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A NOTE ON THE PAPER "STABILITY AND STABILIZABILITY OF INFINITE DIMENSIONAL SYSTEMS", BY A. J. PRITCHARD AND J. ZABCZYK, SIAM REVIEW, 23 (1981), 25–52

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Abstract. The paper provides details of the proof of a stability result for abstract hyperbolic systems presented in a survey paper by A. J. Pritchard and the author.

The following result, in a slightly weaker formulation, was presented in [2] *and proved there in a rather concise way. It is my intention to provide here its detailed proof. I am grateful to professor Kai Liu whose questions motivated me to write the note.*

Let A be a negative, in general, unbounded operator on a Hilbert space Z and α a positive number. Consider the second order system:

$$\ddot{z} + \alpha \dot{z} - Az = 0, \tag{1}$$

$$\dot{z}(0) \in Z, \qquad z(0) \in \mathcal{D}((-A)^{1/2}). \tag{2}$$

To have the semigroup formulation of (1) and (2) one defines the state space $\mathcal{H} = \mathcal{D}((-A)^{1/2} \times Z$ with the norm:

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \left(|x|^2 + |(-A)^{1/2} y|^2 \right)^{1/2}, \qquad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{H}$$

and the scalar product:

$$\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \right\rangle = \langle x, \overline{x} \rangle + \langle (-A)^{1/2} y, (-A)^{1/2} \overline{y} \rangle.$$

It is well known, see e.g. [2], that the following operator $\mathcal{A}$ defined on the domain $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times \mathcal{D}((-A)^{1/2})$ by the formula

$$\mathcal{A} = \begin{bmatrix} 0, & I \\ A, & -\alpha \end{bmatrix}$$

defines a C_0-semigroup $S(t)$, $t \geq 0$ on $\mathcal{H}$. The solution to (1)–(2) is given by the formula

$$\begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} = S(t) \begin{bmatrix} z(0) \\ \dot{z}(0) \end{bmatrix}, \qquad t \geq 0.$$

Let

$$\underline{\omega}(A) = \sup\{\lambda,\ \lambda \in \sigma(A)\} < 0$$

where $\sigma(A)$ is the spectrum of the operator A.

Theorem 1. 1) *The operator*

$$P = \begin{bmatrix} I - \frac{\alpha^2}{2}A^{-1}, & -\frac{\alpha}{2}A^{-1} \\ \frac{\alpha}{2}I, & I \end{bmatrix}$$

is the unique non-negative solution of the Liapunov equation

$$2\left\langle P\mathcal{A}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right\rangle = -\alpha \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2, \qquad \begin{matrix} x \in \mathcal{D}(A) \\ y \in \mathcal{D}(-A)^{1/2}. \end{matrix} \tag{3}$$

2) *The following estimate holds:*

$$\gamma_- \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2 \le \left\langle P\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right\rangle \le \gamma_+ \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2, \qquad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{H}, \tag{4}$$

where

$$\gamma_- = \frac{\sqrt{1+a}}{1+\sqrt{1+a}}, \qquad \gamma_+ = 1 + \frac{1+\sqrt{1+a}}{a}$$

and

$$a = \frac{4|\underline{\omega}(A)|}{\alpha^2}.$$

Moreover, γ_- and γ_+ are the best constants in (4).

3) *For arbitrary $t \ge 0$*

$$\|S(t)\| \le \sqrt{\frac{\gamma_-}{\gamma_+}}\, e^{-\frac{\alpha}{2\gamma_+}t}.$$

Proof of 1). In the following calculations we use elementary properties of sel-adjoint operators, see [1].

To check that P is symmetric take $\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} \in \mathcal{H}$. Then

$$\begin{aligned}
&\left\langle P\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}\right\rangle \\
&= \left\langle \begin{bmatrix} I - \frac{\alpha^2}{2}A^{-1}, & -\frac{\alpha}{2}A^{-1} \\ \frac{\alpha}{2}I, & I \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}\right\rangle \\
&= \left\langle \begin{bmatrix} \left(I - \frac{\alpha^2}{2}A^{-1}\right)x - \frac{\alpha}{2}A^{-1}y \\ \frac{\alpha}{2}x + y \end{bmatrix}, \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}\right\rangle \\
&= \left\langle (-A)^{1/2}\left[\left(I - \frac{\alpha^2}{2}A^{-1}\right)x - (-A)^{1/2}\frac{\alpha}{2}A^{-1}y\right], (-A)^{1/2}\overline{x}\right\rangle + \left\langle \frac{\alpha}{2}x + y, \overline{y}\right\rangle \\
&= \left\langle (-A)^{1/2}x + \frac{\alpha^2}{2}(-A)^{1/2}(-A)^{-1}x + (-A)^{1/2}\frac{\alpha}{2}(-A)^{-1}y, (-A)^{1/2}\overline{x}\right\rangle \\
&\quad + \left\langle \frac{\alpha}{2}x + y, \overline{y}\right\rangle \\
&= \left\langle (-A)^{1/2}x + \frac{\alpha^2}{2}(-A)^{-1/2}x + \frac{\alpha}{2}(-A)^{-1/2}y, (-A)^{1/2}\overline{x}\right\rangle \\
&\quad + \left\langle \frac{\alpha}{2}x + y, \overline{y}\right\rangle = \left\langle (-A)^{1/2}x, (-A)^{1/2}\overline{x}\right\rangle \\
&\quad + \frac{\alpha^2}{2}\langle x, \overline{x}\rangle + \frac{\alpha}{2}\langle y, \overline{x}\rangle + \frac{\alpha}{2}\langle x, \overline{y}\rangle + \langle y, \overline{y}\rangle.
\end{aligned}$$

This implies symmetricity of P.

