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Abstract

We study pairs $(X, \mathcal{V})$ where X is a vector field on a smooth manifold M and $\mathcal{V} \subset TM$ is a vector distribution, both satisfying certain regularity conditions. We construct basic invariants of such objects and solve the equivalence problem. For a given pair $(X, \mathcal{V})$ we construct a canonical connection on a certain frame bundle. The results are applied to the problem of time-scale preserving equivalence of ordinary differential equations. The framework of pairs $(X, \mathcal{V})$ is shown to include sprays, Hamiltonian systems, Veronese webs and other structures.

1 Introduction

In this paper we study pairs $(X, \mathcal{V})$ where X is a vector field on a manifold M and $\mathcal{V} \subset TM$ is a vector distribution (a sub-bundle of the tangent bundle TM), both satisfying certain regularity conditions. Such pairs appear to encode a large variety of geometric objects as geodesic sprays on Riemannian (pseudo-Riemannian) manifolds and, more generally, manifolds with affine connections, spray spaces, control systems, systems of ordinary differential equations, Veronese and Kronecker webs.

The aim of the article is twofold. The first aim is to study general geometric objects attributed to the pair $(X, \mathcal{V})$ and identify the invariants. The second aim is to test the proposed approach on systems of ordinary differential equations of order ≥ 2, where pairs $(X, \mathcal{V})$ appear as a tool. We use time scale preserving equivalence. This is done in the second part of the article, where we solve the corresponding equivalence problem. The approach was partially developed in the thesis [19].

The equivalence problem for ODEs is a classical one and was mainly attacked using contact or point transformations (see Cartan [5], Chern [6], Bryant [3], Fels [13], Doubrov, Komrakov and Morimoto [11], Dunajski and Tod [12], Godlinski and Nurowski [16] for a very partial list of contributions). We do not address this version of the problem, which is more complicated and gives less hopes for a simple and complete solution (i.e., identifying complete sets of independent invariants).

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Time scale preserving equivalence problem was less studied, even if it is more natural from the point of view of applications. The problem was formally solved by Chern [7] (systems of order two) and [8] (systems of higher order), using Cartan's method of equivalence (his second paper was totally ignored in the literature). Our approach (Section 3) treats ODE's as pairs $(X, \mathcal{V})$ with a flag of integrable distributions and can be thought, roughly, as dual to Chern's [8] (more details on dual approaches and formulas for invariants will be provided in a future paper). We additionally give relations between invariants of systems of order two (Section 3.4).

The setting considered here includes control systems and variational equations appearing there (cf. e.g. [21, 4, 2]). It also includes equivalence problems of special geometric structures, one of them being a Veronese web (Gelfand and Zakharevich [14, 26], Touriel [25]). We show that a Veronese web can be coded as a pair $(X, \mathcal{V})$ and its invariants can be found as a special case of our invariants (Section 4).

The simplest invariants, the curvature operators which we construct in Section 2.2, generalise the curvatures used in [17] for an analysis of the variational equation. In the special case of geodesic spray on the tangent bundle of a Riemann or Finsler manifold, or a general spray (cf. Shen [22]), there is only one curvature operator K_0, equivalent to the Riemann curvature (Section 2.3). It is the curvature which appears in the generalised Jacobi equation (see [22], Chapter 8). Our formalism applied to Hamiltonian systems (Section 2.3) gives a symmetric curvature operator K_0 which is equivalent to the curvature introduced in a different way by Agrachev and Gamkrelidze [2] and used for estimation of conjugate points.

We outline our approach. The subject of the study is a pair $(X, \mathcal{V})$ of a vector field X and a distribution $\mathcal{V} \subset TM$ on a smooth manifold M. We attach to it a sequence of distributions defined inductively, using Lie bracket, by

$$\mathcal{V}^0 := \mathcal{V}, \quad \mathcal{V}^{i+1} := \mathcal{V}^i + [X, \mathcal{V}^i].$$

We impose natural *regularity conditions* on $(X, \mathcal{V})$, assuming that the distributions have maximal possible ranks. More precisely, we assume that $\dim M = (k+1)m+1$, $k, m \geq 1$, and

(R1) $\quad \operatorname{rk} \mathcal{V}^i = (i+1)m, \text{ for } i = 0, \ldots, k,$

(R2) $\quad \mathcal{V}^k \oplus \operatorname{span}\{X\} = TM.$

Our formalism also works with modified assumptions, where $\dim M = (k+1)m$ and (R2) is replaced with

(R2') $\quad \mathcal{V}^k = TM, \quad \text{and} \quad X(x) \neq 0, \text{ for } x \in M.$

A pair $(X, \mathcal{V})$ satisfying (R1) and (R2) is called *dynamic pair* or *regular pair*. We show that there is a natural class of frames in TM (called *normal frames*), attached in an invariant way to a regular pair $(X, \mathcal{V})$. This class of frames defines a canonical G-structure P on M, where $G = Gl(m)$. The G-structure has the following property.

> *Two regular pairs $(X, \mathcal{V})$ and $(X', \mathcal{V}')$ are locally equivalent if and only if the corresponding G-structures are isomorphic.*

Above, two pairs $(X, \mathcal{V})$ and $(X', \mathcal{V}')$ are called (locally) equivalent if there is a (local) diffeomorphism $\Phi\colon M \to M$ such that $\Phi_* X = X'$ and $\Phi_* \mathcal{V} = \mathcal{V}'$.

In order to solve the equivalence problem in a more explicit way we use the principal G-bundle P, defined by the G-structure, and define a canonical frame on P. Our main results in Section 2 (Theorems 2.9 and 2.11) say the following.

> *Two regular pairs $(X, \mathcal{V})$ and $(X', \mathcal{V}')$ are locally equivalent if and only if the corresponding canonical frames are.*

These theorems give a tool for solving the equivalence problem, since there is a standard procedure for determining if two frames are equivalent. Additionally, the canonical frames enable us to define a principal connection on P. The connection can be used for the analysis of geometric properties of the pair $(X, \mathcal{V})$.

Important geometric objects attached to $(X, \mathcal{V})$ are *curvature operators*, defined in Section 2.2 (see also [19] and [17]). In the classical case of semi-sprays defined by differential equations of order two there is only one such operator.

In the second part (Section 3) we study systems of ordinary differential equations

$$x^{(k+1)} = F(t, x, \ldots, x^{(k)})),$$

where $x \in \mathbb{R}^m$ and F is of class C^∞. Then we take $M = J^k(\mathbb{R}, \mathbb{R}^m)$ - the manifold of k-jets of parametrised curves in $\mathbb{R}^m$. The vector field X is the total derivative

$$X_F = \partial_t + \sum_{i=0}^{k-1} \sum_{j=1}^{m} x^j_{i+1} \partial_{x^j_i} + \sum_{j=1}^{m} F^j \partial_{x^j_k}.$$

A *vertical distribution* $\mathcal{V} = \mathcal{V}_F$ on E_F can be defined as the restriction to E_F of the kernel of the canonical projection $J^{k+1}(1, m) \to J^{k-1}(1, m)$. In coordinates

$$\mathcal{V}_F = \operatorname{span}\{\partial_{x^j_k} \mid j = 1, \ldots, m\}.$$

At the end (Section 4) we study Veronese webs, i.e. one-parameter families of foliations of special type, introduced by Gelfand and Zakharevich [14] and strongly related to bi-hamiltonian systems. We prove that Veronese webs can be treated as ODEs for which our *curvature operators* vanish. In this way we relate Veronese webs with dynamic pairs and we solve the equivalence problem. It appears that Veronese webs play the same role for time-scale preserving contact transformations as paraconformal structures (called also $Gl(2)$-structures) play for the general contact transformations [3, 12, 16, 20].

Other examples of pairs $(X, \mathcal{V})$ that can be interesting for applications are:

(1) Given a Riemannian or a pseudo-Riemannian manifold N, we take X to be the geodesic spray on $M = \mathbb{R} \times TN$ and $\mathcal{V}$ the vertical distribution on the bundle $\mathbb{R} \times TN \to N$.

(2) For a vector field X on a fibre bundle $E \to B$, we may take $M = E$ and $\mathcal{V}$ - the vertical distribution tangent to fibres.

(3) Given a control system $\dot{x} = f(x) + \sum_i u_i g_i(x)$ on a manifold M, we take $X = f$ and $\mathcal{V} = \operatorname{span}\{g_1, \ldots, g_m\}$.

2 Vector fields with distributions

2.1 Dynamic pairs $(X, \mathcal{V})$ and their normal frames

Let M be a smooth differentiable manifold of dimension n. Consider a C^∞ dynamic pair $(X, \mathcal{V})$, i.e., a vector field X on M and a distribution $\mathcal{V} \subset TM$ of rank m, satisfying the regularity conditions (R1) and (R2). Then $\mathcal{V}(x)$ is an m-dimensional subspace of the tangent space T_xM, for every $x \in M$ and, locally, there exist m smooth vector fields $V_1, \ldots, V_m$ such that

$$\mathcal{V}(x) = \mathrm{span}\{\, V_1(x), \ldots, V_m(x)\,\}.$$

If $V_1, \ldots, V_m$ span $\mathcal{V}$ on an open subset $U \subset M$ then we will briefly denote them as the row vector $V = (V_1, \ldots, V_m)$ and call V a *local frame* of $\mathcal{V}$ on U.

Recall that the pair $(X, \mathcal{V})$ defines a sequence of distributions

$$\mathcal{V} = \mathcal{V}^0 \subset \mathcal{V}^1 \subset \cdots \subset \mathcal{V}^k \subset TM,$$

where

$$\mathcal{V}^i(x) = \mathrm{span}\{\mathrm{ad}_X^s V_j(x) \mid 0 \le s \le i, 1 \le j \le m\}.$$

Above $\mathrm{ad}_X Y = [X, Y]$, $\mathrm{ad}_X^2 Y = [X, [X, Y]]$ etc., where $[X, Y]$ denotes the Lie bracket of vector fields. Denote by $\mathcal{X}$ the 1-dimensional distribution spanned by X.

Definition 2.1 *A local section Y of $\mathcal{V}$ is called* normal *if*

$$\mathrm{ad}_X^{k+1} Y = 0 \mod \mathcal{V}^{k-1} \oplus \mathcal{X}. \tag{1}$$

A local frame $V = (V_1, \ldots, V_m)$ of $\mathcal{V}$ on $U \subset M$ is called normal frame *of $\mathcal{V}$ if all V_i are normal sections of $\mathcal{V}$.*

Note that it follows from (R1) and (R2) that the vector fields X and $\mathrm{ad}_X^i V_j(x)$, $0 \le i \le k$, $1 \le j \le m$, are linearly independent and they span TM. Thus, denoting

$$\mathrm{ad}_X^i V = (\mathrm{ad}_X^i V_1, \ldots, \mathrm{ad}_X^i V_m)$$

we have

$$\mathrm{ad}_X^{k+1} V = V H_0 + (\mathrm{ad}_X V) H_1 + \cdots + (\mathrm{ad}_X^k V) H_k \mod \mathcal{X}, \tag{2}$$

where H_i are $m \times m$ matrices of functions. Then V is a normal frame iff $H_k = 0$.

Let $V = (V_1, \ldots, V_m)$ and $W = (W_1, \ldots, W_m)$ be local frames of $\mathcal{V}$ on U. Then there exists a unique $Gl(m)$-valued function $G\colon U \to Gl(m)$ such that, in the matrix notation,

$$W = VG. \tag{3}$$

Proposition 2.2 *(a) Given a regular pair $(X, \mathcal{V})$ and a frame V_{x_0} in $\mathcal{V}(x_0)$, there is a normal frame $V = (V_1, \ldots, V_m)$ of $\mathcal{V}$ in neighbourhood of x_0 such that $V(x_0) = V_{x_0}$.*
(b) If V and W are two normal frames of $\mathcal{V}$ on U then equation (3) *holds, where $G\colon U \to Gl(m)$ satisfies*

$$X(G) = 0. \tag{4}$$

(c) If V is a frame of $\mathcal{V}$ on U and H_k is defined via (2), then $W = VG$ is a normal frame of $\mathcal{V}$ on U if and only if

$$X(G) = -\frac{1}{k+1} H_k G. \tag{5}$$

Proof. (a) Let $V = (V_1, \ldots, V_m)$ be a local frame of $\mathcal{V}$ and let $V(x_0) = V_{x_0}$. We will find functions $G = (G_i^j)_{i,j=1,\ldots,m}$ such that $W_i = \sum_{j=1}^m G_i^j V_j$, for $i = 1, \ldots, m$, are the desired vector fields, i.e.,

$$\mathrm{ad}_X^{k+1} W = 0 \mod \mathcal{V}^{k-1} \oplus \mathcal{X}. \tag{6}$$

In the matrix notation $W = VG$, thus $[X, W] = [X, VG] = [X, V]G + VX(G)$ and, inductively, $\mathrm{ad}_X^i W = \mathrm{ad}_X^i (VG) = \sum_{j=0}^i \binom{i}{j} \mathrm{ad}_X^j (V) X^{i-j}(G)$. This implies

$$\mathrm{ad}_X^{k+1} W = (\mathrm{ad}_X^{k+1} V)G + (k+1)(\mathrm{ad}_X^k V)X(G) \mod \mathcal{V}^{k-1}.$$

It follows from (R1) and (R2) that (2) holds, thus

$$\mathrm{ad}_X^{k+1} V = (\mathrm{ad}_X^k V)H \mod \mathcal{V}^{k-1} \oplus \mathcal{X}, \tag{7}$$

for a certain square matrix $H_k = H = (H_i^j)$. Since $\mathrm{ad}_X^k W = (\mathrm{ad}_X^k V)G \mod \mathcal{V}^{k-1}$, equation (6) is equivalent to the following equation for the unknown function G:

$$HG + (k+1)X(G) = 0. \tag{8}$$

This is a linear first order differential equation for G, thus it can be solved, locally, so that $G(x_0) = \mathrm{Id}$. If G is a solution, we have $\mathrm{ad}_X^{k+1} W = 0 \mod \mathcal{V}^{k-1} \oplus \mathcal{X}$ and $W(x_0) = V(x_0)\mathrm{Id} = V_{x_0}$.

(b) As above, we have $W = VG$. Since all elements $Y = V_i$ of V satisfy (1), the matrix $H = H_k$ in equation (7) is zero. Equation (6) implies that the matrix valued function G satisfies (8), thus $X(G) = 0$.

(c) This follows from the proof of (a), as $H = H_k$ and equations (5) and (8) coincide. □

Note that the "initial condition" $V(x_0) = V_{x_0}$ in statement (a) can be imposed on any local hypersurface transversal to X in M. From statement (b) we get

Corollary 2.3 *If V and W are normal frames of $\mathcal{V}$ on $U \subset M$ then $W = VG$ and, for all i,*

$$\mathrm{ad}_X^i W = \mathrm{ad}_X^i (VG) = (\mathrm{ad}_X^i V)G. \tag{9}$$

2.2 Curvature operators

We will define the most basic invariants of dynamic pairs, called curvature operators.

Proposition 2.2 says that if both V and $W = VG$ are normal frames of $\mathcal{V}$ then $X(G) = 0$. Hence, $\mathrm{ad}_X^i W = (\mathrm{ad}_X^i V)G$ for any i. It follows that distributions

$$\mathcal{H}^i = \mathrm{span}\{V_1^i, \ldots, V_m^i\}, \quad \text{with} \quad V_j^i := \mathrm{ad}_X^i V_j,$$

do not depend on the choice of a normal frame $V = (V_1, \dots, V_m)$, for $i = 1, \dots, k$. $\mathcal{H}^i$ will be called i-th *quasi-connection*, for reasons which will become clear in the second part of the paper. We also denote $\mathcal{H}^0 = \mathcal{V}$ and stress that $\operatorname{rk}\mathcal{H}^i = m$. Condition (R1) implies

$$\mathcal{V}^i = \mathcal{V} \oplus \mathcal{H}^1 \oplus \cdots \oplus \mathcal{H}^i,$$

for $i = 1, \dots, k$. Condition (R2) gives

$$TM = \mathcal{V} \oplus \mathcal{H}^1 \oplus \cdots \oplus \mathcal{H}^k \oplus \mathcal{X}.$$

The last relation defines projections

$$\pi^0 \colon TM \to \mathcal{V} \quad \text{and} \quad \pi^i \colon TM \to \mathcal{H}^i, \quad i = 1, \dots, k.$$

Notice that the operators

$$A_i \colon \mathcal{V} \to \mathcal{H}^i, \quad A_i = \pi^i \circ \operatorname{ad}_X^i, \quad 0 \le i \le k,$$

are vector bundle isomorphisms, similarly as

$$\pi^i \circ \operatorname{ad}_X : \mathcal{H}^k \to \mathcal{H}^i, \quad 0 \le i \le k-1.$$

In particular, $A_0 = \operatorname{Id} : \mathcal{V} \to \mathcal{V}$.

Definition 2.4 *An i-th* curvature operator $K_i \in \operatorname{End}(\mathcal{V})$ *of a dynamic pair* $(X, \mathcal{V})$ *is*

$$K_i = (-1)^i (A_i)^{-1} \circ \pi^i \circ \operatorname{ad}_X \circ A_k \colon \mathcal{V} \to \mathcal{V}, \quad 0 \le i \le k-1.$$

The alternating sign is chosen for consistency with classical interpretations of curvatures and, in particular, for simplicity of the variational equation [17].

Equivalently, K_i can be defined as follows. Let $x \in M$. In a fixed basis of the space $\mathcal{V}(x)$ the operator $K_i(x)$ is represented by $m \times m$ matrix, also denoted $K_i(x)$. If $V = (V_1, \dots, V_m)$ is a normal frame, then matrices of the curvature operators are defined in the basis $V_1(x), \dots, V_m(x)$ of $\mathcal{V}(x)$ by the following equation:

$$\operatorname{ad}_X^{k+1} V + (-1)^{k-1} (\operatorname{ad}_X^{k-1} V) K_{k-1} + \cdots - (\operatorname{ad}_X V) K_1 + V K_0 = 0 \mod \mathcal{X}. \tag{10}$$

Due to the transformation rule (9), when normal generators V is transformed to normal generators VG, the matrices K_i are transformed via

$$K_i \longmapsto G^{-1} K_i G. \tag{11}$$

The operators

$$K_i(x) \colon \mathcal{V}(x) \to \mathcal{V}(x)$$

are invariantly assigned to the dynamic pair $(X, \mathcal{V})$.

