In search of a structure of fractals by using membranes as hyperedges

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Abstract. The internal structure of the iterations of Koch curve and Sierpiński gasket—the known fractals [4]—is described in terms of multi-hypergraphical membrane systems related to membrane structures [13] and whose membranes are hyperedges of multi-hypergraphs used to define gluing patterns for the components of the iterations of the considered fractals.

1 Introduction

One finds in [10] a more or less explicit conclusion that the birth of functional analysis was accompanied by the emergence of various mathematical structures (from vector space, abstract metric spaces and topological spaces to Hilbert spaces, including spaces of functions) which were an antidotum against 'capricious' intuitiveness of symbolic 'calculations' of early calculus.

This conclusion inspired the author of the present paper to search for structures of fractals and self-similarity against their intuitive explanations[1] proposed e.g. in [9]:

'Local' statements of self-similarity say something like 'almost any small pattern observed in one part of the object can be observed throughout the object, at all scales'. Global statements say something like 'the whole object consists of several smaller copies of itself glued together'; more generally, there may be a whole family of objects, each of which can be described as several objects in the family glued together.

Viewed from another angle, a theory of global self-similarity is a theory of recursive decomposition.

One should point out here that in a large extent the concepts of fractals and self-similarity have been already described precisely in terms of iterated function systems with their attractors constructed by using the tools of functional analysis (Hahn–Banach fix point theorem) [4] and domain theory (Tarski fix point theorem) [3]. But a translation from the language of the above intuitive explanation to the language of some derived concepts from the precise description of

[1] the explanations suggested by the visual presentations of the iterations of some fractals seen in the books and many articles about fractals.

fractals and self-similarity (e.g. the trees induced by iterated function systems, cf. [3]) is not effortless and not yet ready.

Thus searching for structure of fractals and self-similarity is approached by various mathematicians, cf. [7], [9], not necessarily motivated explicitly by a need of the above translation.

The goal of the paper is to propose an approach to searching for structure of fractals which could provide the above translation. We describe in Section 3 the internal structure of the iterations of Koch curve and Sierpiński gasket—the known fractals [4]—in terms of multi-hypergraphical membrane systems related to membrane structures [13] and whose membranes are hyperedges of multi-hypergraphs used to define gluing patterns for the components of the iterations of the considered fractals.

2 Multi-hypergraphical membrane systems

We introduce the following new concepts.

By a *directed multi-hypergraph* we mean a structure $\mathcal{G}$ given by its *set* $E(\mathcal{G})$ *of hyperedges*, its *set* $V(\mathcal{G})$ *of vertices* and the *source* and *target* mappings

$$s_{\mathcal{G}} : E(\mathcal{G}) \to \mathcal{P}(V(\mathcal{G})), \quad t_{\mathcal{G}} : E(\mathcal{G}) \to \mathcal{P}(V(\mathcal{G}))$$

such that $V(\mathcal{G})$ together with

$$\big\{(\mathcal{V}_1, \mathcal{V}_2) \,|\, s_{\mathcal{G}}(e) = \mathcal{V}_1 \text{ and } t_{\mathcal{G}}(e) = \mathcal{V}_2 \text{ for some } e \in E(\mathcal{G})\big\}$$

form a directed hypergraph as in [5], where $\mathcal{P}(X)$ denotes the set of all subsets of a set X.

We say that two directed multi-hypergraphs $\mathcal{G}, \mathcal{G}'$ are *isomorphic* if there exist two bijections $h : V(\mathcal{G}) \to V(\mathcal{G}')$, $h' : E(\mathcal{G}) \to E(\mathcal{G}')$ such that

$$s_{\mathcal{G}'}(h'(e)) = \{h(v) \,|\, v \in s_{\mathcal{G}}(e)\} \text{ and } t_{\mathcal{G}'}(h'(e)) = \{h(v) \,|\, v \in t_{\mathcal{G}}(e)\}$$

for all $e \in E(\mathcal{G})$.

Membrane structures in [13] are simply finite trees with nodes labelled by multisets, where the finite trees have a natural visual presentation by Venn diagrams and the tree nodes are called *membranes*.

We introduce (*directed*) *multi-hypergraphical membrane systems* to be finite trees with nodes labelled by (directed) multi-hypergraphs.

