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IM PAN Preprint 734 (2011)

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Published as manuscript

Received 6 December 2011

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Abstract

A Gowers' Pigeon Principle is proved without using ultrafilters.

The purpose of the paper is to give a proof of Gowers theorem ([5]) being a generalization of Hindman theorem ([6]). The presented proof is purely combinatorial and does not use the theory of ultrafilters as in the original Gowers' proof. The method is similar to Baumgartner's simplification of the Hindman's original proof (see [2]); in fact our proof stands in the same relation to Baumgartner's as the 'tricky' (via ultrafilters) proof of Hindman theorem to the proof of Gowers theorem (see e.g. [7]). In forthcoming paper [4] the authors extend some of ideas to study distortion problem in Banach spaces. It is worth to mention that Gowers theorem was motivated by this problem as well.

The paper is organized in two sections. Section **1.** consists of useful notations and basic facts. The second section includes crucial **Lemma 4** and the proof of the **Theorem**. In fact the proof of the theorem is very similar to [2] and the most essential (or original) is a proof of **Lemma 4**.

1. Basic facts and notations.

Throughout the paper we use the following letters i, j, k, l, m, n for non-negative natural numbers and by ω we denote their universe. We prefer to use Von Neumann's definition of numbers; thus for example $k+1 = \{0, 1, \ldots, k\}$ and $i < 2$ means $i \in \{0, 1\}$. Gowers' Pigeon Principle is a Ramsey-type theorem about particular families of functions (sequences) and that is why we need specific notations; some of them we borrow from [7]. For k and function $p : \omega \to k+1$ we denote its support by $\mathrm{supp}(p) = \{n < \omega : p(n) \neq 0\}$ and its range $p[\omega]$ by $\mathrm{rng}(p)$. Define

$$\mathrm{FIN}_k = \{p \in^\omega (k+1) : |\mathrm{supp}(p)| < \omega \ \&\ k \in \mathrm{rng}(p)\}.$$

In particular FIN_0 is a singleton of the null function $\mathrm{FIN}_0 = \{\{(i, 0) : i < \omega\}\}$. Note also that FIN_k is a subset of a linear space

$$c_{00} = \{p \in^\omega \mathbb{R} : |\mathrm{supp}(p)| < \omega\}.$$

We equip the collections FIN_k with a partial ordering defined as

$$p < q \quad \text{if} \quad \max \mathrm{supp}(p) < \min \mathrm{supp}(q)$$

and two operations

- **sum** p+q: $(p+q)(n) = p(n) + q(n)$, $n < \omega$, *if* $\mathrm{supp}(p) \cap \mathrm{supp}(q) = \emptyset$;
- ***tetris*** $T : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1} : T(p)(n) = \max\{p(n) - 1, 0\},\ n < \omega$.

If we write $p+q$ it is always implicite assumed that $\mathrm{suppp}(p) \cap \mathrm{suppp}(q) = \emptyset$. Moreover, for a subset C of c_{00} we use the following notation

$$C + C = \{p + q \in c_{00} : p, q \in C \quad (\text{if } p + q \text{ is defined})\}.$$

Let us now call a (finite or infinite) family $B \subseteq \mathrm{FIN}_k$ ***block sequence*** if $b_0 < b_1$ for any different elements b_0, b_1 of B. For a block sequence B we define $\langle B \rangle$ as the smallest subfamily of FIN_k consisting of B and (almost) closed on the summing and tetris operation, e.i.

$$\langle B \rangle = \bigcup_{f \in \mathcal{N}} \left(T^{(f(0))}[B] + T^{(f(1))}[B] + \cdots + T^{(f(|f|-1))}[B]\right),$$

where $\mathcal{N}$ is a set of sequences defined as follows

$$f \in \mathcal{N} \text{ if and only if } f \text{ is a finite sequence such that } 0 \in \mathrm{rng}(f) \subseteq k+1,$$