If the generators V are not normal, the formula (2) holds. The linear operators defined by the matrices $H_i(x)$ are not invariantly assigned to the pair. However, the curvature matrices can be computed in terms of matrices H_i. In particular, we have

Proposition 2.5 *If* $k = 1$ *and* $\mathrm{ad}_X^2 V = (\mathrm{ad}_X V)H_1 + VH_0$ *then*

$$K_0 = -H_0 + \frac{1}{2}X(H_1) - \frac{1}{4}H_1^2 \tag{12}$$

and

$$\mathcal{H}^1 = \mathrm{span}\left\{\mathrm{ad}_X V - \frac{1}{2}VH_1\right\}. \tag{13}$$

Proof. We use statement (c) in Proposition 2.2, which states that $W = VG$ is a normal frame, if $X(G) = -\frac{1}{2}H_1G$. Then $X(X(G)) = -\frac{1}{2}X(H_1)G + \frac{1}{4}H_1^2G$. Since $\mathrm{ad}_X^2(VG) = (\mathrm{ad}_X^2V)G + 2(\mathrm{ad}_XV)X(G) + VX^2(G)$, using the assumed formula for ad_X^2V and eliminating $X(G)$ and $X^2(G)$ gives

$$\mathrm{ad}_X^2(VG) = VGG^{-1}\left(H_0 - \frac{1}{2}X(H_1) + \frac{1}{4}H_1^2\right)G.$$

This leads to the formula for K_0 in the normal frame $W = VG$, according to (10) (we may assume that $G(x) = \mathrm{Id}$, at a fixed point x). The formula for the connection $\mathcal{H}^1$ follows from $\mathrm{ad}_X(VG) = (\mathrm{ad}_XV)G + VX(G)$. □

2.3 Examples

Hamiltonian vector field. Let $M = \mathbb{R} \times N$, where N is a symplectic manifold of dimension $2m$ with symplectic form σ. Let H be a time dependent Hamiltonian vector field on N and, thus, a vector field on M. Take $X = \partial_t + H$, where t is the canonical time coordinate on $\mathbb{R}$, and on $\mathbb{R} \times N$. Let $\mathcal{V}_t \subset TN$, $t \in \mathbb{R}$, be a family of Lagrangian distributions on N, i.e., $\dim \mathcal{V}_t = m$ and $\sigma(v, w) = 0$ for any vectors $v, w \in \mathcal{V}_t$. Denote by $\mathcal{V}$ the resulting distribution on $M = \mathbb{R} \times N$. Assume that $(H, \mathcal{V})$ is regular, that is $\mathrm{span}\{\mathcal{V}_t, [H, \mathcal{V}_t]\}_x = T_xN$, for any $(t, x) \in M$. The pair $(X, \mathcal{V})$ is a regular dynamic pair on M with $k = 1$. Given a normal frame $V = (V_1, \ldots, V_m)$ in $\mathcal{V}$, we have

$$[X, [X, V_i]] = [H, [H, V_i]] = \sum_j (K_0)_i^j V_j, \tag{14}$$

where K_0 is the curvature matrix defined according to formula (10): $\mathrm{ad}_X^2V = VK_0$ (the coefficient at X in (10) vanishes because of the term ∂_t in X). One can check, consulting [1], that the above curvature K_0 coincides with the curvature introduced in a completely different way in [2]. The following proposition is easy to prove.

Proposition 2.6 *(a) The distribution*

$$\mathcal{H}^1 = \mathrm{span}\{[H, V_1], \ldots, [H, V_m]\}$$

is Lagrangian with respective to σ*, on each fiber* $\{t\} \times N$*.*
(b) The matrix $g = (g_{ij})$ *given by*

$$g_{ij}(x) = \sigma([H, V_i], V_j)(x), \quad x \in M = \mathbb{R} \times N, \quad i, j = 1, \ldots, m.$$

is symmetric, nondegenerate, and defines a pseudo-Riemannian metric on $\mathcal{V}$*.*
(c) The matrix K_0 *is symmetric and defines a selfadjoint (relative to* g*) operator* $K_0 : \mathcal{V} \to \mathcal{V}$*.*

Proof. Since $\mathcal{V}$ is Lagrangian, we have $\sigma(V_i, V_j) = 0$. Lie differentiating this equality with respect to H gives

$$0 = \sigma([H, V_i], V_j) + \sigma(V_i, [H, V_j])$$

and proves the symmetry of g in (b), since σ is antisymmetric. Differentiating twice gives

$$0 = \sigma(\mathrm{ad}_H^2 V_i, V_j) + 2\sigma([H, V_i], [H, V_j]) + \sigma(V_i, \mathrm{ad}_H^2 V_j).$$

The side terms are zero which follows from (14) and the fact that $\mathcal{V}$ is Lagrangian. Thus $\sigma([H, V_i], [H, V_j]) = 0$, which shows (a). Differentiating three times yields

$$0 = \sigma([H, \mathrm{ad}_H^2 V_i], V_j) + 3\sigma(\mathrm{ad}_H^2 V_i, [H, V_j]) + 3\sigma([H, V_i], \mathrm{ad}_H^2 V_j) + \sigma(V_i, [H, \mathrm{ad}_H^2 V_j])$$

and, applying (14) and the summation convention,

$$\begin{aligned} 0 &= \sigma([H, (K_0)_i^s V_s], V_j) + 3\sigma((K_0)_i^s V_s, [H, V_j]) \\ &\quad + 3\sigma([H, V_i], (K_0)_j^s V_s) + \sigma(V_i, [H, (K_0)_j^s V_s]) \\ &= \sigma((K_0)_i^s [H, V_s], V_j) + 3\sigma((K_0)_i^s V_s, [H, V_j]) \\ &\quad + 3\sigma([H, V_i], (K_0)_j^s V_s) + \sigma(V_i, (K_0)_j^s [H, V_s]), \end{aligned}$$

where in the second equality we use the fact that $\sigma(H((K_0)_i^s) V_s, V_j) = 0$, as $\mathcal{V}$ is Lagrangian. Using antisymmetry of σ and the definition of g_{ij} we get

$$0 = (K_0)_i^s g_{sj} - 3(K_0)_i^s g_{js} + 3(K_0)_j^s g_{is} - (K_0)_j^s g_{si} = -2(K_0)_i^s g_{sj} + 2(K_0)_j^s g_{is},$$

since g is symmetric. This proves (c).

Since $\mathcal{V} = \mathrm{span}\{V_1, \ldots, V_m\}$ and $\mathcal{H}^1 = \mathrm{span}\{[H, V_1], \ldots, [H, V_m]\}$ are transversal Lagrangian subspaces in TN and σ is nondegenerate on TN, it follows that the matrix g in (b) is nondegenerate. □

Remark. A canonical example of the pair as above is a time dependent vector field H on the cotangent bundle $N = T^*\tilde{N}$ of a differentiable manifold $\tilde{N}$, where $\mathcal{V}_t$ is the vertical distribution of the bundle $\pi : T^*\tilde{N} \to \tilde{N}$, i.e., $\mathcal{V}_t(x) = T_{\pi(x)}\tilde{N}$.

Another example (see [2]) is given by a Hamiltonian function $h : T^*\tilde{N} \to \mathbb{R}$. The corresponding Hamiltonian vector field H and the vertical distribution never form a regular pair with $k = 1$ since the dimension of $M = T^*\tilde{N}$ can not be equal to $2m + 1$. However, we can have a regular pair if we restrict our considerations to a level submanifold of the Hamiltonian. Namely, take $M = \{h = c\} \subset T^*\tilde{N}$ and the vector field X equal to H restricted to M, $X = H|_M$. The distribution $\mathcal{V}$ on M is defined as the vertical distribution of the cotangent bundle intersected with the tangent space to M: $\mathcal{V}(x) = T(T^*_{\pi(x)}\tilde{N}) \cap T_x M$, for $x \in M$. Then, if $\dim \tilde{N} = m + 1$, we have $\dim \mathcal{V}(x) = m$, $\dim M = 2m + 1$ and assuming regularity of the pair $(H, \mathcal{V})$ makes sense (typical examples are regular). In this case the equality in formula (14) holds, modulo H, and all statements of Proposition 14 hold true, for the canonical symplectic form σ on $T^*\tilde{N}$ replaced with the symplectic form $\hat{\sigma} = \sigma|_M$ considered on the quotient space $T_x M/\mathrm{span}\{H(x)\}$.

Geodesic spray. Consider a geodesic equation on a manifold N of dimension m with local coordinates (x^i). In local coordinates we have

$$(x^i)'' = -\sum_{p,q=1}^{m} \Gamma^i_{pq}(x^p)'(x^q)',$$

where Γ^i_{pq} are Christoffel symbols of a connection ∇. Note that the equation does not depend on the torsion of the connection and thus we will assume that ∇ is symmetric. Let $J^2(\mathbb{R}, N)$ be the space of 2-jets of functions $\mathbb{R} \to N$. On $J^2(\mathbb{R}, N)$ there are local coordinates (t, x^i_0, x^i_1, x^i_2), induced by the coordinates (x^i), where $i = 1, \dots, m$. The geodesic equation is uniquely defined by the submanifold

$$E = \left\{ (t, x^i_0, x^i_1, x^i_2) \mid x^i_2 = -\sum_{p,q} \Gamma^i_{pq} x^p_1 x^q_1 \right\} \subset J^2(\mathbb{R}, N).$$

There is a canonical projection $J^2(\mathbb{R}, N) \to TN \times \mathbb{R}$ which, restricted to E, defines the diffeomorphism $E \simeq TN \times \mathbb{R}$. In particular $(x^i_0) = (x^i)$ are local coordinates on N whereas (x^i_1) are the corresponding linear coordinates on the fibres of $TN \to N$.

Let

$$X = \partial_t + \sum_i x^i_1 \partial_{x^i_0} - \sum_{i,p,q} \Gamma^i_{pq} x^p_1 x^q_1 \partial_{x^i_1}$$

be the total derivative. We take the vertical distribution tangent to the fibres of TN, that is $\mathcal{V} = \mathrm{span}\{\partial_{x^i_1} \mid i = 1, \dots, m\}$. Clearly, conditions (R1) and (R2) are satisfied for such a pair $(X, \mathcal{V})$, with the parameter $k = 1$. Direct computations give

$$\begin{aligned}
\mathrm{ad}_X \partial_{x^i_1} &= -\partial_{x^i_0} + \sum_{j,p} 2\Gamma^j_{ip} x^p_1 \partial_{x^j_1} \\
\mathrm{ad}^2_X \partial_{x^i_1} &= \sum_{j,p} 2\Gamma^j_{ip} x^p_1 \mathrm{ad}_X \partial_{x^j_1} \\
&+ \sum_{j,p,q,r} \Big(2\partial_{x^q_0}(\Gamma^j_{ip}) x^p_1 x^q_1 - 2\Gamma^j_{ir}\Gamma^r_{pq} x^p_1 x^q_1 - \partial_{x^i_0}(\Gamma^j_{pq}) x^p_1 x^q_1 \Big) \partial_{x^j_1}.
\end{aligned}$$

Since $k = 1$, there is only one curvature operator K_0. From Proposition 2.5 we get

$$K_0 = \left(\sum_{p,q,r} \Big(\partial_{x^i_0}(\Gamma^j_{pq}) - \partial_{x^q_0}(\Gamma^j_{ip}) + \Gamma^j_{ir}\Gamma^r_{pq} - \Gamma^r_{ip}\Gamma^j_{rq} \Big) x^p_1 x^q_1 \right)_{i,j=1,\dots m}.$$

We see that K_0 is a quadratic function in the coordinates on fibres x^p_1, x^q_1 and we recognise that the coefficients $R^j_{iqp} = \partial_{x^i_0}(\Gamma^j_{pq}) - \partial_{x^q_0}(\Gamma^j_{ip}) + \sum_r (\Gamma^j_{ir}\Gamma^r_{pq} - \Gamma^r_{ip}\Gamma^j_{rq})$ are components of the curvature tensor R of the connection ∇. Denoting $Y = y = (x^i_1)$ and $x = (x^i_0)$, we get

$$K_0(x, y)(Z) = R_x(Z, Y)Y.$$

A simple calculation using Bianchi identity for R gives the converse formula

$$\begin{aligned}
R_x(Y, Z)W &= \frac{1}{3}(K_0(x, z + w)(Y) - K_0(x, y + w)(Z) \\
&\quad - K_0(x, z)(Y) - K_0(x, w)(Y) + K_0(x, y)(Z) + K_0(x, w)(Z)),
\end{aligned}$$

where $x = (t, (x_0^i))$ and we identify the elements $y = Y, z = Z, w = W$. We also have

$$\mathcal{H}^1 = \operatorname{span}\left\{ -\partial_{x_0^i} + \sum_{j,p} \Gamma_{ip}^j x_1^p \partial_{x_1^j} \right\}.$$

It follows that the quasi-connection $\mathcal{H}^1$ coincides, in the Ehresmann sense, with the connection ∇. More general cases will be considered in Chapter 3.

ODE with constant coefficients. Consider the following linear system of ODEs

$$x^{(k+1)} + A_{k-1}x^{(k-1)} + \cdots + A_1 x' + A_0 x = 0.$$

where $x = (x^1, \ldots, x^m)^T \in \mathbb{R}^m$ and $A_i \in \mathbb{R}^{m\times m}$ are constant matrices. Let $J^k(1,m)$ be the space of k-jets of functions from $\mathbb{R}$ to $\mathbb{R}^m$. The standard global coordinate functions on $J^k(1,m)$ are denoted (t, x_i^j), where $i = 0, \ldots, k$ and $j = 1, \ldots, m$. We set $x_i = (x_i^1, \ldots, x_i^m)^T$. Let

$$X = \partial_t + \sum_{i=1}^{k}\sum_{j=1}^{m} x_i^j \partial_{x_{i-1}^j} + \sum_{j=1}^{m} F^j \partial_{x_k^j}$$

be the corresponding total derivative, where we abbreviate

$$(F^1, \ldots, F^m)^T = \Big(A_{k-1}x_{k-1} + \cdots + A_1 x_1 + A_0 x_0 \Big).$$

Taking $\mathcal{V} = \operatorname{span}\{\partial_{x_k^j} \mid j = 1, \ldots, m\}$ one can prove that $(\partial_{x_k^1}, \ldots, \partial_{x_k^m})$ is a normal frame of the pair $(X, \mathcal{V})$. Moreover, a simple induction gives that the corresponding curvature operators are given by the matrices $K_i = (-1)^i A_i$.

2.4 Canonical bundle

In order to identify further invariants of dynamic pairs, and solve the equivalence problem, we introduce a sub-bundle $\mathcal{F}_N(X, \mathcal{V})$ of the tangent frame bundle, called canonical or normal frame bundle of $(X, \mathcal{V})$. Assume $(X, \mathcal{V})$ satisfy (R1) and (R2).

Definition 2.7 *Fix $x \in M$. A frame F_x in T_xM of the form*

$$F_x = (V^0, V^1, \ldots, V^k, X(x)), \quad \text{where} \quad V^i = (V_1^i, \ldots, V_m^i),$$

is called normal at x *if there is a local normal frame $V = (V_1, \ldots, V_m)$ in $\mathcal{V}$ such that $V_j^i = (\operatorname{ad}_X^i V_j)(x)$. Equivalently, F_x is called normal if there is a local frame V of $\mathcal{V}$, possibly not normal, such that*

$$V^0 = V(x), \quad V^i = A_i V(x), \qquad i = 1, \ldots, k, \tag{15}$$

where the operators $A_i = \pi^i \circ \operatorname{ad}_X^i : \mathcal{V} \to \mathcal{H}_i$ are the vector bundle isomorphisms from Section 2.2. Both definitions coincide since they coincide for a local normal frame V and, given the pair $(X, \mathcal{V})$, the second one depends on the value $V(x)$, only. A local normal frame in TM *is a smooth local field of normal frames $x \mapsto F_x$.*

Denote by $\mathcal{F}_N(x)$ the set of all normal frames in T_xM. The set

$$\mathcal{F}_N = \mathcal{F}_N(X, \mathcal{V}) := \bigcup_{x \in M} \mathcal{F}_N(x)$$

forms a bundle over M, called *normal frame bundle* corresponding to the pair $(X, \mathcal{V})$ or *canonical bundle* of $(X, \mathcal{V})$. This is a smooth sub-bundle,

$$\mathcal{F}_N \subset \mathcal{F},$$

of the bundle $\pi\colon \mathcal{F} \to M$ of all frames on M.

There is a natural right action of $Gl(m)$ on $\mathcal{F}_N$ given by

$$(F_x, A) \longmapsto R_A F_x = (V^0 A, V^1 A, \ldots, V^k A, X(x))$$

where $A \in Gl(m)$ and $V^i A = \left(\sum_{j=1}^m A_1^j V_j^i, \ldots, \sum_{j=1}^m A_m^j V_j^i \right)$. This action is briefly denoted

$$R_A F_x = F_x \mathbb{A}, \quad \text{where } \ \mathbb{A} = \mathrm{diag}(A, A, \ldots, A, 1).$$

With respect to this action $\mathcal{F}_N$ is a principal $Gl(m)$-bundle. This means that $Gl(m)$ acts transitively and freely on each fiber $\mathcal{F}_N(x)$. To check this pick two normal frames $F(x)$ and $F'(x)$ at x, given by local normal frames V and V' of $\mathcal{V}$. Then, by Proposition 2.2, there is a matrix valued function G such that $V = V'G$, where $X(G) = 0$. Thus $\mathrm{ad}_X V = (\mathrm{ad}_X V')G$ and, generally, $\mathrm{ad}_X^i V = (\mathrm{ad}_X^i V')G$. Thus, the corresponding local normal frames F and F' are related by the block diagonal matrix

$$\mathbb{G} = \mathrm{diag}(G, \ldots, G, 1), \quad F = F'\mathbb{G}.$$

If $x \in M$ is fixed, the group of block diagonal matrices $\mathbb{G}(x)$ is isomorphic to the group $Gl(m)$, with the isomorphism given by $\mathbb{G}(x) \mapsto G(x)$. We conclude that

Proposition 2.8 *$\mathcal{F}_N = \mathcal{F}_N(X, \mathcal{V})$ is a principal $Gl(m)$-bundle and a sub-bundle of the bundle $\mathcal{F}$ of frames on M, i.e., a $Gl(m)$-structure. Two regular pairs $(X, \mathcal{V})$ and $(X', \mathcal{V}')$ are locally equivalent if and only if the corresponding canonical bundles $\mathcal{F}_N(X, \mathcal{V})$ and $\mathcal{F}_N(X', \mathcal{V}')$ are isomorphic as $Gl(m)$-structures.*

Proof. The second statement follows directly from the definition of $\mathcal{F}_N(X, \mathcal{V})$ and from equivariance of the Lie bracket with respect to diffeomorphisms. Note that any isomorphism of $\mathcal{F}_N(X, \mathcal{V})$ and $\mathcal{F}_N(X', \mathcal{V}')$ is given by a diffeomorphism Φ of M such that $\Phi_* \mathcal{F}_N(X, \mathcal{V}) = \mathcal{F}_N(X', \mathcal{V}')$ which implies $\Phi_* \mathcal{V} = \mathcal{V}'$. □

2.5 Canonical frame and connection, equivalence

Assume that a pair $(X, \mathcal{V})$ satisfies conditions (R1) and (R2). To any such pair we will assign, in an invariant way, a frame on the canonical bundle. Moreover, we will show that the normal frame bundle $\pi\colon \mathcal{F}_N \to M$ with the structural group

$$\{\mathrm{diag}(A, \ldots, A, 1) \in Gl(n) \mid A \in Gl(m)\} \simeq Gl(m),$$

where $n = (k+1)m + 1$, possesses a canonical principal connection understood as a smooth distribution $\mathcal{D}$ on $\mathcal{F}_N$, which is transversal to the fibres and which satisfies

$$\mathcal{D}(R_A F) = (R_A)_* \mathcal{D}(F),$$

for any $F \in \mathcal{F}_N$ and any $A \in Gl(m)$. Here R_A is the transformation of $\mathcal{F}_N$ induced by the right action: $R_A F = F\mathbb{A}$, $\mathbb{A} = \{\mathrm{diag}(A, \ldots, A, 1)$.

Note that any distribution $\mathcal{D}$ on $\mathcal{F}_N$ of rank $n = \dim M$ which is transversal to fibers defines a frame on $\mathcal{F}_N$. Namely, let F be a point in $\mathcal{F}_N$ which is an n-tuple of vectors $F = (V^0, V^1, \ldots, V^k, X)$ on M, where $V^i = (V^i_1, \ldots, V^i_m)$. Then there exists a unique tuple of vectors lifted to $\mathcal{D}$,

$$\mathbf{F} = (\mathbf{V}^i_j, \mathbf{X} \mid j = 1, \ldots, m, \quad i = 0, \ldots, k),$$

i.e., such that $\mathbf{V}^i_j, \mathbf{X} \in \mathcal{D}(F)$ and

$$\pi_*(\mathbf{V}^i_j) = V^i_j, \qquad \pi_*(\mathbf{X}) = X. \tag{16}$$

We will briefly denote $\mathbf{V}^i = (\mathbf{V}^i_1, \ldots, \mathbf{V}^i_m)$ and $\mathbf{V} = (\mathbf{V}^0, \ldots, \mathbf{V}^k)$.

