which demands that all canonical diagrams

$$\mathcal{P}(C) \to \prod_{\alpha=0}^{n} \mathcal{P}(C_\alpha) \rightrightarrows \prod_{\alpha,\beta=0}^{n} \mathcal{P}(C_\alpha \cap C_\beta) \tag{47}$$

are equalizers.

Definition. We call a pattern $\mathcal{P}$ on X *global* if for any closed subset $C \subset X$ the restriction morphism $\mathcal{P}(X) \to \mathcal{P}(C)$ is surjective.

Definition. For a continuous map $f : X \to Y$ the preimage $f^{-1}(D)$ of any closed subset $D \subset Y$ is closed in X and f defines the *direct image* functor f_* of patterns: $(f_*\mathcal{P})(D) := \mathcal{P}(f^{-1}(D))$.

We call (*globally*) *algebraized space* a pair consisting of a topological space X equipped with a (global) pattern $\mathcal{A}_X$ of algebras.

Definition. A morphism of (globally) algebraized spaces consists of a continuous map of topological spaces $f : X \to Y$ and a morphism of patterns of algebras $\mathcal{A}_Y \to f_*\mathcal{A}_X$.

In this framework the aforementioned Gelfand-Naimark duality between gluing of compact Hausdorff spaces and the generalized Chinese remainder theorem for C*-algebras reads now as follows.

Theorem 3 *The Gelfand-Naimark duality induces a full embedding of the category opposite to unital commutative C*-algebras equipped with lattices of closed *-ideals into the category of compact Hausdorff globally algebraized spaces.*

Note that in the above theorem the Gelfand-Naimark duality between pairs (A, L) and $(X, \mathcal{A}_X)$ dualizes a unital commutative C*-algebra A to a compact Hausdorff space X and the lattice L of closed *-ideals in A to a global pattern of algebras $\mathcal{A}_X$.

Note that every lattice of closed *-ideals in a C*-algebra is distributive. Therefore, according to the general ideology of noncommutative topology, a pair consisting of a unital associative C*-algebra A and a lattice L of closed *-ideals in A should be regarded as a *"noncommutative compact Hausdorff globally algebraized space"*.

4.1 C*-algebras and patterns

4.1.1 Continuous fields of C-algebras*

In functional analysis of function C*-algebras, in opposite to algebraic geometry, the notion of sheaf plays no a significant role. The appropriate replacement is then the notion of sections of *continuous fields of C*-algebras* [7]. They are contravariant functors on the category of closed subsets transforming closed embeddings into surjective restriction homomorphisms. It is easy to observe that they satisfy the unique gluing property with respect to finite closed coverings of closed subsets , i.e. they are patterns in our terminology. This property was used in computation of K-theory of an important class of Toeplitz algebras on Lie groups, with use of the Mayer-Vietoris sequence [11].

4.1.2 Continuous functions vanishing at infinity

Another example of patterns arising in theory of function C*-algebras comes from continuous functions vanishing at infinity. For any locally compact space one has a non-unital C*-algebra $\mathcal{C}_0(X)$ of continuous functions vanishing at infinity. It is widely accepted that $\mathcal{C}_0(X)$ is an appropriate C*-algebraic replacement of the locally compact space X, mostly in view of the Gelfand-Naimark duality in the unital-versus-compact case. A beautiful part of functional analysis was created to extend the Gelfand-Naimark duality in this way. However, an idea that relaxing compactness to local compactness can be dualized to forgetting about the unit of the C*-algebra is specious, at least if one wants to preserve the usual relation between continuous functions and topology.

First, about C*-algebras and locality. Although continuous functions form a sheaf under restriction to open subsets, the vanishing at infinity property does not survive the restriction. This means that given two open subsets $U \subset V \subset X$ there is no a restriction homomorphism from $\mathcal{C}_0(V)$ to $\mathcal{C}_0(U)$. Strange enough, there is a well defined injective homomorphism of non-unital algebras in the opposite direction $\mathcal{C}_0(U) \to \mathcal{C}_0(V)$, given by the extension by zero. Moreover, given open subsets $U_0, \ldots, U_n$ one has a canonical equalizer diagram

$$\mathcal{C}_0(U) \to \prod_{\alpha=0}^{n} \mathcal{C}_0(U_\alpha) \rightrightarrows \prod_{\alpha,\beta=0}^{n} \mathcal{C}_0(U_\alpha \cup U_\beta), \tag{48}$$

whose arrows are defined as collections of extensions by zero with respect to inclusions $U = U_0 \cap \ldots \cap U_n \subset U_\alpha$, $U_\alpha \subset U_\alpha \cup U_\beta$ and $U_\beta \subset U_\alpha \cup U_\beta$.