$T^{(i)}$ is an i$^{\text{th}}$ iteration of tetris operation and $|f|$ stands for a length of a given sequence f. We say that $\langle B \rangle$ is generated by block sequence B. Note that $\mathrm{FIN}_k = \langle \{\{(i,k)\} : i < \omega\} \rangle$. On the family of block sequences of FIN_k we define the following partial ordering

$$B_0 \preceq B_1 \quad \text{iff} \quad B_0 \subseteq \langle B_1 \rangle.$$

We say that a family $\mathcal{F} \subseteq \langle B \rangle$ is B-**dense** if $\mathcal{F} \cap \langle B' \rangle \neq \emptyset$ for all $B' \preceq B$. Put

$$\mathcal{D}_B = \{\mathcal{F} \subseteq \langle B \rangle : \mathcal{F} \text{ is } B\text{-dense}\} \quad \text{and} \quad \mathcal{D}[B] = \bigcup \{\mathcal{D}_{B'} : B' \preceq B\}.$$

We end this section with three simple lemmas concerning above families. The proofs are almost the same as in [2] and they are easy consequences of definitions but we include them for completeness. Below we consider only infinite block sequences $B \subseteq \mathrm{FIN}_k$.

Lemma 1 If $\mathcal{F} \in \mathcal{D}_B$ and $S \subseteq \mathrm{FIN}_k$ is finite then $\{p \in \mathcal{F} : \forall_{s \in S}\ s < p\} \in \mathcal{D}_B$.

Proof:
Suppose $\mathcal{F}' = \{p \in \mathcal{F} : \forall_{s \in S}\ s < p\} \notin \mathcal{D}_B$ for some finite $S \subseteq \mathrm{FIN}_k$. We can find then $B' \preceq B$ such that $\langle B' \rangle \cap \mathcal{F}' = \emptyset$. However it implies that an infinite block sequence $\{b \in B' : \forall_{s \in S}\ s < b\} \preceq B$ witnesses that $\mathcal{F} \notin \mathcal{D}_B$.

Lemma 2 The family $\mathcal{D}[B]$ forms (nonpricipal) coideal (see [7]), i.e.

- $B \in \mathcal{D}_B \subseteq \mathcal{D}[B]$;
- if $\mathcal{F} \in \mathcal{D}[B]$ and $\mathcal{F} \subseteq \mathcal{F}'$ then $\mathcal{F}' \in \mathcal{D}[B]$;
- if $\mathcal{F}_0 \cup \mathcal{F}_1 \in \mathcal{D}[B]$ then $\mathcal{F}_i \in \mathcal{D}[B]$ for some $i < 2$.

Proof:
Only the last point is not immediate. Suppose $\mathcal{F}_0 \notin \mathcal{D}[B]$ and $\mathcal{F}_0 \cup \mathcal{F}_1 \in \mathcal{D}_{B'}$ for some $B' \preceq B$. Then $\langle B' \rangle \cap \mathcal{F}_0 = \emptyset$ and for arbitrary $B'' \preceq B'$ it holds $(\mathcal{F}_0 \cup \mathcal{F}_1) \cap \langle B'' \rangle \neq \emptyset$ and $\mathcal{F}_0 \cap \langle B'' \rangle = \emptyset$. Therefore $\mathcal{F}_1 \in \mathcal{D}_{B'} \subseteq \mathcal{D}[B]$.

Lemma 3 Let $\mathcal{F} \in \mathcal{D}_B$ for some block sequence $B \subseteq \mathrm{FIN}_k$ and let $j < k$. If for every $i < j$ it holds

$$\varphi(\mathcal{F}, B, i) \equiv \bigvee_{p,q \in B \cap \mathcal{F}} T^{(i)}(p) + q \in \mathcal{F},$$

then $\varphi(\mathcal{F}', B', j)$ is satisfied for some $\mathcal{F}' \subseteq \mathcal{F}$ and $B' \leqslant B$ with $\mathcal{F}' \in \mathcal{D}_{B'}$.

Proof:

If we enumerate $B = \{b_0 < b_1 < b_2 < \ldots\}$ then it is easy to check that $\mathcal{F}' = (T^{(j)}[\mathcal{F}] + \mathcal{F}) \cap \mathcal{F} \in \mathcal{D}_{B'}$, where $B' = \{b'_0 < b'_1 < b'_2 < \ldots\}$ and $b'_n = b_{2n} + T^{(j)}(b_{2n+1})$, $n < \omega$, do the work.