We consider directed multi-hypergraphical membrane systems of a special feature described formally in the following way.

A *multi-hyperedge membrane system* $\mathcal{S}$ is given by:

- the *underlying tree* $\mathbb{T}_{\mathcal{S}}$ of $\mathcal{S}$ which is a finite graph given by its set $V(\mathbb{T}_{\mathcal{S}})$ of *vertices*, its set $E(\mathbb{T}_{\mathcal{S}}) \subseteq V(\mathbb{T}_{\mathcal{S}}) \times V(\mathbb{T}_{\mathcal{S}})$ of *edges*, and its *root* r which is a distinguished vertex such that for every vertex v different from r there exists a unique path from v into r in $\mathbb{T}_{\mathcal{S}}$, where for every vertex v we define $\mathrm{rel}(v) = \{v' \,|\, (v', v) \in E(\mathbb{T}_{\mathcal{S}})\}$ and in trivial case $V(\mathbb{T}_{\mathcal{S}}) = \{r\}$ we assume $E(\mathbb{T}_{\mathcal{S}}) = \varnothing$;

– a family $(G_v \,|\, v \in V(\mathbb{T}_{\mathcal{S}}))$ of finite directed multi-hypergraphs for G_v given by its set $V(G_v)$ of *vertices*, its set $E(G_v)$ of *edges*, its *source function* $s_v : E(G_v) \to \mathcal{P}(V(G_v))$, and its *target function* $t_v : E(G_v) \to \mathcal{P}(V(G_v))$ such that the following conditions hold:
  1) $E(G_v) = \mathrm{rel}(v)$,
  2) $V(G_v)$ is empty for every *elementary* vertex v, i.e. such that $\mathrm{rel}(v)$ is empty.

The above multi-hypergraphical membrane systems can be drawn by using Venn diagrams with discs or boxes d_v corresponding to vertices v of $\mathbb{T}_{\mathcal{S}}$.

One can expect the applications of multi-hypergraphical membrane systems for modelling various hierarchically organized systems of nested modules (hyperedges) interconnected by many input and output lines (vertices), where the module interactions are described by source and target functions. These systems of modules appear in computer science, where the modules are complex actions, instructions, transitions (e.g. of structured Petri nets [2]), etc., from state charts [6], models of systemC components [17], the systems discussed in [1], to the semantics of some extensions of formal systems in [12], [17], and hierarchical specifications [15].

3 Koch curve and Sierpiński gasket

We describe in this section the iterations of Koch curve and Sierpiński gasket [8], [4], [14] in terms of multi-hypergraphical membrane systems.

For natural numbers $n > 0$ and $i \in \{\mathrm{Koch}, \mathrm{Sierp}\}$ we define multi-hyperedge membrane systems $\mathcal{S}_n^i$ in the following way:

– the underlying tree $\mathbb{T}_n^i$ of $\mathcal{S}_n^i$ is such that
  - the set $V(\mathbb{T}_n^i)$ of vertices is the set of all strings (sequences) of length not greater than n of digits in $D^{\mathrm{Sierp}} = \{1, 2, 3\}$ for $i = \mathrm{Sierp}$, and in $D^{\mathrm{Koch}} = \{1, 2, 3, 4\}$ for $i = \mathrm{Koch}$,
  - the set $E(\mathbb{T}_n^i)$ of edges of $\mathbb{T}_n^i$ is such that $E(\mathbb{T}_n^i) = \{(\mathit{\Gamma} j, \mathit{\Gamma}) \,|\, \{\mathit{\Gamma} j, \mathit{\Gamma}\} \subset V(\mathbb{T}_n^i)$ and $j \in D^i\}$ with source and target functions being the projections on the first and the second component, respectively, where $\mathit{\Gamma} j$ is the string obtained by juxtaposition a new digit j on the right end of $\mathit{\Gamma}$,