In addition, on $\mathcal{F}_N$ there are defined fundamental vector fields which are tangent to the fibres of $\mathcal{F}_N$ and which come from the action of the structural group $Gl(m)$. The fundamental vector fields with the Lie bracket form a Lie algebra isomorphic to the Lie algebra $\mathfrak{g}$ of the structural group, where

$$\mathfrak{g} = \{\mathrm{diag}(a, a, \ldots, a, 0) \mid a \in \mathfrak{gl}(m)\} \subset \mathfrak{gl}(n)$$

is naturally isomorphic to $\mathfrak{gl}(m)$. Vector fields corresponding to matrices e^s_t with 1 at the position $(s,t) = (\mathrm{row}, \mathrm{column})$ and 0 elsewhere will be denoted $\mathbf{G}^t_s$ and the collection of all such vector fields will be shortly denoted $\mathbf{G} = (\mathbf{G}^t_s)_{s,t=1,\ldots,m}$. Clearly the tuple $(\mathbf{F}, \mathbf{G})$ is a frame on $\mathcal{F}_N$ defined uniquely by $\mathcal{D}$.

Vice versa, any frame $(\mathbf{F}, \mathbf{G})$ on $\mathcal{F}_N$, where $\mathbf{G}$ consists of the fundamental vector fields, defines a unique Ehresmann connection $\mathcal{D} = \mathrm{span}\{\mathbf{X}, \mathbf{V}^i_j : 0 \le i \le k, 1 \le j \le m\}$ on $\mathcal{F}_N$. In our case both the connection and the frame will be $Gl(m)$ invariant.

We will show that one can choose $\mathcal{D}$ in a canonical way. Then two pairs $(X, \mathcal{V})$ and $(X', \mathcal{V}')$ will be equivalent if and only if the corresponding connections are equivalent (or if and only if the corresponding frames are equivalent).

Let $(\theta^j_i, \alpha, \omega^s_t)$ be the coframe on $\mathcal{F}_N$ dual to the frame $(\mathbf{V}^i_j, \mathbf{X}, \mathbf{G}^t_s)$. We put

$$T^{ijr}_{pql} = \theta^r_l([\mathbf{V}^i_p, \mathbf{V}^j_q]), \qquad \hat{T}^{ir}_{pl} = \theta^r_l([\mathbf{X}, \mathbf{V}^i_p]),$$

$$S^{ij}_{pq} = \alpha([\mathbf{V}^i_p, \mathbf{V}^j_q]), \qquad \hat{S}^i_p = \alpha([\mathbf{X}, \mathbf{V}^i_p])$$

and

$$R^{ijs}_{pqt} = \omega^s_t([\mathbf{V}^i_p, \mathbf{V}^j_q]), \qquad \hat{R}^{is}_{pt} = \omega^s_t([\mathbf{X}, \mathbf{V}^i_p]).$$

These functions are called structural coefficients of the frame. We will see in the next section that they are coefficients of the torsion and curvature of the connection $\mathcal{D}$. Some of them can be normalised so that the connection and frame are unique.

Namely, the main result of this part of the paper says the following:

Theorem 2.9 *Assume that* $(X, \mathcal{V})$ *satisfies (R1) and (R2), with a given* $k \geq 1$.
(a) There exists a unique principal connection on $\mathcal{F}_N$ *such that the corresponding frame* $(\mathbf{V}, \mathbf{X}, \mathbf{G})$ *satisfies the following conditions:*

$$\hat{T}^{kr}_{pk} = 0, \qquad p, r = 1, \ldots, m, \tag{17}$$
$$T^{01r}_{pq1} = 0, \qquad p, q, r = 1, \ldots, m, \tag{18}$$
$$T^{0ir}_{pq0} = 0, \qquad p, q, r = 1, \ldots, m, \quad i = 1, \ldots, k. \tag{19}$$

It then also satisfies

$$\hat{S}^i_p = 0, \qquad p = 1, \ldots, m, \quad i = 0, \ldots, k-1, \tag{20}$$

and

$$\hat{T}^{ir}_{pl} = \delta^r_p \delta^{i+1}_l, \quad p, r = 1, \ldots, m, \quad i = 0, \ldots, k-1, \ l = 0, \ldots, k. \tag{21}$$

(b) Two pairs $(X, \mathcal{V})$ *and* $(X', \mathcal{V}')$ *on* M *are diffeomorphic if and only if the corresponding frames on* $\mathcal{F}_N(X, \mathcal{V})$ *and* $\mathcal{F}_N(X', \mathcal{V}')$ *are diffeomorphic.*
(c) The symmetry group of $(X, \mathcal{V})$ *is at most* $(k+1)m + m^2 + 1$*-dimensional, and it is maximal if and only if* $(X, \mathcal{V})$ *is locally equivalent to a pair defined by the system*

$$x^{(k+1)} + K_{k-1} x^{(k-1)} + \cdots + K_1 x' + K_0 x = 0,$$

where $x \in \mathbb{R}^m$ *and matrices* K_i *are diagonal and constant.*

Remark. Conditions (17), (18), (19), (20), (21) can be equivalently stated as

$$[\mathbf{X}, \mathbf{V}^k_p] = 0 \mod \mathbf{V}^0, \ldots, \mathbf{V}^{k-1}, \mathbf{G}, \mathbf{X}, \tag{22}$$
$$[\mathbf{V}^0_p, \mathbf{V}^1_q] = 0 \mod \mathbf{V}^2, \ldots, \mathbf{V}^k, \mathbf{G}, \mathbf{X}, \tag{23}$$
$$[\mathbf{V}^0_p, \mathbf{V}^i_q] = 0 \mod \mathbf{V}^1, \ldots, \mathbf{V}^k, \mathbf{G}, \mathbf{X}, \quad i = 2, \ldots, k, \tag{24}$$
$$[\mathbf{X}, \mathbf{V}^i_p] = \mathbf{V}^{i+1}_p \mod \mathbf{G}, \quad i = 0, \ldots, k-1. \tag{25}$$

Namely, (17) is equivalent to (22); (18) and (19) are (jointly) equivalent to (23) and (24); (20) and (21) are (jointly) equivalent to (25). We can also see from (29) below that the equalities

$$[\mathbf{G}^t_s, \mathbf{X}] = 0, \qquad [\mathbf{G}^t_s, \mathbf{V}^i_j] = \delta^t_j \mathbf{V}^i_s. \tag{26}$$

hold modulo the vertical distribution span$\{\mathbf{G}\}$. From the invariance of the distribution $\mathcal{D} = \text{span}\{\mathbf{X}, \mathbf{V}^i_j\}$ under the action of $Gl(m)$ it follows that they hold exactly.

Lemma 2.10 *There is a unique vector field* $\mathbf{X}$ *on* $\mathcal{F}_N$ *satisfying* (17) *and* $\pi_* \mathbf{X} = X$. *If* $t \mapsto F(t) \in \mathcal{F}_N$ *is an integral curve of* $\mathbf{X}$ *then* $F(t)$ *is a normal frame along* $x(t) = \pi(F(t))$*, i.e. there is a local normal frame* $\hat{F}$ *in* TM *such that* $F(t) = \hat{F}(x(t))$.

Proof. Let us fix an open subset $U \subset M$ and a section $U \ni x \mapsto F_x \in \mathcal{F}_N$ of normal frames, $F_x = (V^0(x), \ldots, V^k(x), X(x))$. Then a point $\nu \in \mathcal{F}_N$ is encoded by its projection $x = \pi(\nu) \in U$ and an element $G = G(\nu) \in Gl(m)$ such that

$$\nu = (V^0(x)G, V^1(x)G, \ldots, V^k(x)G, X(x)).$$

In this way we get a local trivialisation $\mathcal{F}_N = U \times Gl(m)$. Using this trivialisation we introduce natural coordinates on fibers of $\mathcal{F}_N$ (the coordinates depend on the initial choice of the section $x \mapsto F_x$ but our construction will not). The vertical vector fields $\mathbf{G}^t_s$, in these coordinates, are given by

$$\mathbf{G}^t_s = \sum_r G^r_s \partial_{G^r_t}, \tag{27}$$

where $G^r_s(\nu)$ is the $(r,s) = (row, column)$ coordinate of the above defined matrix $G = G(\nu)$. The lifted vector fields $\mathbf{X}$ and $\mathbf{V}^i_j$ (see (16)) can be written in coordinates as

$$\mathbf{X} = X + \sum_{s,t} \alpha^s_t \mathbf{G}^t_s, \qquad \mathbf{V}^i_j = \sum_p G^p_j V^i_p + \sum_{s,t} \beta^{is}_{jt} \mathbf{G}^t_s, \tag{28}$$

for some functions α^s_t and β^{is}_{jt}. By direct checking we see that

$$[\mathbf{G}^t_s, \mathbf{X}] = 0 \mod \mathbf{G} \quad \text{and} \quad [\mathbf{G}^t_s, \mathbf{V}^i_j] = \delta^t_j \mathbf{V}^i_s \mod \mathbf{G}. \tag{29}$$

Let $H = (H^i_j)$ denote the matrix of functions which satisfies

$$[X, V^k_j] = \sum_s H^s_j V^k_s \mod \mathcal{V}^{k-1} \oplus \mathcal{X}. \tag{30}$$

This relation and (29) give

$$[\mathbf{X}, \mathbf{V}^k_j] = \sum_s (\alpha^s_j + \hat{H}^s_j) \mathbf{V}^k_s \mod \mathbf{V}^0, \ldots, \mathbf{V}^{k-1}, \mathbf{X}, \mathbf{G},$$

where $\hat{H}^i_j = \sum_{p,q} (G^{-1})^p_j H^q_p G^i_q$ are coefficients of the matrix $\hat{H} = G^{-1} H G$. Consequently, we have $\hat{T}^{ks}_{jk} = \alpha^s_j + \hat{H}^s_j$ and condition (17) is satisfied if and only if $\alpha^s_t = -H^s_j$. In this way the first part of the lemma is proved and we get the explicit formula

$$\mathbf{X} = X - \sum_{s,t} \hat{H}^s_t \mathbf{G}^t_s. \tag{31}$$

The second part follows directly from the first part if we choose the section of normal frames F with $V^i = (\mathrm{ad}^i_X V)$, where $V = (V_1, \ldots, V_m)$ is a local normal frame of $\mathcal{V}$. Then $H^i_j = 0$ and $\mathbf{X} = X$. Thus any integral curve in $\mathcal{F}_N$ of $\mathbf{X}$ is a section of a local normal frame in TM coming from a local normal frame VG (the matrix G is kept constant along such a curve). $\square$

Proof of Theorem 2.9. (a) We will use the local section $U \ni x \mapsto F_x \in \mathcal{F}_N$ of normal frames, $F_x = (V^0(x), \ldots, V^k(x), X(x))$, and local coordinates defined in the proof of Lemma 2.10. Here $V^i = (V^i_1, \ldots, V^i_m)$. The lifted vector fields can be written

$$\mathbf{X} = X - \sum_{s,t} \hat{H}^s_t \mathbf{G}^t_s, \qquad \mathbf{V}^i_j = \sum_p G^p_j V^i_p + \sum_{s,t} \beta^{is}_{jt} \mathbf{G}^t_s. \tag{32}$$

Assume that

$$[V^0_p, V^l_q] = \sum_{i,j} b^{lj}_{pqi} V^i_j + c^l_{pq} X. \tag{33}$$

We will show that β^{is}_{jt} are uniquely determined by conditions (18) and (19), namely

$$\beta^{is}_{jt} = \sum_{p,q,r} a^{ir}_{pq} G^p_j G^q_t (G^{-1})^s_r, \quad \text{where} \tag{34}$$

$$a^{0r}_{pq} = -b^{1r}_{pq1} \quad \text{and} \quad a^{ir}_{pq} = b^{ir}_{pq0}, \quad \text{for} \;\; i \geq 1. \tag{35}$$

Taking into account that

$$\sum_s (G^{-1})^s_r \mathbf{G}^t_s = \partial_{G^r_t}$$

we will see that the resulting frame on $\mathcal{F}_N$ consists of vertical vector fields $\mathbf{G}^t_s$ and of

$$\mathbf{X} = X - \sum_{s,t} \hat{H}^s_t \mathbf{G}^t_s, \qquad \mathbf{V}^i_j = \sum_p G^p_j (V^i_p + \sum_{q,r} a^{ir}_{pq} \hat{G}^q_r), \tag{36}$$

where

$$\hat{G}^q_r = \sum_t G^q_t \partial_{G^r_t}$$

can be identified with right invariant vector fields on $Gl(m)$ corresponding to matrices e^r_q, with 1 at the position (q, r) and 0 elsewhere.

To prove (34) note that from (32) and (33) we obtain

$$[\mathbf{V}^0_p, \mathbf{V}^l_q] = \sum_{s,t,i,j} G^s_p G^t_q b^{lj}_{sti} V^i_j + \sum_{s,j} \beta^{0s}_{pq} G^j_s V^l_j - \sum_{s,j} \beta^{ls}_{qp} G^j_s V^0_j \mod \mathbf{G}, \mathbf{X}. \tag{37}$$

Thus (18) is equivalent to the equations

$$\sum_{s,t} G^s_p G^t_q b^{1j}_{st1} = -\sum_s \beta^{0s}_{pq} G^j_s, \tag{38}$$

for $p, q, j = 1, \ldots, m$, and (19) is equivalent to the equations:

$$\sum_{s,t} G^s_p G^t_q b^{lj}_{st0} = \sum_s \beta^{ls}_{qp} G^j_s,$$

for $p, q, j = 1, \ldots, m$ and $l = 1, \ldots, k$. It follows from these equations that (β^{ls}_{pq}) are uniquely determined by (18) and (19) and are given by (34).

It is evident from (36) that so constructed frame is invariant under the right action of $Gl(m)$ on $\mathcal{F}_N$. This means that the distribution $\mathcal{D} = \text{span}\{\mathbf{X}, \mathbf{V}^i_j\}$, where $i = 0, \ldots, k$ and $j = 1, \ldots, m$, defines a principal connection on $\mathcal{F}_N$. Thus statement (a) of the theorem is proved. Note that condition (25) is a direct consequence of (36), thus so are conditions (20) and (21) .

(b) Our construction is invariant with respect to the action of diffeomorphisms, thus $(X, \mathcal{V})$ and $(X', \mathcal{V}')$ are equivalent if and only if the corresponding frames are.

(c) The first part of statement (c) follows directly from statement (a), since the dimension of the symmetry group of a frame is bounded from above by the dimension of the ambient manifold (see [18], Chapter 1, Theorem 3.2). In our case this dimension is $\dim \mathcal{F}_N = (k + 1)m + m^2 + 1$.

The symmetry group of $(X, \mathcal{V})$ is maximal if and only if the structural functions of the frame $(\mathbf{G}^s_t, \mathbf{X}, \mathbf{V}^i_j)$ are constant. We have (26) and, by definition, Lie brackets of vertical vector fields $\mathbf{G}^t_s$ behave as generators of $\mathfrak{gl}(m)$, thus have constant structural functions. Therefore, in order to finish the proof we have to check when the structural functions of $[\mathbf{X}, \mathbf{V}^i_j]$ and of $[\mathbf{V}^i_p, \mathbf{V}^l_q]$ are constant as functions on $\mathcal{F}_N$.

The coefficients β^i_j in (34) are homogeneous of order one with respect to G. It implies that the structural functions of the bracket $[\mathbf{V}^i_p, \mathbf{V}^l_q]$ are homogeneous of order either one (functions next to $\mathbf{G}^s_t$ and $\mathbf{V}^i_j$), or two (a function next to $\mathbf{X}$). Thus, in order to be constant they have to vanish. Condition (25), already proved, says:

$$[\mathbf{X}, \mathbf{V}^i] = \mathbf{V}^{i+1} \mod \mathbf{G}, \quad i = 0, \ldots, k-1.$$

Moreover,

$$[\mathbf{X}, \mathbf{V}^k] = -\mathbf{V}^0 \hat{K}_0 + \cdots + (-1)^k \mathbf{V}^{k-1} \hat{K}_{k-1} \mod \mathbf{X}, \mathbf{G}$$

where, in coordinates, $\hat{K}_i = G^{-1} K_i G$. By homogeneity argument, we see that in the most symmetric case coefficients next to $\mathbf{X}$ and $\mathbf{G}$ vanish, whereas K_i have to be diagonal and constant so that $G^{-1} K_i G = K_i$.

In conclusion we see that structural functions of a pair $(X, \mathcal{V})$ with maximal symmetry group coincide with the structural functions of the pair $(X, \mathcal{V})$ corresponding to the system

$$x^{(k+1)} + K_{k-1} x^{(k-1)} + \cdots + K_1 x' + K_0 x = 0,$$

with diagonal and constant K_i. (The canonical frame of the pair $(X, \mathcal{V})$, which corresponds to the linear system, is given by $\mathbf{G}^s_t$, $\mathbf{X} = X$ and $\mathbf{V}^i_j = G^s_j \mathrm{ad}^i_X \partial_{x^s_0}$, since vector fields $\mathrm{ad}^i_X \partial_{x^s_0}$ are constant and all their Lie brackets vanish.) □

There is also another way of defining the canonical frame on the bundle $\mathcal{F}_N$, with condition (19) replaced by a condition on the structural functions $\hat{R}^{is}_{pt} = \omega^s_t([\mathbf{X}, \mathbf{V}^i_p])$. It will be more convenient in certain situations.

Theorem 2.11 *Assume that $(X, \mathcal{V})$ satisfies (R1) and (R2).*
There exists a unique principal connection on $\mathcal{F}_N$ such that the corresponding frame $(\mathbf{V}, \mathbf{X}, \mathbf{G})$ satisfies (17)*,* (18) *and the following condition:*

$$\hat{R}^{is}_{pt} = 0, \quad p, s, t = 1, \ldots, m, \quad i = 0, \ldots, k-1. \tag{39}$$

Two pairs $(X, \mathcal{V})$ and $(X', \mathcal{V}')$ are diffeomorphic if and only if the corresponding frames are diffeomorphic.

Note that condition (39), taking into account the form (32) of $\mathbf{V}^i_j$, is equivalent to

$$[\mathbf{X}, \mathbf{V}^i] = \mathbf{V}^{i+1}, \qquad i = 0, \ldots, k-1. \tag{40}$$

Proof. The proof is similar to the proof of Theorem 2.9. We get from (17) and Lemma 2.10, in coordinates used in its proof, that $\mathbf{X} = X - \sum_{s,t} \hat{H}^s_t \mathbf{G}^t_s$. Then we use equation (37) with $l = 1$ and we see that condition (18) is equivalent to (38), and to (34) with $i=0$. In this way we normalise $\mathbf{V}^0=(\mathbf{V}^0_1,\ldots,\mathbf{V}^0_m)$ so that (32) gives

$$\mathbf{V}^0_j=\sum_p G^p_j(V^0_p-\sum_{q,r} b^{1r}_{pq1}\hat{G}^q_r), \tag{41}$$

where $\hat{G}^q_r=\sum_t G^q_t\partial_{G^r_t}$ and the coefficients b^{1r}_{pq1} are determined from the relation

$$[V^0_p,V^1_q]=\sum_j b^{1j}_{pq1}V^1_j \mod X,\ V^i_j,\quad i=0,2,3,\ldots,k,\ j=1,\ldots,m, \tag{42}$$

where $V^0(x),\ldots,V^k(x),X(x)$ is a section of the normal frame bundle $\mathcal{F}_N$. Then (40) defines $\mathbf{V}^i$ uniquely. The resulting frame defines a principal connection.

Note that we can take $V^i_j=\mathrm{ad}^i_X V_j$, where $(V_1,\ldots,V_m)$ is a local normal frame of $\mathcal{V}$. In this case we can apply formula (31) with $\hat{H}=0$, thus $\mathbf{X}=X$. We get

$$\mathbf{V}^i_j=\sum_p G^p_j\left(V^i_p-\sum_{q,r}X^i(b^{1r}_{pq1})\hat{G}^q_r\right),\quad i=1,\ldots,k, \tag{43}$$

which follows from $\mathbf{X}=X$, (40) and (41). □

Definition 2.12 *The frame of Theorem 2.9 will be called* first canonical frame*, denoted* $\mathbf{V}$*, and the frame of Theorem 2.11 the* second canonical frame*, denoted* $\tilde{\mathbf{V}}$*. By definition, the vector fields* $\mathbf{X}$ *and* $\mathbf{G}$ *in both canonical frames coincide.*

The following direct corollaries of the proofs of Theorems 2.9, 2.11 and Definition 2.7 will be useful.