2. Lemma, theorem and proof.

The following proof is basically an adaptation of Baumgartner's proof of Hindman Theorem to the Gowers Theorem. An idea of the proof using **Lemma 4** is from combinatorial point of view a rather standard concept of increasing dimension of Ramsey-type theorems; from the geometrical angle it is a folklore gliding hump argument in infinite-dimensional Banach space theory. Interested reader is advised to consult with a proof Positive Stepping Up Lemma from [3] (especially a notion of an end-homogenous set) and for example the proof that c_0 is prime from or more generally Pelczynski-Bessaga Selection Principle from e.g. [1]. All block sequences below are assumed to be infinite. We start with aforementioned 'gliding hump' lemma.

Lemma 4 Let $\mathcal{F} \in \mathcal{D}_B$ for some block sequence $B \subseteq \mathrm{FIN}_k$. Then there exist $s \in \mathcal{F}$ such that

$$C(\mathcal{F}, s) = \{p \in \mathcal{F} : \langle \{s, p\} \rangle \subseteq \mathcal{F}\} \in \mathcal{D}[B].$$

Proof:

If we could find for every $n < \omega$ a function $p_n \in B$ such that $p_n > p_i$ for $i < n$ and $\langle \{p_i\}_{i \leqslant n} \rangle \cap \mathcal{F} = \emptyset$ then we would have that $\langle \{p_i\}_{i<\omega} \rangle \cap \mathcal{F} = \emptyset$, a contradiction with $\mathcal{F} \in \mathcal{D}_B$. Therefore there exist $p_0, \ldots, p_{n_0}$ such that for any $p \in \mathcal{F}$ we have $\langle \{p_i\}_{i \leqslant n_0} \cup \{p\} \rangle \cap \mathcal{F} \neq \emptyset$. This gives us the following finite decomposition of $\mathcal{F}' = \{p \in \mathcal{F} : \forall_{i \leqslant n_0}\, p_i < p\} \in \mathcal{D}_B$ (by **Lemma 1**):

$$\mathcal{F}' = \bigcup \{\{p \in \mathcal{F}' : \langle \{p, q\} \rangle \cap \mathcal{F} \neq \emptyset\} : q \in \langle \{p_i\}_{i \leqslant n} \rangle\}.$$

Next by **Lemma 2** there exists $s \in \langle \{p_i\}_{i \leqslant n_0} \rangle$ and $B' \leqslant B$ such that $\mathcal{F}'' = \{p \in \mathcal{F}' : \langle \{s, p\} \rangle \cap \mathcal{F} \neq \emptyset\} \in \mathcal{D}_{B'}$. After repeating this procedure in order to obtain consecutive vectors $s_0 < s_1 < \ldots$ and using denseness we can assume in fact that the above s is in $\mathcal{F}$. Finally, an application of **Lemma 3** to $\mathcal{F}'' \in \mathcal{D}_{B'}$ we get the desired objects (note that for $k = 1$ the above lemma is a crucial step in Baumgartner's proof [2]).

Theorem (Gowers' pigeon principle)
For every finite coloring of FIN_k there is an block sequence B such that $\langle B \rangle$ is monochromatic.

Proof:

Note that it suffices to prove the theorem for two colors. Therefore, assume that $\mathrm{FIN}_k = \mathcal{F} \cup \mathcal{F}'$ is a partition such that there is no infinite block sequence $B \subseteq \mathrm{FIN}_k$ such that $\langle B \rangle \subseteq \mathcal{F}'$. However it means that $\mathcal{F}$ is FIN_k-dense.

Now, starting from $\mathcal{F}_0 = \mathcal{F}$ and applying **Lemma 1** and **Lemma 4** recursively, we obtain sequences $(\mathcal{F}_i)_{i<\omega}$, $\mathcal{S} = \{s_n : n < \omega\} \subseteq \mathrm{FIN}_k$ with $s_i \in \mathcal{F}_i$ for $i < \omega$ and $\mathrm{FIN}_k = B_0 \geqslant B_1 \geqslant \ldots$ such that

$$\mathcal{F}_{i+1} = \{p \in C(\mathcal{F}_i, s_i) : \forall_{j \leqslant i}\, s_j < p\} \in \mathcal{D}_{B_{i+1}} \quad \text{for } i < \omega.$$

Then observe that $\mathcal{S}$ is a block sequence contained in $\mathcal{F}$. Moreover, the definition of sets of type $C(\mathcal{F}, s)$ and $\mathcal{F} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \ldots$ ensure that $\langle \mathcal{S} \rangle \subseteq \mathcal{F}$.

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