The operator P is non-negative as it follows from the identities:

$$\begin{aligned}\left\langle P\begin{bmatrix}x\\y\end{bmatrix},\begin{bmatrix}x\\y\end{bmatrix}\right\rangle &= |(-A)^{1/2}x|^2+\tfrac{\alpha^2}{2}|x|^2+\alpha\langle x,y\rangle+|y|^2\\ &= |(-A)^{1/2}x|^2+\tfrac{1}{2}\left[|\alpha x|^2+2\langle\alpha x,y\rangle+|y|^2\right]+\tfrac{1}{2}|y|^2\\ &= |(-A)^{1/2}x|^2+\tfrac{1}{2}\left(|\alpha x+y|^2+|y|^2\right)\geq 0.\end{aligned}$$

To prove that P solves the Liapunov equation (3) note that for $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\in\mathcal{D}(A)$

$$\begin{aligned}&\left\langle P\mathcal{A}\begin{bmatrix}x\\y\end{bmatrix},\begin{bmatrix}x\\y\end{bmatrix}\right\rangle=\left\langle P\begin{bmatrix}y\\Az-\alpha y\end{bmatrix},\begin{bmatrix}x\\y\end{bmatrix}\right\rangle\\ &=\left\langle(-A)^{1/2}y,(-A)^{1/2}x\right\rangle+\tfrac{\alpha^2}{2}\langle y,x\rangle+\tfrac{\alpha}{2}\langle Ax-\alpha y,x\rangle+\tfrac{\alpha}{2}\langle y,y\rangle+\langle Ax-\alpha y,y\rangle\\ &=\left\langle(-A)^{1/2}y,(-A)^{1/2}x\right\rangle+\tfrac{\alpha^2}{2}\langle y,x\rangle+\tfrac{\alpha}{2}\langle Ax,x\rangle-\tfrac{\alpha^2}{2}\langle y,x\rangle\\ &\quad+\tfrac{\alpha}{2}\langle y,y\rangle+\langle Ax,y\rangle-\alpha\langle y,y\rangle\\ &=-\langle y,Ax\rangle+\tfrac{\alpha^2}{2}\langle y,x\rangle+\tfrac{\alpha}{2}\langle Ax,x\rangle-\tfrac{\alpha^2}{2}\langle y,x\rangle+\tfrac{\alpha}{2}\langle y,y\rangle+\langle Ax,y\rangle-\alpha\langle y,y\rangle\\ &=-\tfrac{\alpha}{2}\langle(-A)x,x\rangle-\tfrac{\alpha}{2}|y|^2=-\tfrac{\alpha}{2}\left(|(-A)^{1/2}x|^2+|y|^2\right)=-\tfrac{\alpha}{2}\left\|\begin{bmatrix}x\\y\end{bmatrix}\right\|^2.\end{aligned}$$

The proof that P is a non-negative solution to (3) is therefore complete.

The uniqueness will follow from the fact $S(t)$ is exponentially stable as then

$$P=\alpha\int_0^{+\infty}S^*(t)\,S(t)\,dt.$$

The exponential stability of $S(t)$ will follow from 3).

We pass to the proof of 2). We will find the maximal $\gamma\geq 0$ such that for all $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\in\mathcal{H}$:

$$\left|(-A)^{1/2}x\right|^2+\tfrac{\alpha^2}{2}|x|^2+\alpha\langle x,y\rangle+|y|^2\geq\gamma\left(|(-A)^{1/2}x|^2+|y|^2\right).\tag{5}$$

If $\gamma=1$, inequality (5) becomes:

$$\tfrac{\alpha^2}{2}|x|^2+\alpha\langle x,y\rangle\geq 0,$$

equivalent to

$$\left|\frac{\alpha}{\sqrt{2}}x+\frac{\sqrt{2}}{2}y\right|^2-\left|\frac{\sqrt{2}}{2}y\right|^2\geq 0,$$

which, obviously, does not hold for all $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\in\mathcal{H}$. Therefore $\gamma\in[0,1[$ and (5) becomes

$$(1-\gamma)\left|(-A)^{1/2}x\right|^2+\tfrac{\alpha^2}{2}|x|^2+\alpha\langle x,y\rangle+(1-\gamma)|y|^2\geq 0.\tag{6}$$