The Čech-Stone compactification $X \hookrightarrow \beta X$ and the Gelfand-Naimark duality put the problem of geometric description of the extension by zero into the right perspective. The extension by zero $\mathcal{C}_0(U) \to \mathcal{C}_0(V)$ is equivalent to a surjective restriction homomorphism of unital quotient algebras

$$\mathcal{C}(\beta X \setminus U) = \mathcal{C}(\beta X)/\mathcal{C}_0(U) \to \mathcal{C}(\beta X)/\mathcal{C}_0(V) = \mathcal{C}(\beta X \setminus V), \tag{49}$$

when we regard $\mathcal{C}_0(U)$ as a closed *-ideal in the C*-algebra $\mathcal{C}(\beta X) \cong \mathcal{C}_b(X)$ of continuous functions on βX isomorphic to the C*-algebra of bounded functions on X. This restriction homomorphism is Gelfand-Naimark dual to the closed inclusion $C := \beta X \setminus U \subset \beta X \setminus V =: D$, and makes (48) the equalizer diagram verifying the pattern property of the assignment $C \mapsto \mathcal{I}(C) := \mathcal{C}_0(\beta X \setminus C)$ on the finite closed covering

$$\beta X \setminus U = \bigcup_{\alpha=0}^{n} (\beta X \setminus U_\alpha). \tag{50}$$

This means that the pattern $\mathcal{I}$ is a (sub)pattern of ideals in the constant pattern $\mathcal{C}(\beta X)$ of algebras. The pattern $C \mapsto \mathcal{C}(C)$ is then the pattern of quotient algebras.

4.1.3 Pattern cohomology on topological spaces

Patterns admit an analog of the Čech cohomology with respect to finite closed coverings. Assume that there is given such a covering $X = C_0 \cup \ldots \cup C_n$ of a topological space X and a pattern $\mathcal{P}$. Mimicking the Čech complex construction we define the *pattern cohomology*

$$\mathrm{H}^p(C_0,\ldots,C_n;\mathcal{P}) := \mathrm{H}^p(\prod_{i_0<\ldots<i_\bullet} \mathcal{P}(C_{i_0}\cap\ldots\cap C_{i_\bullet})). \tag{51}$$

The pattern property implies that

$$\mathrm{H}^0(C_0,\ldots,C_n;\mathcal{P}) = \mathcal{P}(X). \tag{52}$$

If the pattern takes values in a distributive lattice of subgroups of an abelian group A in such a way that

$$\mathcal{P}(C_{i_0}\cap\ldots\cap C_{i_p}) = \mathcal{P}(C_{i_0})+\ldots+\mathcal{P}(C_{i_p}) \tag{53}$$

$$\mathcal{P}(C_{i_0}\cup\ldots\cup C_{i_p}) = \mathcal{P}(C_{i_0})\cap\ldots\cap\mathcal{P}(C_{i_p}). \tag{54}$$

then by Theorem 2 we obtain for $p>0$

$$\mathrm{H}^p(C_0,\ldots,C_n;\mathcal{P}) = 0. \tag{55}$$

In particular, for the constant pattern $A(C) := A$

$$\mathrm{H}^p(C_0,\ldots,C_n;A) = \begin{cases} A \text{ if } p=0,\\ 0 \text{ if } p\neq 0\end{cases} \tag{56}$$

and for its subpattern $\mathcal{I}$ taking values in a distributive lattice of subgroups of A, satisfying (53) and (54), and such that $\mathcal{I}(X)=0$

$$\mathrm{H}^p(C_0,\ldots,C_n;\mathcal{I}) = 0 \tag{57}$$

for all p. The short exact sequence of patterns

$$0\to\mathcal{I}\to A\to A/\mathcal{I}\to 0 \tag{58}$$

induces then a long exact sequence of pattern cohomology, which implies that

$$\mathrm{H}^p(C_0,\ldots,C_n;A/\mathcal{I}) = \begin{cases} A \text{ if } p=0,\\ 0 \text{ if } p\neq 0\end{cases} \tag{59}$$

This shows, in particular, that the cohomological behavior of the operation of taking remainders modulo ideals (restrictions to closed subsets) of an arithmetical ring A expressed in terms of the globally algebraized space structure defined on the Zariski topology of Spec(A) is similar to the cohomological behavior of localizations (restrictions to open subsets) of A expressed in terms of the locally ringed space structure on Spec(A).

4.1.4 Sheaves versus patterns

Due to Cartan, Leray's *"faisceaux continus"* on locally compact spaces are equivalent to sheaves. Actually, given a sheaf $\mathcal{F}$ on a locally compact space X one can assign to every closed subset $C\subset X$ the stalk of $\mathcal{F}$ along C. This assignment is different from our "pattern". For a sheaf of continuous functions the stalk at a point consists of germs, while the evaluation of the pattern on a point consists of values. If the space X is not discrete the kernel of the surjective evaluation map from the stalk to the set of values is usually big.

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