Corollary 2.13 *The coefficients* (35)*, needed for determining the first normal frame* (36)*, and the coefficients* (43)*, needed for determining the second normal frame* (41)*, can be computed using any local frame* V *of* $\mathcal{V}^k$ *by defining the section of* $\mathcal{F}_N$ *via*

$$F_x=(V^0(x),\ldots,V^k(x),X(x)),\quad \text{where}\quad V^i(x)=A_iV(x).$$

Corollary 2.14 *If* $k=1$*, the operator* $A_1=\pi^1\circ\mathrm{ad}_X:\mathcal{V}\to\mathcal{H}^1$ *has the form* $A_1(Y)=\mathrm{ad}_X(Y)-\frac{1}{2}VH_1\hat{Y}$*, for* $Y=\sum_j V_jf_j$ *and* $\hat{Y}=(f_1,\ldots,f_m)$*,* $f_j\in C^\infty(M)$*.*

Example. For illustration we continue the example of geodesic equation from Section 2.3 (see Section 3.4 for a general case). We will use the above corollaries. In order to solve the equivalence problem one has to construct $\mathbf{V}^0$, $\mathbf{V}^1$ and $\mathbf{G}$ on $\mathcal{F}_N=\mathbb{R}\times TN\times Gl(m)$ (we locally trivialise the canonical bundle). The vector fields in $\mathbf{G}$ are standard, given by (27).

Take $V_j=V^0_j=\partial_{x^j_1}$ and compute the corresponding components $V^1_j=A_1V_j$ of the normal frame. From the formulas for ad_XV_j and $\mathrm{ad}^2_XV_j$ computed earlier we see that the matrix H_1 is $(H_1)^j_i=2\sum_p\Gamma^j_{ip}x^p_1$. This and $\mathrm{ad}_XV_j=-\partial_{x^j_0}+2\sum_{i,p}\Gamma^i_{jp}x^p_1\partial_{x^i_1}$ give

$$V^1_j=A_1V_j=-\partial_{x^j_0}+\sum_{p,i}\Gamma^i_{jp}x^p_1\partial_{x^i_1}.$$

We have

$$[X,V_j^1] = -\sum_{i,u}\Gamma^i_{ju}x_1^u V_j^0 \mod \mathcal{V}^1, \qquad [V_p^0,V_q^1] = \sum_j \Gamma^j_{pq}V_j^0,$$

i.e. the matrix $H=(H^i_j)$ defined via (30) is $H^i_j = -\sum_{i,u}\Gamma^i_{ju}x_1^u$ and the coefficients defined in the proof of Theorem 2.9 via formula (33) are $b^{1j}_{pq1}=0$ and $b^{1j}_{pq0}=\Gamma^j_{pq}$. Hence, formulas (36) and (35) give

$$\begin{aligned}
\mathbf{X} &= X + \sum_{p,q,r,s,t,u}(G^{-1})^p_t\Gamma^q_{pu}x_1^u G^s_q G^r_s \partial_{G^r_t}\\
\mathbf{V}_i^0 &= \sum_j G^j_i \partial_{x_1^j},\\
\mathbf{V}_i^1 &= \sum_j G^j_i\Big(-\partial_{x_0^j} + \sum_{p,q}\Gamma^p_{jq}x_1^q\partial_{x_1^p} + \sum_{p,q,r}\Gamma^p_{jr}G^r_q\partial_{G^p_q}\Big).
\end{aligned}$$

It is straightforward to verify that

$$[\mathbf{V}_i^0,\mathbf{V}_j^0]=0, \qquad [\mathbf{V}_i^0,\mathbf{V}_j^1]=0, \qquad [\mathbf{X},\mathbf{V}_j^0]=\mathbf{V}_j^1, \qquad [\mathbf{X},\mathbf{V}_j^1] = -\hat{K}^i_{0j}\mathbf{V}_i^1 \mod \mathbf{G},$$

with $\hat{K}_0 = G^{-1}K_0G$ and K_0 computed earlier, and

$$[\mathbf{V}_i^1,\mathbf{V}_j^1] = \sum_{s,t,p,q} G^s_iG^t_jR^p_{stq}x_1^q\partial_{x_1^p} + \sum_{s,t,p,q,r} G^s_iG^t_jG^r_qR^p_{stq}\partial_{G^p_r},$$

where

$$R^p_{stq} = \partial_{x_0^t}(\Gamma^p_{sq}) - \partial_{x_0^s}(\Gamma^p_{tq}) + \sum_r\Big(\Gamma^p_{tr}\Gamma^r_{sq} - \Gamma^p_{sr}\Gamma^r_{tq}\Big)$$

are components of the curvature tensor of ∇. Due to formula (26) we see that there are no more nontrivial structural functions on $\mathcal{F}_N$.

2.6 Torsion and curvature

We can describe the principal connection given by Theorem 2.9 (or Theorem 2.11) by a connection form. The corresponding curvature and torsion of the connection can be used as an alternative description of the invariants of the pair $(X,\mathcal{V})$.

Denote $\phi = (\theta^j_i,\alpha)$ and $\omega=(\omega^s_t)$. Then ω is a 1-form on $\mathcal{F}_N$ with values in the Lie algebra $\mathfrak{g}\simeq\mathfrak{gl}(m)$, called the connection form. It defines the connection $\mathcal{D}$ by $\mathcal{D}(F)=\ker\omega(F)$. The 1-form ϕ on $\mathcal{F}_N$ with values in $\mathbb{R}^n$ is called the canonical soldering form, where $n=(k+1)m+1$ is the dimension of the manifold M. The following Cartan structural equations are satisfied

$$d\phi + \omega\wedge\phi = \Theta, \qquad d\omega + \omega\wedge\omega = \Omega$$

and define torsion Θ and curvature Ω of the connection, both being 2-forms with values in $\mathbb{R}^n$ and $\mathfrak{g}$, respectively.

The structural functions from Section 2.5 can be described in terms of the torsion and curvature. Namely, since Θ has values in $\mathbb{R}^n$, where $n=(k+1)m+1$, we can decompose $\Theta=(\Theta_i^j,\hat{\Theta})_{i=0,\dots,k}^{j=1,\dots,m}$. Similarly, we can write $\Omega=(\Omega_t^s)_{s,t=1,\dots,m}$. Then

$$\Theta_i^j=\sum_{\substack{p<q\\ s,t}} T_{pqi}^{stj}\theta_t^q\wedge\theta_s^p+\sum_{l,r}\hat{T}_{ri}^{lj}\theta_l^r\wedge\alpha,\quad \hat{\Theta}=\sum_{\substack{p<q\\ i,j}} S_{pq}^{ij}\theta_j^q\wedge\theta_i^p+\sum_{i,p}\hat{S}_p^i\theta_i^p\wedge\alpha$$

and

$$\Omega_t^s=\sum_{\substack{p<q\\ i,j}} R_{pqt}^{ijs}\theta_j^q\wedge\theta_i^p+\sum_{i,p}\hat{R}_{pt}^{is}\theta_i^p\wedge\alpha.$$

The coincidence of the coefficients in these formulas and the structural functions introduced in Section 2.5 follows from the general formula $d\beta(Y,Z)=-\beta([Y,Z])$ for a 1-form β belonging to a coframe and Y, Z being frame vector fields in $\mathcal{D}=\ker\omega$.

Note that the structural functions satisfy relations which follow from the Bianchi identities

$$D\Theta=\Omega\wedge\theta\quad\text{and}\quad D\Omega=0,$$

where D denotes the covariant derivative. We will not use them in full generality in the present paper, but only restrict to the simplest cases, important for our applications. Namely, in the next section we will consider cases $k=1$ and $k=2$ with additional integrability conditions. We will work in terms of canonical frame rather than connection and use the identities in the form of Jacobi identity for the vector fields in the canonical frame.

Remark. Uniqueness in Theorem 2.9 is obtained by normalising torsion, setting part of its coefficients to zero (conditions (17), (18) and (19)). Uniqueness in Theorem 2.11 is reached by normalising torsion (conditions (17), (18)) and curvature (condition (39)). Note that (39) (respectively, (40)) means that the vertical part of $[\mathbf{X},\mathbf{V}_p^i]$ is zero (the horizontal parts of $[\mathbf{X},\mathbf{V}_p^i]$ and $\mathbf{V}_p^{i+1}$ are the same, by (32)).

The freedom of choosing torsion normalisation conditions in Theorem 2.9 can be explained following Sternberg [23]. Note that both Θ and Ω vanish if one of their arguments is in the vertical distribution span $\mathbf{G}$. Thus at a fixed point $F\in\mathcal{F}_N$ they can be considered as elements of $\hom(\mathcal{D}(F)\wedge\mathcal{D}(F),\mathbb{R}^n)$ and $\hom(\mathcal{D}(F)\wedge\mathcal{D}(F),\mathfrak{g})$, respectively. Moreover, using the isomorphism $\phi|_{\mathcal{D}(F)}\colon\mathcal{D}(F)\to\mathbb{R}^n$ we obtain that $\Theta\in\hom(\mathbb{R}^n\wedge\mathbb{R}^n,\mathbb{R}^n)$ and $\Omega\in\hom(\mathbb{R}^n\wedge\mathbb{R}^n,\mathfrak{g})$. If we fix a point in $F\in\mathcal{F}_N$ then it follows that the set of sub-spaces of $T_F\mathcal{F}_N$ which are transversal to the fibre is an affine space modeled on the linear space $\hom(\mathbb{R}^n,\mathfrak{g})$. If two connections differ at F by an element $\eta\in\hom(\mathbb{R}^n,\mathfrak{g})$ then their torsions at F differ by

$$\delta\eta\in\hom(\mathbb{R}^n\wedge\mathbb{R}^n,\mathbb{R}^n).$$

where $\delta\colon\hom(\mathbb{R}^n,\mathfrak{g})\to\hom(\mathbb{R}^n\wedge\mathbb{R}^n,\mathbb{R}^n)$ is Spencer operator

$$\delta\eta(Y_1,Y_2)=\eta(Y_1)Y_2-\eta(Y_2)Y_1.$$

The approach of Sternberg [23], page 318, says that in order to choose a canonical connection one should fix a subspace $\mathcal{N}\subset\hom(\mathbb{R}^n,\mathfrak{g})$ such that $\mathcal{N}\oplus\operatorname{Im}\delta=\hom(\mathbb{R}^n,\mathfrak{g})$ and then consider connections with torsion in $\mathcal{N}$. Such a connection is unique provided that $\ker\delta=\{0\}$. This is the case in our situation:

Lemma 2.15 *The kernel of the Spencer operator is trivial,* $\ker\delta = \{0\}$.

Proof. Let $\eta \in \hom(\mathbb{R}^n, \mathfrak{g})$ and $\delta\eta = 0$. There exists $a \in \hom(\mathbb{R}^n, \mathfrak{gl}(m))$ such that $\eta = \mathrm{diag}(a, \dots, a, 0)$ and $\eta(Y_1)Y_2 - \eta(Y_2)Y_1 = 0$ for any two vectors $Y_1, Y_2 \in \mathbb{R}^n$. Both Y_1 and Y_2 are column vectors of length $(k+1)m + 1$, where $k \geq 1$. Take $Y_1 = (y_1, 0, 0, \dots, 0)^T$ and $Y_2 = (0, y_2, 0, \dots, 0)^T$ for some $y_1, y_2 \in \mathbb{R}^m$. Then we get $a(y_1)y_2 = 0$ (and also $a(y_2)y_1 = 0$) for arbitrary y_1 and y_2. Thus $a = 0$ and consequently $\eta = 0$. □

An upshot of Lemma 2.15 is that in order to solve the problem of equivalence for dynamic pairs one should fix, once and for all, a subspace $\mathcal{N} \subset \hom(\mathbb{R}^n, \mathfrak{g})$ transversal to $\mathrm{Im}\,\delta$ and then assign to a dynamic pair a unique connection on $\mathcal{F}_N$ with torsion having values in $\mathcal{N}$. Choosing normalisation conditions (17), (18), (19) was a choice of the subspace $\mathcal{N} \subset \hom(\mathbb{R}^n, \mathfrak{g})$. Of course, there is a freedom in choosing another transversal subspace $\mathcal{N}$ and the only criterion for choosing one seems simplicity of the resulting invariants.

3 Ordinary differential equations

Consider a system of m ordinary differential equations of order $k + 1 \geq 2$,

$$x^{(k+1)} = F(t, x, x^{(1)}, \dots, x^{(k)}), \tag{F}$$

where $x = (x^1, \dots, x^m) \in \mathbb{R}^m$ and $F = (F^1, \dots, F^m)$ is a smooth map $\mathbb{R}^n \to \mathbb{R}^m$, $n = 1 + (k+1)m$. Two systems (F) and (F') of this form will be called *equivalent* (alternatively, *time-scale preserving equivalent* or *affine-contact equivalent*), if there exists a smooth diffeomorphism $\mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ of the form

$$t \longmapsto t' = t + c, \quad x \longmapsto x' = \Phi(t, x), \tag{44}$$

with $c \in \mathbb{R}$, which maps the set of solutions to (F) onto the set of solutions of (F').

In the present section we will focus on the problem whether two systems of the form (F) are time-scale preserving equivalent and on determining invariants. We start with providing a geometric background to the definition of equivalence.

3.1 Jet space and its affine distribution

Let $J^k(1, m)$ denote the space of k jets of smooth functions $\mathbb{R} \to \mathbb{R}^m$. The space $J^k(1, m)$ is endowed with the natural coordinate system $(t, y) := (t, x_0, \dots, x_k)$, where $x_i = (x_i^1, \dots, x_i^m)$. Recall that any parametrised curve $x : I \to \mathbb{R}^m$, with $I \subset \mathbb{R}$ an open interval, has its k-jet extension $j^k x : I \to J^k(1, m)$ defined by $(j^k x)(t) = (t, x(t), x^{(1)}(t), \dots, x^{(k)}(t))$. On each such curve we identify $x_i(t) = x^{(i)}(t)$.

For any given $(t, y) \in J^k(1, m)$ there is a smooth curve $x : I \to \mathbb{R}^m$ such that $(j^k x)(t) = (t, y)$. The vector tangent to this curve at (t, y) is of the form

$$v = \partial_t + \sum_{i=0}^{k-1} \sum_{j=1}^{m} x_{i+1}^j \partial_{x_i^j} + \sum_{j=1}^{m} u_j \partial_{x_k^j},$$

where u_j are arbitrary numbers and, again, we identify $x_i = x^{(i)}(t)$. All such vectors form an affine subspace of the tangent space to $M = J^k(1,m)$ at the point $(t,y) \in M$. This subspace is denoted $\mathcal{A}_k(t,y) \subset T_{t,y}M$ and we have

$$\mathcal{A}_k(t,y) = D_k(t,y) + \mathrm{span}\{\partial_{x_k^1}, \ldots, \partial_{x_k^m}\},$$

where D_k denotes the vector field

$$D_k(t,y) = \partial_t + \sum_{i=0}^{k-1} \sum_{j=1}^{m} x_{i+1}^j \partial_{x_i^j}.$$

We will call $\mathcal{A}_k$ *canonical affine distribution* on the space $J^k(1,m)$ of k-jets of parametrised curves in $\mathbb{R}^m$. We may write

$$\mathcal{A}_k = D_k + \mathcal{V}_k,$$

where $\mathcal{V}_k$ denotes the involutive distribution $\mathcal{V}_k = \mathrm{span}\{\partial_{x_k^1}, \ldots, \partial_{x_k^m}\}$.

Proposition 3.1 *A diffeomorphism $\Psi : J^k(1,m) \to J^k(1,m)$ which preserves $\mathcal{A}_k$ is the k-jet extension of a diffeomorphism* (44)*, which means that it is of the form*

$$(t, x_0, \ldots, x_k) \longmapsto (t+c, \Phi(t,x_0), (D_k\Phi)(t,x_0,x_1), \ldots, (D_k^{k-1}\Phi)(t,x_0,\ldots,x_k)). \quad (45)$$

In particular Ψ preserves the 1-form dt. Vice versa, any Ψ as in (45) *preserves $\mathcal{A}_k$.*

Proof. Suppose, $\Psi : J^k(1,m) \to J^k(1,m)$ preserves $\mathcal{A}_k$, i.e., $\Psi_*\mathcal{A}_k = \mathcal{A}_k$. Denoting with the same same symbol $\mathcal{A}_k$ the set (the sheaf) of vector fields belonging to the distribution $\mathcal{A}_k$ we have

$$[\mathcal{A}_k, \mathcal{A}_k] = \mathcal{V}_k^1,$$
$$[\mathcal{A}_k, [\mathcal{A}_k, \mathcal{A}_k]] = \mathcal{V}_k^2,$$
$$\vdots$$
$$[\mathcal{A}_k, [\cdots [\mathcal{A}_k, \mathcal{A}_k]]] = \mathcal{V}_k^k,$$

where $[\cdot,\cdot]$ denotes the Lie bracket and $\mathcal{V}_k^1, \ldots, \mathcal{V}_k^k$ denote the involutive distributions

$$\mathcal{V}_k^i = \mathrm{span}\{\partial_{x_s^j} \mid s = k-i, k-i+1, \ldots, k, j = 1, \ldots, m\}.$$

Since Ψ preserves $\mathcal{A}_k$, it also preserves these distributions and the corresponding foliations. In particular, it preserves $\mathcal{V}_k^{k-1}$ and $\mathcal{V}_k^k$, which means that t is transformed into t' and (t,x) is transformed into (t',x'). The 1-form $\alpha = dt$ is determined by the conditions $\alpha(\mathcal{V}_k^k) = 0$ and $\alpha(Y) = 1$, for $Y \in \mathcal{A}_k$, thus it is also preserved by Ψ. This implies that t is mapped into $t+c$, $c \in \mathbb{R}$ and, therefore, $(t',x') = (t+c, \Phi(t,x))$.

Since through any point in $J^k(1,m)$ there passes a k-jet extension of a curve in $\mathbb{R}^m$, and for any curve $s \mapsto \gamma(s) = (s, x(s))$ we have

$$x_i'(\Psi(\gamma(s))) = \left(\frac{d}{ds}\right)^i x'(\Psi(\gamma(s))) = (D^i\Phi)(\gamma(s)),$$

it follows that $x_i' = (D^i\Phi)(t,x)$ and, thus, Ψ is of the desired form.

The converse implication is straightforward. □

Remark. In geometric theory of ODEs one uses a Cartan distribution $\mathcal{C}_k$, which is the vector distribution spanned by the more subtle object, the affine distribution $\mathcal{A}_k$. $\mathcal{C}_k$ gives all vectors tangent to k-jet extensions of unparametrised curves. We have

$$\mathcal{C}_k = \mathrm{span}\{D_k, \mathcal{V}_k\} \quad \text{and} \quad \mathcal{A}_k = \{Y \in \mathcal{C}_k : dt(Y) = 1\}.$$

3.2 Dynamic pair of (F) and equivalence

The system (F) can be equivalently defined by a submanifold $E_F \subset J^{k+1}(1,m)$,

$$E_F = \{(t, x_0, \ldots, x_{k+1}) \in J^{k+1}(1,m) \mid x_{k+1} - F(t, x_0, \ldots, x_k) = 0\}.$$

The functions $t, x_0, \ldots, x_k$ restricted to E_F define a system of coordinates on E_F, since the projection $\pi : J^{k+1}(1,m) \to J^k(1,m)$ restricted to E_F is a diffeomorphism, $\pi|_{E_F} : E_F \to J^k(1,m)$.