– the family $\big(G_{\mathit{\Gamma}}^i \,|\, \mathit{\Gamma} \in V(\mathbb{T}_n^i)\big)$ of directed multi-hypergraphs of $\mathcal{S}_n^i$ is such that for every non-elementary vertex $\mathit{\Gamma} \in V(\mathbb{T}_n^i)$, i.e. with $\mathrm{rel}(\mathit{\Gamma}) \neq \varnothing$, $G_{\mathit{\Gamma}}^i$ is determined in the following way:
  - for $i = \mathrm{Koch}$ if $\mathit{\Gamma}$ is the empty string, then the directed multi-hypergraph $G_{\mathit{\Gamma}}^i$ is such that $V(G_{\mathit{\Gamma}}^i)$ is a five element set $\{v_0, \ldots, v_4\}$, $E(G_{\mathit{\Gamma}}^i) = \{\mathit{\Gamma} j \,|\, j \in D^i\}$, and the source and target functions of $G_{\mathit{\Gamma}}^i$ are given by

$$s_{G_{\mathit{\Gamma}}^i}(\mathit{\Gamma} j) = \{v_{j-1}\}, \quad t_{G_{\mathit{\Gamma}}^i}(\mathit{\Gamma} j) = \{v_j\} \text{ for all } j \in \{1, \ldots, 4\},$$

where

$$v_0 = (0,0),\ v_1 = (\tfrac{1}{3}, 0),\ v_2 = (\tfrac{1}{2}, \tfrac{2}{2\sqrt{3}}),\ v_3 = (\tfrac{2}{3}, 0),\ v_4 = (1,0),$$

- for $i = \text{Sierp}$ if Γ is the empty string, then the directed multi-hypergraph G^i_Γ is such that $V(G^i_\Gamma)$ is a six element set $\{v_0, \ldots, v_5\}$, $E(G^i_\Gamma) = \{\Gamma j \mid j \in D^i\}$, and the source and target functions of G^i_Γ are given by

$$s_{G^i_\Gamma}(\Gamma 3) = \{v_1, v_2\}, \qquad t_{G^i_\Gamma}(\Gamma 3) = \{v_0\},$$
$$s_{G^i_\Gamma}(\Gamma j) = \{v_{j+2}, v_{j+3}\}, \qquad t_{G^i_\Gamma}(\Gamma j) = \{v_j\} \text{ for } j \in \{1, 2\},$$

where

$$v_0 = (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}),$$
$$v_1 = (\tfrac{1}{4}, \tfrac{\sqrt{3}}{4}), \quad v_2 = (\tfrac{3}{4}, \tfrac{\sqrt{3}}{4}),$$
$$v_3 = (0, 0), \quad v_4 = (\tfrac{1}{2}, 0), \quad v_5 = (1, 0),$$

- if a non-elementary vertex Γ of $\mathbb{T}^i_n$ is of the form[2] $k\Omega$ for $k \in D^i$ and a string Ω of digits in D^i, then

$$V(G^i_\Gamma) = \left\{f^i_k(v) \mid v \in V(G^i_\Omega)\right\}, \quad E(G^i_\Gamma) = \{\Gamma j \mid j \in D^i\},$$

and

$$\delta_{G^i_\Gamma}(\Gamma j) = \left\{f^i_k(v) \mid v \in \delta_{G^i_\Omega}(\Omega j)\right\} \quad \text{for all } j \in D^i \text{ and } \delta \in \{s, t\}$$

where f^i_k is the k-th function of the iterated function system given in [14] for Koch curve in the case $i = \text{Koch}$ and for Sierpiński gasket in the case $i = \text{Sierp}$, respectively.

Lemma. *For all natural numbers $n > 0$ and $i \in \{\text{Koch}, \text{Sierp}\}$ the multi-hyperedge membrane system $\mathcal{S}^i_n$ is such that for every non-elementary vertex Γ of $\mathbb{T}^i_n$ the directed multi-hypergraph G^i_Γ is isomorphic to G^i_Λ for empty string Λ—the root of $\mathbb{T}^i_n$.*

Proof. We prove the lemma by induction on n and by using the property of the functions of the iterated function systems for Koch curve and Sierpiński gasket that they are injections.