For fixed x the minimal value, with respect to y, of the expression

$$\alpha\langle x,y\rangle+(1-\gamma)|y|^2=(1-\gamma)\left|y+\frac{\alpha}{2(1-\delta)}x\right|^2-\frac{\alpha^2}{4(1-\gamma)}|x|^2$$

is

$$-\frac{\alpha^2}{4(1-\gamma)}|x|^2.$$

Thereofre the required $\gamma \in [0,1[$ should be such that for $x \in D(-A)^{1/2}$,

$$(1-\gamma)\left|(-A)^{1/2}x\right|^2 \geq \frac{\alpha^2}{2}\left(\frac{1}{2(1-\gamma)} - 1\right)|x|^2.$$

Equivalently, one is looking for the maximal $\gamma \in [0,1[$ such that

$$\inf_{x\neq 0}\frac{\left|(-A)^{1/2}x\right|^2}{|x|^2} = |\underline{\omega}(A)| \geq \frac{\alpha^2}{4}\left(\frac{1}{(1-\gamma)^2} - 2\frac{1}{1-\gamma}\right),$$

or,

$$a \geq \frac{1}{(1-\gamma)^2} - 2\frac{1}{1-\gamma}.$$

This easily gives:

$$\gamma_- = \frac{\sqrt{1+a}}{1+\sqrt{1+a}}.$$

In a similar way the expression for γ_+ can be obtained. This time one is looking for a minimal number $\gamma > 0$ such that for all $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathcal{H}$:

$$\left|(-A)^{1/2}x\right|^2 + \frac{\alpha^2}{2}|x|^2 + \alpha\langle x,y\rangle + |y|^2 \leq \gamma\left(|(-A)^{1/2}x|^2 + |y|^2\right). \tag{7}$$

It is clear that one should look for $\gamma > 1$. For fixed $x \in D(-A)^{1/2}$, the maximal value, with respect to y, of the expression

$$-(\gamma-1)|y|^2 + \alpha\langle x,y\rangle = -(\gamma-1)\left|y - \frac{\alpha}{2(\gamma-1)}x\right|^2 + \frac{\alpha^2}{4(\gamma-1)}|x|^2$$

is

$$\frac{\alpha^2}{4(\gamma-1)}|x|^2.$$

Therefore, the required $\gamma > 1$ should be such that

$$(\gamma-1)\left|(-A)^{1/2}x\right|^2 \geq \frac{\alpha^2}{2}\left(1 + \frac{1}{2(1-\gamma)}\right)|x|^2.$$

Equivalently, one is looking for minimal $\gamma > 1$ such that

$$\inf_{x\neq 0}\frac{\left|(-A)^{1/2}x\right|^2}{|x|^2} = |\underline{\omega}(A)| \geq \frac{\alpha^2}{4}\left(\frac{1}{(\gamma-1)^2} + 2\frac{1}{\gamma-1}\right),$$

or

$$a \geq \frac{1}{(\gamma-1)^2} + 2\frac{1}{1-\gamma}.$$

This easily gives

$$\gamma_+ = 1 + \frac{1+\sqrt{1+a}}{a}.$$

To prove the final part of the theorem denote by $z(t)$, $t \geq 0$, the strong solution of the problem

$$\frac{d}{dt}z(t) = \mathcal{A}z(t), \qquad z(0) = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{D}(\mathcal{A}).$$

Then, from the Liapunov equation,

$$\frac{d}{dt}\langle Pz(t), z(t)\rangle = -\alpha\|z(t)\|^2, \qquad t \geq 0.$$

On the other hand

$$\gamma_-\|z(t)\|^2 \le \langle Pz(t), z(t)\rangle \le \gamma_+\|z(t)\|^2,$$

and therefore

$$\frac{d}{dt}\langle Pz(t), z(t)\rangle = -\alpha\|z(t)\|^2 \le -\frac{\alpha}{\gamma_+}\langle Pz(t), z(t)\rangle.$$

Consequently

$$\langle Pz(t), z(t)\rangle \le e^{-\frac{\alpha}{\gamma_+}t}\langle Pz(0), z(0)\rangle \le e^{-\frac{\alpha}{\gamma_+}t}\gamma_+\|z(0)\|^2.$$

Finally,

$$\|z(t)\|^2 \le \frac{1}{\gamma_-}\langle Pz(t), z(t)\rangle \le \frac{\gamma_+}{\gamma_-}e^{-\frac{\alpha}{\gamma_+}t}\|z(0)\|^2$$

and

$$\|S(t)\| \le \sqrt{\frac{\gamma_+}{\gamma_-}}\, e^{-\frac{\alpha}{2\gamma_+}t}, \qquad t \ge 0.$$

□

References

[1] N. Dunford and J. T. Schwartz, *Linear operators, Part II*, Interscience Publishers, 1963.

[2] A. J. Pritchard and J. Zabczyk, *Stability and Stabilizability of Infinite Dimensional Systems*, SIAM Review, 23 (1981), 25–52.

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