The canonical affine distribution $\mathcal{A}_{k+1} = D_{k+1} + \mathcal{V}_{k+1}$ on $M = J^{k+1}(1,m)$, intersected with the tangent space to the submanifold E_F, defines a unique vector

$$X_F(t,y) = \mathcal{A}_{k+1}(t,y) \cap T_{(t,y)}E_F, \quad \text{for any} \ \ (t,y) \in E_F.$$

This follows from the fact that $\mathcal{V}_{k+1}$ and TE_F are mutually transversal subspaces in $TJ^{k+1}(1,m)$, at any point in E_F. In this way (F) defines a vector field X_F on E_F. X_F is called *total derivative* corresponding to (F) and, in coordinates, it is given by

$$X_F = \partial_t + \sum_{i=0}^{k-1} \sum_{j=1}^{m} x_{i+1}^j \partial_{x_i^j} + \sum_{j=1}^{m} F^j \partial_{x_k^j}. \tag{46}$$

Consider the Lie square $[\mathcal{A}_{k+1}, \mathcal{A}_{k+1}]$, which is a vector distribution on $J^{k+1}(1,m)$ and, in coordinates, $[\mathcal{A}_{k+1}, \mathcal{A}_{k+1}] = \mathrm{span}\{\partial_{x_k^i}, \partial_{x_{k+1}^j} | i,j = 1, \ldots, m\}$. We define the distribution on E_F by intersecting $[\mathcal{A}_{k+1}, \mathcal{A}_{k+1}]$ with the tangent bundle TE_F,

$$\mathcal{V}_F = TE_F \cap [\mathcal{A}_{k+1}, \mathcal{A}_{k+1}].$$

In coordinates,

$$\mathcal{V}_F = \mathrm{span}\{\partial_{x_k^1}, \ldots, \partial_{x_k^m}\}. \tag{47}$$

It is easy to check that the pair $(X_F, \mathcal{V}_F)$ is regular, i.e., it satisfies conditions (R1) and (R2) on $M = E_F$. We will call it the *dynamic pair* of system (F).

Consider two equations (F) and (F').

Proposition 3.2 *The following conditions are equivalent.*
(a) Equations (F) and (F') are time-scale preserving equivalent.
(b) There is a diffeomorphism of $J^{k+1}(1,m)$ which preserves the canonical affine distribution $\mathcal{A}_{k+1}$ and transforms E_F onto $E_{F'}$.
(c) There is a diffeomorphism of $J^{k+1}(1,m)$ preserving the Cartan distribution $\mathcal{C}_{k+1}$ and the 1-form dt, and transforming E_F onto $E_{F'}$.
(d) The dynamic pairs $(X_F, \mathcal{V}_F)$ and $(X_{F'}, \mathcal{V}_{F'})$ are diffeomorphic.

Proof. Equivalence of (a), (b) and (c) follows from Proposition 3.1 and the remark following it. From the definitions $X_F = TE_F \cap \mathcal{A}_{k+1}$, $\mathcal{V}_F = TE_F \cap [\mathcal{A}_{k+1}, \mathcal{A}_{k+1}]$ we see that (b) implies (d).

In order to show $(d) \Rightarrow (b)$ assume that there is a diffeomorphism $\psi : E_F \to E_{F'}$ which transforms the pair $(X_F, \mathcal{V}_F)$ into $(X_{F'}, \mathcal{V}_{F'})$. Since E_F and $J^k(1,m)$ are diffeomorphic via the natural projection $\pi : J^{k+1}(1,m) \to J^k(1,m)$ restricted to E_F, and so are $E_{F'}$ and $J^k(1,m)$, there is a diffeomorphism $\hat{\psi} : J^k(1,m) \to J^k(1,m)$ corresponding to $\psi : E_F \to E_{F'}$. Moreover, after projections both pairs $(X_F, \mathcal{V}_F)$ and $(X_{F'}, \mathcal{V}_{F'})$ have, in natural coordinates in $J^k(1,m)$, the form (46) and (47). Since $\hat{\psi}$ transforms the projected pair into the projected pair, it follows from (46) and (47) that they both span the same affine distribution $\mathcal{A}_k = D_k + \mathcal{V}_k$. Thus $\hat{\psi}$ preserves the canonical affine distribution $\mathcal{A}_k$ in $J^k(1,m)$. We then deduce from Proposition 3.1 that $\hat{\psi}$ is of the form (45). Let Ψ be the 1-prolongation of $\hat{\psi}$, which means that it is of the form (45), with k replaced by $k+1$. Then Ψ automatically preserves $\mathcal{A}_{k+1}$ and it is easy to see that it transforms E_F to $E_{F'}$.

The proposition can also be deduced from the classical Lie-Bäcklund theorem, cf. [15], or from Theorem 1 in [11], with additional condition $\Psi^*(dt) = dt$. □

Equations (F) and (F') satisfying one of the above conditions will simply be called *equivalent*. Taking into account conditions (b) and (c) one could also call them *affine-contact equivalent* or *time-scale preserving contact equivalent*.

Condition (d) implies that we can use Theorems 2.9 or 2.11 in order to solve the equivalence problem for systems (F). We can assign to (F) a canonical connection and a canonical frame on the normal frame bundle of the pair $(X_F, \mathcal{V}_F)$. We obtain

Theorem 3.3 *The following conditions are equivalent.*
(a) Equations (F) *and* (F') *are equivalent.*
(b) The dynamic pairs $(X_F, \mathcal{V}_F)$ *on* E_F *and* $(X_{F'}, \mathcal{V}_{F'})$ *on* $E_{F'}$ *are diffeomorphic.*
(c) The canonical frames $(\mathbf{V}, \mathbf{X}, \mathbf{G})$ *and* $(\mathbf{V}', \mathbf{X}', \mathbf{G}')$ *in Theorem 2.9 (resp. Theorem 2.11), corresponding to dynamic pairs* $(X_F, \mathcal{V}_{F'})$ *and* $(X_{F'}, \mathcal{V}_{F'})$ *and living on the normal frame bundles* $\pi\colon \mathcal{F}_N \to J^k(1,m)$ *and* $\pi\colon \mathcal{F}'_N \to J^k(1,m)$*, are diffeomorphic.*

We also deduce that any system (F) has at most $(k+1)m + m^2 + 1$-dimensional group of time-scale preserving contact symmetries and it has maximal dimension if and only if it is equivalent to a linear system with constant and diagonal coefficients. In this way, the problem of time-scale preserving equivalence of systems of ODEs is reduced to the geometry of pairs $(X, \mathcal{V})$.

3.3 Dynamic pairs of ODEs

Not all dynamic pairs $(X, \mathcal{V})$ correspond to systems of ODEs. In order to characterise such pairs we introduce

Definition 3.4 *Let* X *be a smooth vector field and* $\mathcal{V}$ *be a smooth distribution of constant rank* m *on a manifold* M*. The pair* $(X, \mathcal{V})$ *is of* equation type *if there exists a system* (F) *and a diffeomorphism* $\Phi\colon M \to E_F$ *such that* $\Phi_*(X) = X_F$ *and* $\Phi_*(\mathcal{V}) = \mathcal{V}_F$*. The pair* $(X, \mathcal{V})$ *is* locally of equation type *if for any* $x \in M$ *there exists a neighbourhood* $U \ni x$ *such that* $(X|_U, \mathcal{V}|_U)$ *is of equation type.*

Theorem 3.5 *A pair $(X, \mathcal{V})$ is locally of equation type if and only if it satisfies conditions (R1), (R2) and, additionally,*

(R3) $\mathcal{V}^i$ are integrable for $i = 0, \ldots, k$,

(R4) $\mathrm{ad}_X \mathcal{V}^k = \mathcal{V}^k$.

Moreover, condition (R3) is equivalent to:

(R3') $\mathcal{V}^i$ are integrable for $i = k-1$ and $i = k$.

Proof. It is straightforward to check that conditions (R1)-(R4) are satisfied for an arbitrary equation (F) and the corresponding $(X_F, \mathcal{V}_F)$.

In order to prove the theorem in the opposite direction let us notice that (R1) and (R3) imply that $\mathcal{V}^k$ defines a corank one foliation on M. Thus we can choose a local coordinate t on M such that leaves of $\mathcal{V}^k$ are given by equations: $t = \text{const}$. Additionally, it follows from (R3) that we can choose remaining coordinates such that $\mathcal{V}^i = \{t = c, x_0^j = c_0^j, \cdots, x_{k-i-1}^j = c_{k-i-1}^j \mid j = 1, \ldots, m\}$, for $i = 0, \ldots, k-1$, where c and c_s^j are constants. We have

$$X = f\partial_t + \sum_{i=0}^{k} \sum_{j=1}^{m} f_i^j \partial_{x_i^j}$$

for certain functions f and $f_i = (f_i^1, \ldots, f_i^m)$.

Note that (R4) implies that f is constant on leaves of $\mathcal{V}^k$. If not, then the Lie bracket of X and some vector field tangent to $\mathcal{V}^k$ would be transversal to $\mathcal{V}^k$, and hence it would violate condition (R4). Thus, we can reparametrise t so that $f \equiv 1$.

Similarly, let us notice that f_i depend on t and $x_0, \ldots, x_i$ only. Otherwise, the Lie bracket of X and a vector field in $\mathcal{V}^k$ would stick out of $\mathcal{V}^k$. We will modify coordinates x_i^j in such a way that X is of the form X_F for some system (F). Firstly, we set

$$y_0 = x_0 \quad \text{and} \quad y_1 = f_0(t, y_0, x_1).$$

As a consequence of (R1) we will see that $(t, y_0, y_1, x_2, \ldots, x_k)$ can be taken as new coordinates. Indeed, (R1) implies that the matrix $(\partial_{x_1^s} f_0^t)_{s,t=1,\ldots,m}$ has maximal possible rank m, because $\operatorname{rk} \mathcal{V}^1 - \operatorname{rk} \mathcal{V}^0 = m$. We continue the reasoning and inductively define

$$y_i = f_{i-1}(t, y_0, \ldots, y_{i-1}, x_i).$$

At each step we get new coordinate system $(t, y_0, \ldots, y_i, x_{i+1}, \ldots, x_k)$. Finally we obtain

$$X = \partial_t + \sum_{i=0}^{k-1} \sum_{j=1}^{m} y_{i+1}^j \partial_{y_i^j} + \sum_{j=1}^{m} F^j \partial_{y_k^j}.$$

where $F^j(t, y_0, \ldots, y_k) = f_k^j(t, y_0, \ldots, y_k)$ define the desired system of ODEs.

In order to complete the proof it is sufficient to prove that (R3') implies (R3). We proceed by induction. Assume that $\mathcal{V}^{i+1}, \ldots, \mathcal{V}^k$ are integrable, where $i < k-1$. Let Y_1 and Y_2 be two sections of $\mathcal{V}^i \subset \mathcal{V}^{i+1}$. Then, by assumption, $[Y_1, Y_2]$ is a section of $\mathcal{V}^{i+1}$. Moreover, Jacobi identity implies that $[X, [Y_1, Y_2]]$ is also a section of $\mathcal{V}^{i+1}$, since $[X, Y_1]$ and $[X, Y_2]$ are sections of $\mathcal{V}^{i+1}$ by the definition of $\mathcal{V}^{i+1}$. It follows, that $[Y_1, Y_2]$ is a section of $\mathcal{V}^i$. If not, then by condition (R1), the bracket $[X, [Y_1, Y_2]]$ would be a non-trivial section of $\mathcal{V}^{i+2} \mod \mathcal{V}^{i+1}$. □

Theorem 3.5 implies that the canonical frames in Theorems 2.9, 2.11 satisfy

$$[\mathbf{V}_p^i, \mathbf{V}_q^j] = 0 \mod \mathbf{V}^0, \ldots, \mathbf{V}^r, \mathbf{G}, \quad \text{where} \ \ r = \max\{i, j\}. \tag{48}$$

This fact, Theorems 2.9, 2.11 and the remark following Theorem 2.9 will imply

Corollary 3.6 *The canonical frame of Theorem 2.9, corresponding to dynamic pair $(X_F, \mathcal{V}_F)$ of system (F), satisfies the following conditions*

$$[\mathbf{V}_p^0, \mathbf{V}_q^0] = 0, \tag{49}$$
$$[\mathbf{V}_p^0, \mathbf{V}_q^1] = 0 \mod \mathbf{G}, \tag{50}$$
$$[\mathbf{V}_p^0, \mathbf{V}_q^i] = 0 \mod \mathbf{V}^0, \ldots, \mathbf{V}^{i-1}, \mathbf{G}, \quad 2, \ldots, k, \tag{51}$$
$$[\mathbf{X}, \mathbf{V}_p^i] = \mathbf{V}_p^{i+1} \mod \mathbf{G}, \quad i = 0, \ldots, k-1, \tag{52}$$
$$[\mathbf{X}, \mathbf{V}_p^k] = 0 \mod \mathbf{V}^0, \ldots, \mathbf{V}^{k-1}, \mathbf{G}. \tag{53}$$

It is also uniquely determined by the first three of them. The canonical frame of Theorem 2.11 corresponding to $(X_F, \mathcal{V}_F)$ satisfies (and is uniquely determined by) (49), (53) *and*

$$[\mathbf{V}_p^0, \mathbf{V}_q^i] = 0 \mod \mathbf{V}^0, \ldots, \mathbf{V}^{i-1}, \mathbf{G}, \quad i = 1, \ldots, k, \tag{54}$$
$$[\mathbf{X}, \mathbf{V}_p^i] = \mathbf{V}_p^{i+1}, \qquad i = 0, \ldots, k-1. \tag{55}$$

Proof. Conditions (50), (52), (53), (55), and (54) with $i = 1$, follow from the definitions of canonical frames (Section 2.5). In order to prove the remaining ones we will use identities (26), without mentioning. From (48) we have

$$[\mathbf{V}_p^0, \mathbf{V}_q^0] = T_{pq0}^{00r} \mathbf{V}_r^0 + R_{pqs}^{00t} \mathbf{G}_t^s.$$

Taking Lie bracket of both sides with $\mathbf{X}$ and using $[\mathbf{X}, \mathbf{G}_t^s] = 0$, $[\mathbf{X}, \mathbf{V}_r^0] = \mathbf{V}_r^1 \mod \mathbf{G}$, gives

$$[[\mathbf{X}, \mathbf{V}_p^0], \mathbf{V}_q^0] + [\mathbf{V}_p^0, [\mathbf{X}, \mathbf{V}_q^0]] = T_{pq0}^{00r} \mathbf{V}_r^1 \mod \mathbf{V}^0, \mathbf{G}.$$

Using again $[\mathbf{X}, \mathbf{V}_r^0] = \mathbf{V}_r^1 \mod \mathbf{G}$ on the left-hand side and the identity $[\mathbf{V}_i^0, \mathbf{V}_j^1] = 0$ modulo $\mathbf{V}^0$, $\mathbf{G}$ (satisfied for the first and the second canonical frame) we see that the left-hand side equals to zero, modulo $\mathbf{V}^0, \mathbf{G}$. Thus $T_{pq0}^{00r} = 0$.

We repeat the same procedure, taking this time Lie bracket of the above identity with $\mathbf{V}_r^1$ (now $T_{pq0}^{00r} = 0$). Applying the Jacobi identity on the left-hand side and using the identity $[\mathbf{V}_i^0, \mathbf{V}_j^1] = 0$ modulo $\mathbf{V}^0$, $\mathbf{G}$ we find that this side vanishes modulo $\mathbf{V}^0$, $\mathbf{G}$. The right-hand side equals to $R_{pqs}^{00t}[\mathbf{V}_r^1, \mathbf{G}_t^s] = -R_{pqr}^{00t} \mathbf{V}_t^1$ modulo $\mathbf{V}^0$, $\mathbf{G}$, thus $R_{pqr}^{00t} = 0$ and (49) is proved.

In order to prove (51) note that this condition is satisfied for $i = 1$, for the first and the second canonical frames, by the definitions of these frames. Suppose now that (51) is satisfied for some $i \geq 1$ and take the Lie bracket of both sides with $\mathbf{X}$. We see from (52) and condition (48) that is is also satisfied for $i + 1$. Analogously we prove (54). □

Theorem 3.5 implies that the distribution $\mathcal{V}^k$ is integrable. Let S be a leaf of the corresponding foliation (a hypersurface) and let us choose a normal frame $F_x = (V^0, \ldots, V^k, X(x))$ at each point $x \in S$. Then $(V^0, \ldots, V^k)$ constitutes a frame of manifold S and X is transversal to S, by (R2). We will call $(V^0, \ldots, V^k)$ a normal frame of a pair $(\mathcal{V}, X)$ on S. Such a frame together with the vector field X and the curvature matrices $K_0, \ldots, K_{k-1}$ determine $\mathcal{V}$ in a neighbourhood of S.

Corollary 3.7 *Assume that two dynamic pairs $(X, \mathcal{V})$ and $(X', \mathcal{V}')$ are of equation type, $X = X'$, and there exists a common leaf S of distributions $\mathcal{V}^k$ and $\mathcal{V}'^k$ with a common normal frame $(V^0, \ldots, V^k)$. Additionally, assume that there exist a normal frame of $\mathcal{V}$ and a normal frame of $\mathcal{V}'$ which coincide on S and are such that the associated matrices of curvature operators coincide in a neighbourhood of S. Then $\mathcal{V} = \mathcal{V}'$ in a neighbourhood of S.*

Proof. Let V and V' be normal frames of $\mathcal{V}$ and $\mathcal{V}'$, respectively, such that the associated curvature operators coincide on a neighbourhood of S and $V = V'$ on S. We can assume that $V = V' = V^0$ on S (if not, we take $V := V^0 G$ and $V' := V^0 G'$ where $G(x), G'(x) \in Gl(m)$ are transition matrices from $V^0(x)$ to $V(x)$ and $V'(x)$). Now, we know that both V and V' satisfy equation (10) with the same coefficients K_i. Moreover, $\mathrm{ad}_X^i V = \mathrm{ad}_X^i V' = V^i$ on S for $i = 0, \ldots, k$. Thus, the uniqueness theorem for ODEs implies that $V = V'$ on a neighbourhood of S. Consequently $\mathcal{V} = \mathrm{span}\{V\} = \mathrm{span}\{V'\} = \mathcal{V}'$ on a neighbourhood of S. □

3.4 Systems of order 2

Consider a system of second order ODEs on $\mathbb{R}^m$,

$$x'' = F(t, x, x'). \tag{F}$$

Instead of (F), we can consider the corresponding dynamic pairs $(X_F, \mathcal{V}_F)$ or, equivalently, general dynamic pairs $(X, \mathcal{V})$ satisfying (R1)-(R4), with $k = 1$.