For all natural numbers $n > 0$ and $i \in \{\text{Koch}, \text{Sierp}\}$ we define a *geometrical realization of* $\mathcal{S}^i_n$, denoted by $\text{space}(\mathcal{S}^i_n)$, to be a subset of $\mathbb{R}^2$ ($\mathbb{R}^2$ is a Cartesian product of two copies of the set $\mathbb{R}$ of real numbers) which is the n-th iteration of Koch curve for $i = \text{Koch}$ and the n-th iteration of Sierpiński gasket for $i = \text{Sierp}$, i.e.

$$\text{space}(\mathcal{S}^{\text{Koch}}_1) = \bigcup_{j \in D^{\text{Koch}}} f^{\text{Koch}}_j(\text{interval}),$$
$$\text{space}(\mathcal{S}^{\text{Sierp}}_1) = \bigcup_{j \in D^{\text{Sierp}}} f^{\text{Sierp}}_j(\text{equitriang}),$$
$$\text{space}(\mathcal{S}^i_{n+1}) = \bigcup_{j \in D^i} f^i_j(\text{space}(\mathcal{S}^i_n)) \text{ for } i \in \{\text{Koch}, \text{Sierp}\}$$

[2] the form $k\Omega$ of Γ is understood that the first element of Γ is k followed by the string Ω.

where $f^i_j(X)$ is the image of a set X for f^i_j, interval $= \{(t,0)\,|\,t \in \mathbb{R},\ 0 \le t \le 1\}$, and equitriang is the union of the interior and the frontier of the equilateral triangle in $\mathbb{R}^2$ whose vertices are $(0,0)$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(1,0)$.

Theorem. *For all natural numbers $n > 0$ and $i \in \{\text{Koch}, \text{Sierp}\}$ the set space $(\mathcal{S}^i_n)$ is not an amorphous set of points of $\mathbb{R}^2$ but it is a structured set by its hierarchically organized decomposition into subsets according to the underlying tree $\mathbb{T}^i_n$ of $\mathcal{S}^i_n$, where the components of the decomposition form a family $C^{i,n}_{\Gamma}$ $(\Gamma \in V(\mathbb{T}^i_n)$, Γ is non-empty and is not an elementary vertex of $\mathbb{T}^i_n)$ such that:*

- *if Γ is of the form $j\Omega$ for $j \in D^i$ and a string Ω of digits in D^i, then*
  - *for the empty string Ω the component $C^{i,n}_{j\Omega}$ is $f^i_j(\text{space}(\mathcal{S}^i_{n-1}))$,*
  - *for a non-empty string Ω the component $C^{i,n}_{j\Omega}$ is $f^i_j(C^{i,n-1}_{\Omega})$ for the Ω-th component $C^{i,n-1}_{\Omega}$ of $\text{space}(\mathcal{S}^i_{n-1})$,*
- *for $m_i = \max D^i$ the m_i components $C^{i,n}_{\Gamma 1}, \dots, C^{i,n}_{\Gamma m_i}$ are glued according to the pattern given by G^i_{Γ} understood that*

$$\delta(\Gamma j') \cap \gamma(\Gamma j'') = C^{i,n}_{\Gamma j'} \cap C^{i,n}_{\Gamma j''}$$

for all δ, γ, j', j'' with $\{\delta, \gamma\} \subseteq \{s^i_{G^i_{\Gamma}}, t^i_{G^i_{\Gamma}}\}$, $\{j', j''\} \subseteq D^i$, and $j' \neq j''$.

Proof. The theorem is an immediate consequence of the adopted definitions.

The above multi-hypergraphical membrane systems can be drawn by using Venn diagrams with discs or boxes d_{Γ} corresponding to vertices Γ of $\mathbb{T}^i_n$ such that d_{Γ_j} is an immediate subset of d_{Γ}.

Conclusion

The above lemma and theorem provide the translation claimed in the introduction of the paper for iterations of fractals in the cases of Koch curve and Sierpiński gasket. In this translation the main feature of self-similarity described in its 'local' statement corresponds to the isomorphisms of hypergraphs 'giving' the gluing patterns (see the above theorem) for every level of hierarchical organization of the decomposition, where the levels of hierarchical organization coincide with scale layers.

The iterations of jD-Cantor set $(j \in \{1,2,3\})$ require another approach which is proposed in [11], where multigraphical membrane systems are used with vertices as membranes. Thus one may say that the approach proposed in the present paper is a '*hyperedges as membranes*' approach.

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