Theorem 3.8 *The first and the second canonical frames on $\mathcal{F}_N$, corresponding to dynamic pair $(X_F, \mathcal{V}_F)$ of (F), coincide and they satisfy:*

$$\begin{aligned}
[\mathbf{X}, \mathbf{V}_p^0] &= \mathbf{V}_p^1, \\
[\mathbf{X}, \mathbf{V}_p^1] &= -\hat{K}_{0p}^r \mathbf{V}_r^0 + \hat{R}_{ps}^{1t} \mathbf{G}_t^s, \\
[\mathbf{V}_p^0, \mathbf{V}_q^0] &= 0, \\
[\mathbf{V}_p^0, \mathbf{V}_q^1] &= R_{pqs}^{01t} \mathbf{G}_t^s, \\
[\mathbf{V}_p^1, \mathbf{V}_q^1] &= T_{pq0}^{11r} \mathbf{V}_r^0 + R_{pqs}^{11t} \mathbf{G}_t^s.
\end{aligned}$$

The invariants $\hat{R}^1$, T_0^{11} and R^{11} are determined by $\hat{K}_0$, namely

$$T_{pq0}^{11s} = \frac{1}{3}\left(\mathbf{V}_p^0(\hat{K}_{0q}^s) - \mathbf{V}_q^0(\hat{K}_{0p}^s)\right), \tag{56}$$

$$R_{pqr}^{11s} = \frac{1}{3}\left(\mathbf{V}_r^0\mathbf{V}_p^0(\hat{K}_{0q}^s) - \mathbf{V}_r^0\mathbf{V}_q^0(\hat{K}_{0p}^s)\right), \tag{57}$$

$$\hat{R}_{pq}^{1s} = \frac{1}{2}T_{pq0}^{11s} - \frac{1}{2}\left(\mathbf{V}_p^0(\hat{K}_{0q}^s) + \mathbf{V}_q^0(\hat{K}_{0p}^s)\right), \tag{58}$$

and they satisfy

$$\sum_{cycl\{p,q,r\}} R^{11t}_{pqr} = 0, \tag{59}$$

$$\sum_{cycl\{p,q,r\}} \mathbf{V}^1_p(T^{11t}_{qr}) = 0 \tag{60}$$

($\sum_{cycl}$ denotes the cyclic sum). Moreover, R^{01t}_{pqs} is symmetric in lower indices and

$$\mathbf{V}^0_p(R^{01t}_{qrs}) = \mathbf{V}^0_q(R^{01t}_{prs}), \tag{61}$$

$$\mathbf{V}^0_r(R^{11t}_{pqs}) = \mathbf{V}^1_p(R^{01t}_{rqs}) - \mathbf{V}^1_q(R^{01t}_{rps}), \tag{62}$$

$$\mathbf{X}(R^{01t}_{pqs}) = R^{11t}_{pqs} + \mathbf{V}^0_p(\hat{R}^{1t}_{qs}), \tag{63}$$

$$\mathbf{X}(R^{11t}_{pqs}) = -\hat{K}^r_{0p}R^{01t}_{rqs} + \hat{K}^r_{0q}R^{01t}_{rps} + \mathbf{V}^1_p(\hat{R}^{1t}_{qs}) - \mathbf{V}^1_q(\hat{R}^{1t}_{ps}), \tag{64}$$

$$\sum_{cycl\{p,q,r\}} \big(\mathbf{V}^1_p(R^{11t}_{qrs}) - T^{11u}_{qr}R^{01t}_{ups}\big) = 0. \tag{65}$$

If $\hat{K}_0$ vanishes then $[\mathbf{X}, \mathbf{V}^1_p] = 0$, $[\mathbf{V}^1_p, \mathbf{V}^1_q] = 0$ and the only nonzero invariant R^{01} satisfies the relations

$$\mathbf{X}(R^{01t}_{pqs}) = 0 \quad \text{and} \quad \mathbf{V}^i_p(R^{01t}_{qrs}) = \mathbf{V}^i_q(R^{01t}_{prs}), \quad i = 0, 1.$$

Note that $\hat{K}_0 = G^{-1}K_0G$, in coordinates of the proof of Lemma 2.15.

Proof. A priori, due to Corollary 3.6, the structural equations of the first canonical frame have the form:

$$\begin{aligned}
[\mathbf{X}, \mathbf{V}^0_p] &= \mathbf{V}^1_p + \hat{R}^{0t}_{ps}\mathbf{G}^s_t,\\
[\mathbf{X}, \mathbf{V}^1_p] &= -\hat{K}^r_{0p}\mathbf{V}^0_r + \hat{R}^{1t}_{ps}\mathbf{G}^s_t,\\
[\mathbf{V}^0_p, \mathbf{V}^0_q] &= 0,\\
[\mathbf{V}^0_p, \mathbf{V}^1_q] &= R^{01t}_{pqs}\mathbf{G}^s_t,\\
[\mathbf{V}^1_p, \mathbf{V}^1_q] &= T^{11r}_{pq0}\mathbf{V}^0_r + T^{11r}_{pq1}\mathbf{V}^1_r + R^{11t}_{pqs}\mathbf{G}^s_t.
\end{aligned}$$

In order to identify the relations between the structural functions we will use the Jacobi identities for the following combinations of the vector fields (the remaining are either trivial or follow from the considered ones):

$$(\mathbf{X}, \mathbf{V}^0_p, \mathbf{V}^0_q),\ (\mathbf{X}, \mathbf{V}^0_p, \mathbf{V}^1_q),\ (\mathbf{X}, \mathbf{V}^1_p, \mathbf{V}^1_q),\ (\mathbf{V}^0_p, \mathbf{V}^0_q, \mathbf{V}^1_r),\ (\mathbf{V}^0_p, \mathbf{V}^1_q, \mathbf{V}^1_r),\ (\mathbf{V}^1_p, \mathbf{V}^1_q, \mathbf{V}^1_r).$$

First, let us consider the third equation and take the Lie bracket of both sides with $\mathbf{X}$. We have $[\mathbf{X}, [\mathbf{V}^0_p, \mathbf{V}^0_q]] = 0$ and, applying Jacobi identity, the first and fourth equations and the identities (26), we obtain

$$-R^{01t}_{qps}\mathbf{G}^s_t + R^{01t}_{pqs}\mathbf{G}^s_t + \hat{R}^{0t}_{pq}\mathbf{V}^0_t - \hat{R}^{0t}_{qp}\mathbf{V}^0_t - \mathbf{V}^0_q(\hat{R}^{0t}_{ps})\mathbf{G}^s_t + \mathbf{V}^0_p(\hat{R}^{0t}_{qs})\mathbf{G}^s_t = 0,$$

thus

$$\hat{R}^{0t}_{pq} = \hat{R}^{0t}_{qp}, \tag{66}$$

$$R^{01t}_{pqs} = R^{01t}_{qps} + \mathbf{V}^0_q(\hat{R}^{0t}_{ps}) - \mathbf{V}^0_p(\hat{R}^{0t}_{qs}). \tag{67}$$

Next, let us Lie bracket both sides of the fourth structural equation with $\mathbf{X}$. On the left-hand side we have $[\mathbf{X}, [\mathbf{V}_p^0, \mathbf{V}_q^1]]$ and, applying Jacobi identity, the first, the second and the third equations, and the identities (26) we get:

$$T_{pq0}^{11r}\mathbf{V}_r^0 + T_{pq1}^{11r}\mathbf{V}_r^1 + R_{pqs}^{11t}\mathbf{G}_t^s - \mathbf{V}_q^1(\hat{R}_{ps}^{0t})\mathbf{G}_t^s + \hat{R}_{pq}^{0t}\mathbf{V}_t^1$$
$$-\mathbf{V}_p^0(\hat{K}_{0q}^r)\mathbf{V}_r^0 + \mathbf{V}_p^0(\hat{R}_{qs}^{1t})\mathbf{G}_t^s - \hat{R}_{qp}^{1t}\mathbf{V}_t^0.$$

On the right-hand side we obtain $\mathbf{X}(R_{pqs}^{01t})\mathbf{G}_t^s$. Therefore,

$$\hat{R}_{qp}^{1r} = T_{pq0}^{11r} - \mathbf{V}_p^0(\hat{K}_{0q}^r), \tag{68}$$
$$\hat{R}_{pq}^{0r} = -T_{pq1}^{11r}, \tag{69}$$
$$\mathbf{X}(R_{pqs}^{01t}) = R_{pqs}^{11t} + \mathbf{V}_p^0(\hat{R}_{qs}^{1t}) - \mathbf{V}_q^1(\hat{R}_{ps}^{0t}). \tag{70}$$

In particular, we get that $\hat{R}_{pq}^{0r}$ is anti-symmetric in p and q, because T_{pq1}^{11r} is. But (66) reads that $\hat{R}_{pq}^{0t}$ is symmetric in p and q. Thus

$$\hat{R}_{pq}^{0r} = T_{pq1}^{11r} = 0. \tag{71}$$

This proves that the structural equations are as stated in the theorem. Moreover, from these equations we see that they satisfy the axioms of both, the first and the second canonical frames, thus these frames coincide by their uniqueness.

Additionally, equation $\hat{R}_{pq}^{0r} = 0$ together with (70) proves the relation (63) and simplifies (67) to

$$R_{pqs}^{01t} = R_{qps}^{01t}. \tag{72}$$

Now, let us Lie bracket both sides of the last structural equation with $\mathbf{X}$. On the left-hand side we have $[\mathbf{X}, [\mathbf{V}_p^1, \mathbf{V}_q^1]]$ and, applying Jacobi identity, the second and the fourth structural equations, and (26), we obtain

$$-\hat{K}_{0p}^r R_{rqs}^{01t}\mathbf{G}_t^s + \mathbf{V}_q^1(\hat{K}_{0p}^r)\mathbf{V}_r^0 - \mathbf{V}_q^1(\hat{R}_{ps}^{1t})\mathbf{G}_t^s + \hat{R}_{pq}^{1t}\mathbf{V}_t^1$$
$$+\hat{K}_{0q}^r R_{rps}^{01t}\mathbf{G}_t^s - \mathbf{V}_p^1(\hat{K}_{0q}^r)\mathbf{V}_r^0 + \mathbf{V}_p^1(\hat{R}_{qs}^{1t})\mathbf{G}_t^s - \hat{R}_{qp}^{1t}\mathbf{V}_t^1.$$

On the right-hand side, taking into account the first structural equation and (71), we get:

$$T_{pq0}^{11r}\mathbf{V}_r^1 + \mathbf{X}(T_{pq0}^{11r})\mathbf{V}_r^0 + \mathbf{X}(R_{pqs}^{11t})\mathbf{G}_t^s.$$

Thus we have

$$\mathbf{X}(T_{pq0}^{11r}) = \mathbf{V}_q^1(\hat{K}_{0p}^r) - \mathbf{V}_p^1(\hat{K}_{0q}^r) \tag{73}$$
$$T_{pq0}^{11r} = \hat{R}_{pq}^{1t} - \hat{R}_{qp}^{1t} \tag{74}$$

and (64) (which was to be proved). Combining (68) and (74) we can express T_{pq0}^{11r} and $\hat{R}_{pq}^{1t}$ in terms of $\mathbf{V}_p^0(\hat{K}_{0q}^r)$. Precisely, taking into account that $T_{pq0}^{11r} = -T_{qp0}^{11r}$, we find

$$T_{pq0}^{11r} = \frac{1}{3}(\mathbf{V}_p^0(\hat{K}_{0q}^r) - \mathbf{V}_q^0(\hat{K}_{0p}^r)), \qquad \hat{R}_{pq}^{1r} = 2T_{pq0}^{11r} - \mathbf{V}_p^0(\hat{K}_{0q}^r),$$

which gives (56) and (58) in the formulation of the theorem.

Consider next the third structural equation and bracket it with $\mathbf{V}^1_r$. We have $[\mathbf{V}^1_r, [\mathbf{V}^0_p, \mathbf{V}^0_q]] = 0$ and, after applying Jacobi identity and the fourth structural equation,

$$-R^{01t}_{prq}\mathbf{V}^0_t + \mathbf{V}^0_q(R^{01t}_{prs})\mathbf{G}^s_t + R^{01t}_{qrp}\mathbf{V}^0_t - \mathbf{V}^0_p(R^{01t}_{qrs})\mathbf{G}^s_t = 0.$$

This implies (61) and the relation $R^{01t}_{prq} = R^{01t}_{qrp}$ which, together with (72), implies that R^{01t}_{pqr} is symmetric with respect to indices p, q, and r.

Let us now bracket the last structural equation with the vector field $\mathbf{V}^0_r$. On the left-hand side we have $[\mathbf{V}^0_r, [\mathbf{V}^1_p, \mathbf{V}^1_q]]$ and, applying Jacobi identity,

$$-\mathbf{V}^1_q(R^{01t}_{rps})\mathbf{G}^s_t + R^{01t}_{rpq}\mathbf{V}^1_t + \mathbf{V}^1_p(R^{01t}_{rqs})\mathbf{G}^s_t - R^{01t}_{rqp}\mathbf{V}^1_t.$$

On the right-hand side, taking into account the third structural equation and $T^{11r}_{pq1} = 0$, we obtain:

$$\mathbf{V}^0_r(T^{11t}_{pq0})\mathbf{V}^0_t + \mathbf{V}^0_r(R^{11t}_{pqs})\mathbf{G}^s_t - R^{11t}_{pqr}\mathbf{V}^0_t.$$

Thus we get the relation (62) and $R^{11t}_{pqr} = \mathbf{V}^0_r(T^{11t}_{pq0})$. The latter equation and (56) give the desired formula (57) for R^{11t}_{pqr}.

Finally, we consider the Jacobi identity $\sum[\mathbf{V}^1_p, [\mathbf{V}^1_q, \mathbf{V}^1_r]] = 0$, where we take the cyclic sum over p, q, r. Taking into account the last structural equation we get

$$0 = \sum_{cycl\{p,q,r\}} \left(\mathbf{V}^1_p(T^{11s}_{qr})\mathbf{V}^0_s + \mathbf{V}^1_p(R^{11t}_{qrs})\mathbf{G}^s_t - T^{11u}_{qr}R^{01t}_{ups}\mathbf{G}^s_t - R^{11t}_{qrs}[\mathbf{G}^s_t, \mathbf{V}^1_p]\right)$$

which, taking into account $[\mathbf{G}^s_t, \mathbf{V}^1_p] = \delta^s_p\mathbf{V}^1_t$, implies vanishing of the cyclic sums

$$\sum R^{11t}_{pqr} = 0, \quad \sum \mathbf{V}^1_p(T^{11t}_{qr}) = 0, \quad \sum \left(\mathbf{V}^1_p(R^{11t}_{qrs}) - T^{11u}_{qr}R^{01t}_{ups}\right) = 0,$$

i.e., identities (59), (60) and (65). This ends the proof of the theorem as, if $\hat{K}_0 = 0$, all the invariants vanish except of R^{01t}_{pqr}. □

Let us find the structural functions in terms of function F defining an equation and coordinates (t, x^j, y^j) on the space of 1-jets, where y^j corresponds to the first derivative of x^j. We have

$$X_F = \partial_t + \sum_{j=1}^m y^j\partial_{x^j} + \sum_{j=1}^m F^j(t,x,y)\partial_{y^j}$$

and $\mathcal{V}_F = \operatorname{span}\{V_1, \ldots, V_m\}$, where $V_j = \partial_{y^j}$. Let $V = (V_1, \ldots, V_m)$. We compute

$$\operatorname{ad}^2_{X_F} V = (\operatorname{ad}_{X_F} V)H_1 + VH_0$$

where

$$H_1 = \left(-\partial_{y^s}F^t\right)_{s,t=1,\ldots,m}, \qquad H_0 = \left(\partial_{x^s}F^t - X_F(\partial_{y^s}F^t)\right)_{s,t=1,\ldots,m}.$$

Therefore, by Proposition 2.5, we get

$$K_0 = \left(-\partial_{x^s}F^t + \frac{1}{2}X(\partial_{y^s}F^t) - \frac{1}{4}\sum_{r=1}^m \partial_{y^s}F^r\partial_{y^r}F^t\right)_{s,t=1,\ldots,m}$$

and $\mathcal{H}^1 = \mathrm{span}\{V_1^1, \dots, V_m^1\}$ where

$$V_j^1 = -\partial_{x^j} - \frac{1}{2}\sum_{s=1}^m \partial_{y^j} F^s \partial_{y^s}.$$

We have

$$[V_p, V_q^1] = -\frac{1}{2}\sum_{s=1}^m \partial_{y^p}\partial_{y^q} F^s V_s$$

and using (36) we can write

$$\begin{aligned}
\mathbf{V}_j^0 &= \sum_{s=1}^m G_j^s \partial_{y^s}, \\
\mathbf{V}_j^1 &= -\sum_{s=1}^m G_j^s \partial_{x^s} - \frac{1}{2}\sum_{s,t=1}^m G_j^t \partial_{y^t} F^s \partial_{y^s} - \frac{1}{2}\sum_{r,s,t=1}^m G_j^u G_t^w (G^{-1})_r^s \partial_{y^u}\partial_{y^w} F^r \mathbf{G}_s^t.
\end{aligned}$$

Then

$$\begin{aligned}
[\mathbf{V}_p^0, \mathbf{V}_q^1] &= \sum_{u,w,s,t=1}^m G_p^u G_q^w \tilde{R}_{uwt}^s \mathbf{G}_s^t, \\
[\mathbf{V}_p^1, \mathbf{V}_q^1] &= \sum_{u,w,s,t=1}^m G_p^u G_q^w \tilde{T}_{uw}^t (G^{-1})_t^s \mathbf{V}_s^0 \quad \mathrm{mod}\ \mathbf{G}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{R}_{uwt}^s &= -\frac{1}{2}\partial_{y^u}\partial_{y^w}\partial_{y^t} F^s, \\
\tilde{T}_{uw}^t &= \frac{1}{2}\left(\partial_{x^u}\partial_{y^w} F^t - \partial_{x^w}\partial_{y^u} F^t\right) \\
&+ \frac{1}{4}\sum_{v=1}^m \left(\partial_{y^u} F^v \partial_{y^v}\partial_{y^w} F^t - \partial_{y^w} F^v \partial_{y^v}\partial_{y^u} F^t\right).
\end{aligned}$$

By Theorem 3.8 we know that $\tilde{T}$ is expressed in terms of K_0. As a conclusion we get that all invariants of a system (F) are expressed by K_0, $\tilde{R}$ and their derivatives. This strengthens a result of [7] (problem (B)).

We get the following characterisation of trivial systems.

Corollary 3.9 *A system of second order ODEs is equivalent to the trivial system $x'' = 0$ if and only if K_0 vanishes and F is a polynomial of degree at most 2 in x'.*

Proof. $\tilde{R}_{uwt}^s = -\frac{1}{2}\partial_{y^u}\partial_{y^w}\partial_{y^t} F^s = 0$ means that F is polynomial of degree at most 2 as a function in x'. □

Remark. Let us notice that if (F) is a geodesic equation for a Finsler metric then the quantity $\partial_{y^u}\partial_{y^w}\partial_{y^t} F^s$ is called *Berwald curvature* (cf. [22]). In our setting it appears as a component of curvature: R_{uwt}^{01s}. Vanishing of Berwald curvature is necessary an sufficient condition for Finsler metric to be Riemannian.

3.5 Equations of order 3

Let (F) be an equation of the third order

$$x''' = F(t, x, x', x'').$$

As before we consider time-scale preserving contact transformations and we want to solve the equivalence problem for (F).

Theorem 3.10 *Let (F) be a third order ODE. The first canonical frame on $\mathcal{F}_N$ satisfies:*

$$\begin{aligned}
&[\mathbf{X}, \mathbf{V}^0] = \mathbf{V}^1 - L\mathbf{G},\\
&[\mathbf{X}, \mathbf{V}^1] = \mathbf{V}^2,\\
&[\mathbf{X}, \mathbf{V}^2] = -K_0\mathbf{V}^0 + K_1\mathbf{V}^1 + \hat{R}^2\mathbf{G},\\
&[\mathbf{V}^0, \mathbf{V}^1] = 2\mathbf{V}^0(L)\mathbf{G},\\
&[\mathbf{V}^0, \mathbf{V}^2] = L\mathbf{V}^1 + R^{02}\mathbf{G},\\
&[\mathbf{V}^1, \mathbf{V}^2] = T_0^{12}\mathbf{V}^0 - 2\mathbf{X}(L)\mathbf{V}^1 + 2L\mathbf{V}^2 + R^{12}\mathbf{G}.
\end{aligned}$$

where $L = T_1^{02} = \hat{R}^0 = G\tilde{L}$ and $\tilde{L}$, K_0, K_1 are functions of global coordinates (t, x, y, p) on $J^2(1, 1)$ (G is the fiber coordinate), and

$$\begin{aligned}
\hat{R}^2 &= -\frac{1}{2}\mathbf{V}^0(K_0) + \frac{1}{2}\mathbf{V}^1(K_1) - \mathbf{X}^2(L) - LK_1,\\
T_0^{12} &= \frac{1}{2}\mathbf{V}^0(K_0) + \frac{1}{2}\mathbf{V}^1(K_1) - \mathbf{X}^2(L) - LK_1\\
R^{02} &= \mathbf{V}^1(L) + 2\mathbf{V}^0\mathbf{X}(L) - 2(L)^2, \qquad R^{12} = \mathbf{V}^0(T_0^{12}).
\end{aligned}$$

Moreover $\mathbf{X}(L) = \frac{1}{3}\mathbf{V}^0(K_1)$ *and*

$$\mathbf{V}^0(R^{12}) + 2LR^{02} - 2\mathbf{X}(L) + 2\mathbf{V}^2\mathbf{V}^0(L) = \mathbf{V}^1(R^{02}).$$

Proof. The proof, based on Jacobi identity applied to the canonical frame, is analogous to the proof of Theorem 3.8. □

Corollary 3.11 *All structural functions of the canonical frame are combinations of $\tilde{L}$, K_0, K_1 and their derivatives, where*

$$\begin{aligned}
\tilde{L} &= -\frac{1}{3}\partial_p^2 F,\\
K_1 &= \partial_y F - X(\partial_p F) + \frac{1}{3}(\partial_p F)^2,\\
K_0 &= \partial_x F - X(\partial_y F) + \frac{1}{3}\partial_y F\partial_p F + \frac{4}{3}X^2(\partial_p F) - \frac{2}{3}X(\partial_p F)\partial_p F - \frac{2}{27}(\partial_p F)^3.
\end{aligned}$$

In order to prove the corollary we will compute structural functions in terms of function F. We start with the following analog of Proposition 2.5.

Proposition 3.12 *If*

$$\mathrm{ad}_X^3 V = H_2 \mathrm{ad}_X^2 V + H_1 \mathrm{ad}_X V + H_0 V$$

then

$$\begin{aligned} K_1 &= H_1 - X(H_2) + \frac{1}{3}H_2^2, \\ K_0 &= -\left(H_0 - \frac{1}{3}X^2(H_2) - \frac{2}{9}H_2X(H_2) + \frac{2}{27}H_2^3 + \frac{1}{3}H_2H_1 + \frac{2}{9}X(H_2)H_2\right) \end{aligned} \tag{75}$$

and

$$\begin{aligned} \mathcal{H}^1 &= \mathrm{span}\left\{\mathrm{ad}_X V - \frac{1}{3}VH_2\right\}, \\ \mathcal{H}^2 &= \mathrm{span}\left\{\mathrm{ad}_X^2 V - \frac{2}{3}(\mathrm{ad}_X V)H_2 + V\left(\frac{1}{9}H_2^2 - \frac{1}{3}X(H_2)\right)\right\}. \end{aligned} \tag{76}$$

Proof. Equation (5) reads $X(G) = -\frac{1}{3}H_2G$. Then $X^2(G) = -\frac{1}{3}X(H_2)G + \frac{1}{9}H_2^2G$ and $X^3(G) = -\frac{1}{3}X^2(H_2)G + \frac{1}{9}H_2X(H_2)G + \frac{2}{9}X(H_2)H_2G - \frac{1}{27}H_2^3G$. We directly compute

$$\begin{aligned} \mathrm{ad}_X^3(VG) &= (\mathrm{ad}_X V)H_1G + VH_0G + 3(\mathrm{ad}_X V)X^2(G) + VX^3(G) \\ &= ((\mathrm{ad}_X V)G + VX(G))G^{-1}(H_1G + 3X^2(G)) \\ &+ VGG^{-1}(H_0G + X^3(G) - X(G)G^{-1}H_1G - 3X(G)G^{-1}X^2(G)). \end{aligned}$$

At a point x we can substitute $G(x) = \mathrm{Id}$ and this leads to the formula for K_0 and K_1 in the basis $V(x)$. We also have that $\mathrm{ad}_X(VG) = (\mathrm{ad}_X V)G - \frac{1}{3}VH_2G$ and $\mathrm{ad}_X^2(VG) = (\mathrm{ad}_X^2 V)G - \frac{2}{3}(\mathrm{ad}_X V)H_2G + V\left(\frac{1}{9}H_2^2G - \frac{1}{3}X(H_2)G\right)$, and we get the formulae for $\mathcal{H}^1$ and $\mathcal{H}^2$. □

Let (t, x, y, p) denote global coordinates on $J^2(1,1)$ (y corresponds to x' and p corresponds to x''). Then

$$X_F = \partial_t + y\partial_x + p\partial_y + F(t,x,y,p)\partial_p$$

and $\mathcal{V}_F = \mathrm{span}\{V\}$, where $V = \partial_p$. We check that

$$\mathrm{ad}_{X_F}^3 V = -\partial_p F \mathrm{ad}_{X_F}^2 V - (2X(\partial_p F) - \partial_y F)\mathrm{ad}_{X_F} V + (X(\partial_y F) - X^2(\partial_p F) - \partial_x F)V.$$

Therefore, from Proposition 3.12 we easily derive the formulas for K_0 and K_1 in Corollary 3.11. Moreover, $\mathcal{H}^1 = \mathrm{span}\{V^1\}$ and $\mathcal{H}^2 = \mathrm{span}\{V^2\}$, where

$$\begin{aligned} V^1 &= -\partial_y - \frac{2}{3}\partial_p F\partial_p, \\ V^2 &= \partial_x + \frac{1}{3}\partial_p F\partial_y + \left(\partial_y F + \frac{4}{9}(\partial_p F)^2 - \frac{2}{3}X(\partial_p F)\right)\partial_p. \end{aligned}$$

In order to construct the canonical frame we compute

$$\begin{aligned} &[V, V^1] = -\frac{2}{3}\partial_p^2 FV, \\ &[V, V^2] = -\frac{1}{3}\partial_p^2 FV^1 + \left(\frac{1}{3}\partial_y\partial_p F - \frac{2}{3}X(\partial_p^2 F)\right)V, \end{aligned}$$

and equation (34) implies

$$\begin{aligned}
\mathbf{V}^0 &= GV, \\
\mathbf{V}^1 &= GV^1 - G\frac{2}{3}\partial_p^2 F\mathbf{G}, \\
\mathbf{V}^2 &= GV^2 + G\left(\frac{1}{3}\partial_y\partial_p F - \frac{2}{3}X(\partial_p^2 F)\right)\mathbf{G}.
\end{aligned}$$

Then

$$[\mathbf{V}^0, \mathbf{V}^2] = -G\frac{1}{3}\partial_p^2 F\mathbf{V}^1 \mod \mathbf{G}.$$

This gives the structural function $T_1^{02} = -G\frac{1}{3}\partial_p^2 F$ and proves the first formula in Corollary 3.11. In addition we get

Corollary 3.13 *A third order ODE is time-scale preserving contact equivalent to the trivial equation $x''' = 0$ if and only if it is affine in x'' and $K_0 = K_1 = 0$.*

4 Veronese Webs

We apply our results to get local classification of Veronese webs of corank 1. Such webs were introduced by Gelfand and Zakharevich [14] in connection to bi-hamiltonian systems. It was conjectured in [14], and proved by Turiel in [24], that Veronese webs determine bi-hamiltonian structures. Normal forms of Veronese webs were provided in [25] (see also [26]). Below we show that the framework of dynamic pairs includes Veronese webs (and thus, by results of [14, 24], it includes bi-hamiltonian structures).

Let

$$\mathbb{R} \ni t \mapsto \mathcal{F}_t$$

be a family of corank 1 foliations on a manifold S of dimension $k+1$. Assume that ω_t are smooth one-forms annihilating $\mathcal{F}_t$. We say that a family $\{\mathcal{F}_t\}$ is a *Veronese web* if there exist pointwise linearly independent smooth one-forms

$$\alpha_0, \ldots, \alpha_k$$

such that for every $x \in S$

$$\omega_t(x) = t^k\alpha_0(x) + t^{k-1}\alpha_1(x) + \cdots + t\alpha_{k-1}(x) + \alpha_k(x).$$

If we add a one-form at infinity $\omega_\infty = \alpha_k$ then, for every $x \in S$, we get a Veronese curve in the projectivisation of the cotangent space T_x^*S:

$$\mathbb{R}P^1 \ni (s:t) \longmapsto \mathbb{R}\left(\sum_{i=0}^{k} s^i t^{k-i}\alpha_i(x)\right) \in P(T_x^*S). \tag{77}$$

This curve has a canonical parameter defined by the map $t \mapsto \mathcal{F}_t$.

We say that two Veronese webs $\{\mathcal{F}_t\}$ on a manifold S and $\{\mathcal{F}'_t\}$ on a manifold S' are *equivalent* if there exists a diffeomorphism $\Phi\colon S \to S'$ such that $\Phi(\mathcal{F}_t) = \mathcal{F}'_t$ for any $t \in \mathbb{R}$.

4.1 Dynamic pairs of Veronese webs and equivalence

Let

$$\mathbb{R}P^1 \ni (s:t) \mapsto \mathbb{R}\left(\sum_{i=0}^{k} s^i t^{k-i} Y_i(x)\right) \in P(T_x S)$$

be the Veronese curve in the projective space $P(T_xS)$ dual to the curve (77). By definition, this is a curve $Z_t(x)$ in T_xS such that

$$\mathrm{span}\left\{Z_t(x), \frac{d}{dt}Z_t(x), \ldots, \frac{d^{k-1}}{dt^{k-1}}Z_t(x)\right\} = T_x\mathcal{F}_t = \ker\omega_t(x), \tag{78}$$

where

$$Z_t = t^k Y_0 + t^{k-1}Y_1 + \cdots + tY_{k-1} + Y_k \tag{79}$$

and $Y_0, \ldots, Y_k$ are pointwise linearly independent vector fields on S.

Denote

$$M_{\mathcal{F}} = \bigcup_{x\in S} P\left(\left\{\sum_{i=0}^{k} s^i t^{k-i}\alpha_i(x) \mid (s:t) \in \mathbb{R}P^1\right\}\right) \subset P(T^*S).$$

Then $M_{\mathcal{F}}$ is $k+2$ dimensional manifold, more precisely a circle bundle $\mathrm{pr}\colon M_{\mathcal{F}} \to S$. Note that the fibres of $M_{\mathcal{F}}$ have a canonical parameter given by t. If $x \in S$ and $t \in \mathbb{R}$ then (x,t) is a point in $M_{\mathcal{F}}$. On $M_{\mathcal{F}}$ there is a canonical vertical (i.e. tangent to fibres) vector field, denoted $X_{\mathcal{F}}$. In coordinates

$$X_{\mathcal{F}} = \partial_t.$$

Moreover, $M_{\mathcal{F}}$ itself is equipped with a canonical foliation with leaves given by the equations $\{t = \mathrm{const}\}$. This foliation can be treated as a horizontal connection on the bundle $M_{\mathcal{F}} \to S$. Therefore, in particular, we can lift the vector $Z_t(x)$ to a unique vector $\hat{Z}_t(x)$ at the point $(x,t) \in M_{\mathcal{F}}$. In this way we obtain a global vector field $(x,t) \mapsto \hat{Z}(x,t) \in T_{(x,t)}M_{\mathcal{F}}$ defined on $M_{\mathcal{F}}$. We introduce the rank 1 distribution:

$$\mathcal{V}_{\mathcal{F}}(x,t) = \mathrm{span}\{\hat{Z}(x,t)\}.$$

Lemma 4.1 *The pair $(X_{\mathcal{F}}, \mathcal{V}_{\mathcal{F}})$ on $M_{\mathcal{F}}$ satisfies (R1), (R2) and is of equation type.*

Proof. We begin with the observation that $\mathrm{ad}^i_{X_{\mathcal{F}}}\mathcal{V}_{\mathcal{F}}$ is spanned by $\hat{Y}^0, \hat{Y}^1, \ldots, \hat{Y}^i$, where $\hat{Y}^i$ is the lift of $\partial^i_t Z_t$ to $M_{\mathcal{F}}$. By (79) $\hat{Y}^0, \ldots, \hat{Y}^k$ are independent at any point of $M_{\mathcal{F}}$ and thus (R1) and (R2) are satisfied. To finish the proof it is sufficient to prove that $\mathrm{ad}^{k-1}_{X_{\mathcal{F}}}\mathcal{V}_{\mathcal{F}}$ and $\mathrm{ad}^{k}_{X_{\mathcal{F}}}\mathcal{V}_{\mathcal{F}}$ are integrable (see condition (R3') of Theorem 3.5). Integrability of $\mathrm{ad}^{k-1}_{X_{\mathcal{F}}}\mathcal{V}_{\mathcal{F}}$ immediately follows from the definitions of $X_{\mathcal{F}}$, $\mathcal{V}_{\mathcal{F}}$ and from (78). Namely, $\mathrm{pr}(\mathrm{ad}^{k-1}_{X_{\mathcal{F}}}\mathcal{V}_{\mathcal{F}}(x,t)) = T_x\mathcal{F}_t$. On the other hand $\mathrm{ad}^{k}_{X_{\mathcal{F}}}\mathcal{V}_{\mathcal{F}}$ is the distribution tangent to foliation $\{t = \mathrm{const}\}$ on $M_{\mathcal{F}}$. □

We would like to know which dynamic pairs of equation type define Veronese webs.

Definition 4.2 *Let* X*,* $\mathcal{V}$ *be a smooth vector field and a smooth line field on a manifold* M*. We say that* $(X, \mathcal{V})$ *is of* Veronese type*, if there exists a Veronese web* $\{\mathcal{F}_t\}$ *on a manifold* S *such that* $(X, \mathcal{V})$ *is diffeomorphic to the pair* $(X_{\mathcal{F}}, \mathcal{V}_{\mathcal{F}})$ *on the manifold* $M_{\mathcal{F}}$*. We say that* $(X, \mathcal{V})$ *is* locally of Veronese type*, if for any* $x \in M$ *there exists a neighbourhood* $U \ni x$ *and a Veronese web* $\{\mathcal{F}_t\}$ *on a manifold* S *such that* $(X|_U, \mathcal{V}|_U)$ *is diffeomorphic to the pair* $(X_{\mathcal{F}}|_V, \mathcal{V}_{\mathcal{F}}|_V)$ *for an open subset* $V \subset M_{\mathcal{F}}$*.*

Theorem 4.3 *Let* (F) *be an equation of order* $k+1$*. The corresponding dynamic pair* $(X_{\mathcal{F}}, \mathcal{V}_{\mathcal{F}})$ *on* $M = E_F \simeq J^k(1,1)$ *is locally of Veronese type if and only if all curvature operators* $K_0, \dots, K_{k-1}$ *vanish.*

Proof. First, note that if $(X, \mathcal{V})$ is of Veronese type then in local coordinates on $M_{\mathcal{F}}$ we have $X = \partial_t$ and $\mathcal{V}(t,x)$ is spanned by

$$\hat{Z}(t,x) = t^k Y_0(x) + t^{k-1} Y_1(x) + \cdots + t Y_{k-1}(x) + Y_k(x),$$

see formula (79). Since $\mathrm{ad}_{\partial_t}^{k+1} \hat{Z} = 0$, it follows that $\hat{Z}$ is a normal generator of $\mathcal{V}$ and all curvature operators vanish.

On the other hand, if $(X, \mathcal{V})$ is of equation type and all its curvature operators vanish, then we can choose a section V of $\mathcal{V}$ such that $\mathrm{ad}_X^{k+1} V = 0$. Let us choose an open subset $U \subset M$ with local coordinates such that $X = \partial_t$ on U (we can always locally trivialise X). Then along any integral curve of X contained in U we get the formula

$$V(t) = t^k V_0 + t^{k-1} V_1 + \cdots + t V_{k-1} + V_k,$$

where $V_0, \dots, V_k$ are constant vectors along an integral curve of X. Indeed, the equation $\mathrm{ad}_X^{k+1} V = 0$ means that along an integral curve of X the vector field V is a solution to the equation $\frac{d^{k+1}V}{dt^{k+1}} = 0$ and thus V is polynomial in t.

Take U so that the set of trajectories of X in U forms a Hausdorff manifold and define S to be the quotient space $S = U/X$. This means that a point $x \in S$ is an integral line of X with parameter t belonging to some segment $I_x \subset \mathbb{R}$. If we project $\mathcal{V}(t) = \mathrm{span}\{V(t)\}$ to S for every $t \in I_x$ we get a segment of Veronese curve in $P(T_x S)$. Since a Veronese curve is uniquely determined by a finite number of its points, we can uniquely extend the segment of Veronese curve to the full Veronese curve. The dual Veronese curve in $P(T^* S)$ defines the desired Veronese web. □

Theorems 2.9 and 2.11 applied to the pair $(X_{\mathcal{F}}, \mathcal{V}_{\mathcal{F}})$ give the following:

Theorem 4.4 *The following conditions are equivalent.*
(a) Veronese webs $\{\mathcal{F}_t\}$ *and* $\{\mathcal{F}'_t\}$ *are equivalent.*
(b) The dynamic pairs $(X_{\mathcal{F}}, \mathcal{V}_{\mathcal{F}})$ *and* $(X_{\mathcal{F}'}, \mathcal{V}_{\mathcal{F}'})$ *are diffeomorphic by a diffeomorphism preserving* t*.*
(c) The canonical frames $(\mathbf{X}, \mathbf{V}, \mathbf{G})$ *and* $(\mathbf{X}', \mathbf{V}', \mathbf{G}')$ *on the normal frame bundles* $\pi\colon \mathcal{F}_N \to M_{\mathcal{F}}$ *and* $\pi\colon \mathcal{F}'_N \to M_{\mathcal{F}'}$ *, corresponding to dynamical pairs* $(X_{\mathcal{F}}, \mathcal{V}_{\mathcal{F}})$ *and* $(X_{\mathcal{F}'}, \mathcal{V}_{\mathcal{F}'})$ *via Theorem 2.9 (resp. Theorem 2.11), are diffeomorphic by a diffeomorphism preserving* t*.*

Proof. Assume first that $\{\mathcal{F}_t\}$ and $\{\mathcal{F}'_t\}$ are equivalent Veronese webs and the equivalence is established by $\Phi\colon S \to S'$. Let $\Psi\colon M_{\mathcal{F}} \to M_{\mathcal{F}'}$ be the lift of Φ defined in an obvious way. By definition of equivalence of webs we get $\Phi(\mathcal{F}_t) = \mathcal{F}'_t$ and hence Ψ preserves t. Moreover, Ψ maps fibres of $M_{\mathcal{F}} \to S$ onto fibres of $M_{\mathcal{F}'} \to S'$ and we get that $\Psi_* X_{\mathcal{F}} = X_{\mathcal{F}'}$. It is also a direct consequence of the definitions that $\Psi_*\mathcal{V}_{\mathcal{F}} = \mathcal{V}_{\mathcal{F}'}$ because $\Phi_*(\ker\omega_t) = \ker\omega'_t$, for any t, which implies that $\Phi_*(\mathrm{span}\{Z_t\}) = \mathrm{span}\{Z'_t\}$ and, consequently, $\Psi_*(\mathrm{span}\{\hat{Z}\}) = \mathrm{span}\{\hat{Z}'\}$.

On the other hand, if $\Psi\colon M_{\mathcal{F}} \to M_{\mathcal{F}'}$ establish equivalence of dynamic pairs $(X_{\mathcal{F}}, \mathcal{V}_{\mathcal{F}})$ and $(X_{\mathcal{F}'}, \mathcal{V}_{\mathcal{F}'})$ then $\Psi_* X_{\mathcal{F}} = X_{\mathcal{F}'}$ and thus it transports fibres of $M_{\mathcal{F}} \to S$ onto fibres of $M_{\mathcal{F}'} \to S'$. Hence, Ψ defines a mapping $\Phi\colon S \to S'$. If $\Psi_*\mathcal{V}_{\mathcal{F}} = \mathcal{V}_{\mathcal{F}'}$ then also $\Psi_*\mathrm{ad}^{k-1}_{X_{\mathcal{F}}}\mathcal{V}_{\mathcal{F}} = \mathrm{ad}^{k-1}_{X_{\mathcal{F}'}}\mathcal{V}_{\mathcal{F}'}$. The projection of a leaf of $\mathrm{ad}^{k-1}_{X_{\mathcal{F}}}\mathcal{V}_{\mathcal{F}}$ is a leaf of the foliation $\mathcal{F}_t$, for some t, thus Φ maps leaves of $\{\mathcal{F}_t\}$ onto leaves of $\{\mathcal{F}'_t\}$. If additionally Ψ preserves t we get that $\Phi(\mathcal{F}_t) = \mathcal{F}'_t$ for any t. This proves (a) $\Leftrightarrow$ (b).

(b) $\Leftrightarrow$ (c) follows directly from Theorem 2.9 (resp. Theorem 2.11). $\square$

Corollary 4.5 *A Veronese web has at most $k+2$-dimensional group of symmetries. The group has maximal dimension if and only if the web is flat, i.e., it is given by the kernel of the 1-forms $\omega_t = \sum_{i=0}^{k} t^{k-i}dx^i$, in some coordinates on S.*

Proof. Note that Theorem 2.9 imply that the group of symmetries of a web is at most $k+3$ dimensional. However, statement (c) of Theorem 4.4 says that not all symmetries of the canonical frame of a dynamic pair $(X_{\mathcal{F}}, \mathcal{V}_{\mathcal{F}})$ define symmetries of $\{\mathcal{F}_t\}$. Namely a symmetry must keep t invariant. Therefore we get that the dimension of the symmetry group is bounded from above by $k+2$. Moreover, it follows that if the symmetry group has maximal possible dimension then the structural functions of the canonical frame $(\mathbf{V}, \mathbf{G})$ have to be constant on each leaf $\mathcal{F}_N|_{\{t=\mathrm{const}\}} \subset \mathcal{F}_N$. Therefore the part of the curvature and the torsion of the canonical connection which involves $\mathbf{V}$ vanish. Then, using Jacobi identity applied to $[\mathbf{X}, [\mathbf{V}^i, \mathbf{V}^j]]$ we get that the part of the curvature involving $\mathbf{X}$ and $\mathbf{V}^i$ also vanish. Moreover, taking into account that $K_i \equiv 0$ for an arbitrary Veronese web (Lemma 4.1) we get that the Veronese web is flat. $\square$

4.2 Veronese webs on a plane

Let $\mathcal{F} = \{\mathcal{F}_t\}$ be a Veronese web on $\mathbb{R}^2$ defined by a family of 1-dimensional distributions

$$\mathrm{span}\{tY_0 + Y_1\},$$

where Y_0, Y_1 are smooth vector fields on $\mathbb{R}^2$. Theorems 3.8, 3.5 and 4.3 imply

Theorem 4.6 *The first canonical frame on the bundle $\mathcal{F}_N$, corresponding to the dynamic pair $(X_{\mathcal{F}}, \mathcal{V}_{\mathcal{F}})$, satisfies:*

$$[\mathbf{X}, \mathbf{V}^0] = \mathbf{V}^1, \qquad [\mathbf{X}, \mathbf{V}^1] = 0, \qquad [\mathbf{V}^0, \mathbf{V}^1] = R\mathbf{G}$$

for a certain function R such that $\mathbf{X}(R) = 0$.

Remark. Equation $[\mathbf{X}, \mathbf{V}^0] = \mathbf{V}^1$ implies that the first and the second canonical frames, given by Theorems 2.9 and 2.11, coincide for Veronese webs on a plane.

Since $\mathbf{V}^0$ and $\mathbf{V}^1$ are homogeneous of order one with respect to the fiber coordinate in the normal bundle $\mathcal{F}_N \to M_{\mathcal{F}}$, and $\mathbf{G}$ are homogeneous of order zero, it follows that $R = G^2\tilde{R}$, in coordinates, where $\tilde{R}$ is a function on $M_{\mathcal{F}}$. However, $\mathbf{X}(R) = 0$ implies that $\tilde{R}$ is in fact well defined on $\mathbb{R}^2$. Let us fix a point $x \in \mathbb{R}^2$ and let $\hat{x} = (x, 0) \in M_{\mathcal{F}}(x)$. Let us also choose a point ν in the bundle $\mathcal{F}_N(\hat{x})$. Then we can introduce a coordinate system on $\mathbb{R}^2$ in the following way:

$$(x_0, x_1) \mapsto \mathrm{pr} \circ \exp(x_0\mathbf{V}^0) \circ \exp(x_1\mathbf{V}^1)(\nu), \tag{80}$$

where $\mathrm{pr}\colon \mathcal{F}_N \to \mathbb{R}^2$ is the projection composed of the projections $\mathcal{F}_N \to M_{\mathcal{F}} \to \mathbb{R}^2$. If we change $\nu \mapsto \nu G$ for some $G \in Gl(1) \simeq \mathbb{R}_*$, the coordinates are multiplied by a real number. Therefore we get a canonical local system of coordinates on $\mathbb{R}^2$ with the origin in x, given up to multiplication by a constant. In this coordinates we can express function $\tilde{R}$ and get the function on $\mathbb{R}^2$ intrinsically assigned to the web.

Corollary 4.7 *There is one-to-one correspondence between germs of Veronese webs at $0 \in \mathbb{R}^2$ and germs of functions $\tilde{R}\colon \mathbb{R}^2 \to \mathbb{R}$ at 0 given up to the transformations*

$$\tilde{R}(x_0, x_2) \mapsto G^2\tilde{R}(Gx_0, Gx_1), \quad G \neq 0.$$

Proof. We shall show how to recover the web from the function $\tilde{R}$. First we consider $\mathbb{R}^2 \times Gl(1)$ with coordinates x_0, x_1 and G (this space is the level set $\{t = 0\}$ of the canonical bundle $\mathcal{F}_N$). On $\mathbb{R}^2 \times Gl(1)$ we define $R(x_0, x_1, G) = G\tilde{R}(x_0, x_1)$. It follows from the definition of the canonical coordinate system (formula (80)) and the relation $[\mathbf{G}, \mathbf{V}^i] = \mathbf{V}^i$ that we can assume $\mathbf{V}^0 = G\partial_{x_0}$ and $\mathbf{V}^1 = Ga\partial_{x_0} + Gb\partial_{x_1} + Gc\partial_G$ for some functions a, b, c in variables x_0, x_1. Thus the equation $[\mathbf{V}^0, \mathbf{V}^1] = R\mathbf{G}$ in Theorem 4.6 implies the following:

$$\partial_{x_0}a - c = 0, \qquad \partial_{x_0}b = 0, \qquad G\partial_{x_0}c = R.$$

On the plane $\{x_0 = 0\}$ we have $a = c = 0$ and $b = 1$ (again we use the definition (80) of the coordinate system). Hence we are able to recover a, b and c in a unique way. The web on the (x_0, x_1)-plane is spanned by: $\mathrm{pr}_*(\mathbf{V}^0(x_0, x_1, 1) + t\mathbf{V}^1(x_0, x_1, 1))$ where $\mathrm{pr}\colon \mathbb{R}^2 \times Gl(1) \to \mathbb{R}^2$ is the projection on the first factor. □

4.3 Veronese webs on $\mathbb{R}^3$

Let $\mathcal{F} = \{\mathcal{F}_t\}$ be a Veronese web on $\mathbb{R}^3$ given by the kernel of ω_t, where

$$\omega_t = t^2\alpha_0 + t\alpha_1 + \alpha_2$$

and α_0, α_1, α_2 are smooth one-forms on $\mathbb{R}^3$. Theorems 3.10, 3.5 and 4.3 imply:

Theorem 4.8 *The first canonical frame on the bundle $\mathcal{F}_N$, corresponding to the dynamic pair $(X_\mathcal{F}, \mathcal{V}_\mathcal{F})$, satisfies the relations:*

$$\begin{aligned}
&[\mathbf{X}, \mathbf{V}^0] = \mathbf{V}^1 - T\mathbf{G}, \qquad [\mathbf{X}, \mathbf{V}^1] = \mathbf{V}^2, \qquad [\mathbf{X}, \mathbf{V}^2] = 0,\\
&[\mathbf{V}^0, \mathbf{V}^1] = 2\mathbf{V}^0(T)\mathbf{G},\\
&[\mathbf{V}^0, \mathbf{V}^2] = T\mathbf{V}^1 + (\mathbf{V}^1(T) - 2T^2)\mathbf{G},\\
&[\mathbf{V}^1, \mathbf{V}^2] = 2T\mathbf{V}^2,
\end{aligned}$$

for a certain function T such that

$$\mathbf{X}(T) = 0 \quad \text{and} \quad \mathbf{V}^0\mathbf{V}^2(T) + \mathbf{V}^2\mathbf{V}^0(T) = \mathbf{V}^1\mathbf{V}^1(T) - 4T\mathbf{V}^1(T) + 2T^3.$$

The second canonical frame gives more elegant structural functions.

Theorem 4.9 *The second canonical frame satisfies:*

$$\begin{aligned}
&[\mathbf{X}, \tilde{\mathbf{V}}^0] = \tilde{\mathbf{V}}^1, \qquad [\mathbf{X}, \tilde{\mathbf{V}}^1] = \tilde{\mathbf{V}}^2, \qquad [\mathbf{X}, \tilde{\mathbf{V}}^2] = 0,\\
&[\tilde{\mathbf{V}}^0, \tilde{\mathbf{V}}^1] = T\tilde{\mathbf{V}}^0 + \tilde{\mathbf{V}}^0(T)\mathbf{G},\\
&[\tilde{\mathbf{V}}^0, \tilde{\mathbf{V}}^2] = T\tilde{\mathbf{V}}^1 + \tilde{\mathbf{V}}^1(T)\mathbf{G},\\
&[\tilde{\mathbf{V}}^1, \tilde{\mathbf{V}}^2] = T\tilde{\mathbf{V}}^2 + \tilde{\mathbf{V}}^2(T)\mathbf{G},
\end{aligned}$$

for a function T such that

$$\mathbf{X}(T) = 0 \quad \text{and} \quad \tilde{\mathbf{V}}^0\tilde{\mathbf{V}}^2(T) + \tilde{\mathbf{V}}^2\tilde{\mathbf{V}}^0(T) = \tilde{\mathbf{V}}^1\tilde{\mathbf{V}}^1(T). \tag{81}$$

Proof. K_0 and K_1 vanish for Veronese webs. Thus, the definition of the second canonical frame implies:

$$[\mathbf{X}, \tilde{\mathbf{V}}^0] = \tilde{\mathbf{V}}^1, \qquad [\mathbf{X}, \tilde{\mathbf{V}}^1] = \tilde{\mathbf{V}}^2, \qquad [\mathbf{X}, \tilde{\mathbf{V}}^2] = \hat{R}^2\mathbf{G}^1$$

for a function $\hat{R}^2$. Moreover, we have:

$$[\tilde{\mathbf{V}}^0, \tilde{\mathbf{V}}^1] = T_0^{01}\mathbf{V}^0 + R^{01}\mathbf{G}$$

where T_0^{01} and R^{01} are some functions on the canonical bundle. If we compute the Lie brackets $[\mathbf{X}, [\tilde{\mathbf{V}}^0, \tilde{\mathbf{V}}^1]]$ and $[\mathbf{X}, [\mathbf{X}, [\tilde{\mathbf{V}}^0, \tilde{\mathbf{V}}^1]]]$ and apply Jacobi identity we get

$$[\tilde{\mathbf{V}}^0, \tilde{\mathbf{V}}^2] = \mathbf{X}(T_0^{01})\tilde{\mathbf{V}}^0 + T_0^{01}\tilde{\mathbf{V}}^1 + \mathbf{X}(R^{01})\mathbf{G},$$

$$[\tilde{\mathbf{V}}^1, \tilde{\mathbf{V}}^2] = (\hat{R}^2 + \mathbf{X}^2(T_0^{01}))\tilde{\mathbf{V}}^0 + 2\mathbf{X}(T_0^{01})\tilde{\mathbf{V}}^1 + T_0^{01}\tilde{\mathbf{V}}^2 + (\mathbf{X}^2(R^{01}) - \tilde{\mathbf{V}}^0(\hat{R}^2))\mathbf{G}.$$

Now, let us compute the bracket: $[\mathbf{X}, [\tilde{\mathbf{V}}^1, \tilde{\mathbf{V}}^2]]$. Applying Jacobi identity on the one hand we get:

$$\tilde{\mathbf{V}}^1(\hat{R}^2)\mathbf{G} - \hat{R}^2\mathbf{V}^1.$$

On the other hand:

$$\begin{aligned}
&\left(\mathbf{X}(\hat{R}^2) + \mathbf{X}^3(T_0^{01})\right)\tilde{\mathbf{V}}^0 + \left(3\mathbf{X}^2(T_0^{01}) + \mathbf{X}(\hat{R}^2)\right)\tilde{\mathbf{V}}_1 + 3\mathbf{X}(T_0^{01})\tilde{\mathbf{V}}^2\\
&+ \left(T_0^{01}\hat{R}^2 - \mathbf{X}(\tilde{\mathbf{V}}^0(\hat{R}^2)) + \mathbf{X}^3(R^{01})\right)\mathbf{G}
\end{aligned}$$

We compare the coefficients and get: $\mathbf{X}(T_0^{01}) = 0$ (coefficient next to $\tilde{\mathbf{V}}^2$), $\mathbf{X}(\hat{R}^2) = 0$ (coefficient next to $\tilde{\mathbf{V}}^0$). Then we get: $\hat{R}^2 = 0$ (coefficient next to $\tilde{\mathbf{V}}^1$) and finally: $\mathbf{X}^3(R^{01}) = 0$ (coefficient next to $\mathbf{G}$). We substitute this equations to the previous relations and get the simplified versions:

$$[\tilde{\mathbf{V}}^0, \tilde{\mathbf{V}}^2] = T_0^{01}\tilde{\mathbf{V}}^1 + \mathbf{X}(R^{01})\mathbf{G}, \quad [\tilde{\mathbf{V}}^1, \tilde{\mathbf{V}}^2] = T_0^{01}\tilde{\mathbf{V}}^2 + \mathbf{X}^2(R^{01})\mathbf{G}.$$

Then we apply Jacobi identity to $[\tilde{\mathbf{V}}^0, [\tilde{\mathbf{V}}^1, \tilde{\mathbf{V}}^2]]$ and on the one hand we get:

$$-\mathbf{X}^2(R^{01})\tilde{\mathbf{V}}^0 + (T_0^{01})^2\tilde{\mathbf{V}}^1 + \tilde{\mathbf{V}}^0(T_0^{01})\tilde{\mathbf{V}}^2 + \left(\tilde{\mathbf{V}}^0\mathbf{X}^2(R^{01}) + T_0^{01}\mathbf{X}(R^{01})\right)\mathbf{G}$$

whereas on the other hand we get:

$$\begin{aligned}&-\tilde{\mathbf{V}}^2(T_0^{01})\tilde{\mathbf{V}}^0 + \left((T_0^{01})^2 - \mathbf{X}(R^{01}) + \tilde{\mathbf{V}}^1(T_0^{01})\right)\tilde{\mathbf{V}}^1 + R^{01}\tilde{\mathbf{V}}^2\\ &+ \left(\tilde{\mathbf{V}}^1\mathbf{X}(R^{01}) - \tilde{\mathbf{V}}^2(R^{01}) + T_0^{01}\mathbf{X}(R^{01}))\right)\mathbf{G}.\end{aligned}$$

Thus we get: $\mathbf{V}^0(T_0^{01}) = R^{01}$ (coefficient next to $\tilde{\mathbf{V}}^2$). Recall that we have already proved: $\mathbf{X}(T_0^{01}) = 0$. Therefore, it follows that $\mathbf{X}(R^{01}) = \mathbf{V}^1(T_0^{01})$, $\mathbf{X}^2(R^{01}) = \mathbf{V}^2(T_0^{01})$ and $\mathbf{X}^3(R^{01}) = 0$ (the last relation was also obtained before). We see that the only new relation is $\tilde{\mathbf{V}}^0\tilde{\mathbf{V}}^2(T_0^{01}) + \tilde{\mathbf{V}}^2\tilde{\mathbf{V}}^0(T_0^{01}) = \tilde{\mathbf{V}}^1\tilde{\mathbf{V}}^1(T_0^{01})$ (coefficient next to $\mathbf{G}$). If we denote $T := T_0^{01}$ then the theorem is proved. □

Remark. It can be easily verified that the following substitution relates the frames of Theorem 4.8 and Theorem 4.9:

$$\tilde{\mathbf{V}}^0 = \mathbf{V}^0, \qquad \tilde{\mathbf{V}}^1 = \mathbf{V}^1 - T\mathbf{G}, \qquad \tilde{\mathbf{V}}^2 = \mathbf{V}^2.$$

Moreover, the functions T in Theorems 4.8 and 4.9 coincide.

In coordinates $T = G\tilde{T}$, where $\tilde{T}$ is a function on $M_{\mathcal{F}}$. However, similarly to the case of Veronese web on the plane, it follows from $\mathbf{X}(T) = 0$ that $\tilde{T}$ is well defined on $\mathbb{R}^3$. Let us fixed a point $x \in \mathbb{R}^3$ and take $\hat{x} = (x, 0) \in M_{\mathcal{F}}(x)$. Let us also choose a point ν in $\mathcal{F}_N(\hat{x})$. We introduce the following coordinate system on $\mathbb{R}^3$:

$$(x_0, x_2, x_1) \mapsto \mathrm{pr} \circ \exp(x_1\tilde{\mathbf{V}}^1) \circ \exp(x_2\tilde{\mathbf{V}}^2) \circ \exp(x_0\tilde{\mathbf{V}}^0)(\nu),$$

where $\mathrm{pr}\colon \mathcal{F}_N \to \mathbb{R}^3$ is the projection. Note that at the beginning we go along $\tilde{\mathbf{V}}^0$ then along $\tilde{\mathbf{V}}^2$ and finally along $\tilde{\mathbf{V}}^1$.

We are able to compute any derivative of the form $\partial^{a+b+c}\tilde{T}/\partial x_0^a\partial x_2^b\partial x_1^c$ at the origin of our coordinate system. Indeed, this is equivalent to $(\tilde{\mathbf{V}}^0)^a(\tilde{\mathbf{V}}^2)^b(\tilde{\mathbf{V}}^1)^c(\tilde{T})$. Moreover, it follows from the structural equations and (81) that any derivative of the form $(\tilde{\mathbf{V}}^0)^a(\tilde{\mathbf{V}}^2)^b(\tilde{\mathbf{V}}^1)^c(\tilde{T})$ with arbitrary $c \in \mathbb{N}$ can be written as a sum of derivatives $(\tilde{\mathbf{V}}^0)^a(\tilde{\mathbf{V}}^2)^b(\tilde{\mathbf{V}}^1)^c(\tilde{T})$ where $c = 1$ or $c = 0$. Therefore if we know $\tilde{T}$ and $\tilde{S} = \tilde{\mathbf{V}}^1(\tilde{T})$ on the plane $\{x_1 = 0\}$ then we are able to recover all possible derivatives $\partial^{a+b+c}\tilde{T}/\partial x_0^a\partial x_2^b\partial x_1^c$ at the origin of our coordinate system. Hence, if all data are analytic then we can recover $\tilde{T}$ on $\mathbb{R}^3$.

The coordinate system is unique up to the choice of the point $\nu \in \mathcal{F}_N(\hat{x})$. If we change ν then every coordinate function is multiplied by $G \in Gl(1) \simeq \mathbb{R}_*$. We get

Corollary 4.10 *In the analytic category there is one-to-one correspondence between germs of Veronese webs at* $0 \in \mathbb{R}^3$ *and germs at* 0 *of two functions* $\tilde{T}$ *and* $\tilde{S}$ *in two variables:* x_0 *and* x_2*. The functions are given up to the following transformations:*

$$\tilde{T}(x_0, x_2) \mapsto G\tilde{T}(Gx_0, Gx_2), \qquad \tilde{S}(x_0, x_2) \mapsto G\tilde{S}(Gx_0, Gx_2).$$

Proof. The proof is similar to the proof of Corollary 4.7 and we skip it. □

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