Andrzej Pokrzywa – Seminar note on the paper

“Regularization in Hilbert Space under Unbounded Operators and General Source Conditions”

written by

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We consider operators acting in a Hilbert space H. We assume that the domains of these operators are dense in H. The domain, range and kernel of an operator A are denoted by $D(A), \mathrm{ran}A, \ker A$, respectively.

Definitions

Operator A is **closed** if for any sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in D(A)$, $\|x_n - x_0\| \to 0$ and $\|Ax_n - y\| \to 0$ for some $y \in H$ implies $x_0 \in D(A)$ and $Ax_0 = y$.

Operator A is **closable** if there exists a closed extension of A, i.e. there exists a closed operator B such that $D(A) \subset D(B)$ and $Ax = Bx$ for $x \in D(A)$.

An operator B is **adjoint** to A if $\langle Ax, y\rangle = \langle x, By\rangle$ for any $x \in D(A)$. The domain of B is the set of all those $y \in H$ that there exists $z \in H$ such that $\langle Ax, y\rangle = \langle x, z\rangle$. Then $z = By$. And we write $A^* = B$.

Polar decomposition

Because $A^*A \geq 0$ there exists the operator $B \geq 0$ such that $A^*A = B^2$.

Let us define $B^\dagger = 0_{|\ker B} \oplus \mathrm{cl}((B_{|D(B)\cap(\ker B)^\perp})^{-1})$, cl denotes here the closure of an operator.

We have the identity

$$\langle Ax, Ay\rangle = \langle Bx, By\rangle \quad \text{for } x, y \in D(A).$$

Note that

$$\overline{\mathrm{ran}B} = (\ker B)^\perp = (\ker A^*A)^\perp = (\ker A)^\perp = \overline{\mathrm{ran}A^*}.$$

Hence for $u, v \in \mathrm{ran}B$ we have the identity

$$\langle AB^\dagger u, AB^\dagger v\rangle = \langle u, v\rangle.$$

which shows that $AB^\dagger$ is an isometry on $\mathrm{ran}B$. The closure U of $AB^\dagger$ is partial isometry – it isometrically transforms $\overline{\mathrm{ran}B} = \overline{\mathrm{ran}A^*}$ on $\mathrm{ran}U = \overline{\mathrm{ran}AB^\dagger} = \overline{\mathrm{ran}A}$ and vanishes on its kernel – $\ker U = \ker B^\dagger = \ker B = \ker A$.

We have

$$UBx = AB^\dagger Bx = Ax \quad \text{for } x \in D(A).$$

$A = UB$ is called *right polar decomposition* of A.

U^* is also a partial isometry – it isometrically transforms $\overline{\mathrm{ran}A^*}$ on $\overline{\mathrm{ran}A}$ and $\ker U^* = (\mathrm{ran}U)^\perp = (\mathrm{ran}A)^\perp = \ker A^*$.

Moreover the equality

$$\langle U^*Uu, v\rangle = \langle Uu, Uv\rangle = \langle u, v\rangle \quad \text{valid for } u, v \in \overline{\mathrm{ran}B} = \overline{\mathrm{ran}A^*}$$

shows that

$$U^*Uu = \begin{cases} u & \text{for } u \in \overline{\mathrm{ran} A^*}, \\ 0 & \text{for } u \in \ker A. \end{cases}$$

Therefore $A = UBU^*U = CU$, where $C = UBU^*$ is also a selfadjoint nonnegative operator. This is the *left polar decomposition* of A. Note that U is the same in both polar decompositions. The above in particullary implies that $AA^* = UB^2U^* = U(A^*A)U^*$.

Examples.

Let $H = L^2(0,1)$ and A be the differentiation operator $Ax(t) = x'(t)$. $D(A) = \{x \in H \text{ such that } x' \in H\}$. A is closed. What is A^*? The equality

$$\langle Ax, y\rangle = \int x'\bar{y} = -\int x\bar{y}' + (x\bar{y})|_0^1$$

shows that if supp $y \subset (0,1)$ then $A^*y = -y'$, and that for $y \in D(A^*)$ we should have additionally $y(0) = y(1)$.

Thus

$$A^*y = -y', D(A^*) = \{x \in H; x' \in H, x(0) = x(1) = 0\}.$$

We can easily see, that $AA^*y = -y''$ and $A^*Ay = -y''$, However domains of these operators differ.

$$D(AA^*) = \{x \in H; x'' \in H \text{ and } x(0) = x(1) = 0\},$$

$$D(A^*A) = \{x \in H; x'' \in H \text{ and } x'(0) = x'(1) = 0\}.$$

The operator AA^* has eigenfunctions $\sin k\pi x$, with eigenvalues $k^2\pi^2$, $k = 1, 2, \ldots$, The operator A^*A has eigenfunctions $\cos k\pi x$, with eigenvalues $k^2\pi^2$, $k = 0, 1, 2, \ldots$, all these eigenfunctions have norm $\frac{1}{\sqrt{2}}$ except $\cos 0\pi x$, which has norm 1. Thus setting $s_k = s_k(x) = \sqrt{2}\sin k\pi x$, $c_k = c_k(x) = \sqrt{2}\cos k\pi x$, $k = 1, 2, \ldots$ and $c_0 = 1$ we have expansions:

$$AA^* = \sum_{k=1}^{\infty} k^2\pi^2\langle\cdot, s_k\rangle s_k, \quad A^*A = \sum_{k=1}^{\infty} k^2\pi^2\langle\cdot, c_k\rangle c_k$$

Defining

$$B = \sum_{k=1}^{\infty} k\pi\langle\cdot, s_k\rangle s_k, \quad C = \sum_{k=1}^{\infty} k\pi\langle\cdot, c_k\rangle c_k,$$

$$U = \sum_{k=1}^{\infty}\langle\cdot, s_k\rangle c_k, \quad V = -\sum_{k=1}^{\infty}\langle\cdot, c_k\rangle s_k,$$

we have $AA^* = B^2$, $A^*A = C^2$ with nonnegative selfadjoint operators B, C.

We have also $Ac_k = -k\pi s_k = BVc_k = VCc_k$, and $A^*s_k = -k\pi c_k = -CUs_k = -UBs_k$

Now it is easy to see that we have polar decompositions:

$$A = BV = VC, \quad A^* = -CU = -UB.$$

U is an isometry, but its range is not all H, V is partial isometry with kernel spanned by c_0. Here $U^* = -V$.

The operator A has a lot of eigenfunctions, namely any function $e^{\lambda x}$ with complex number λ is its eigenfunction. From this set of eigenfunctions we may get a subset, which forms an orthonormal basis of H, for example $e_k = e_k(x) = e^{2\pi kxi}$, $k = 0, \pm 1, \pm 2, \ldots$.

Hence one may write

$$A = 2\pi i \sum_{-\infty}^{\infty} k\langle \cdot, e_k\rangle e_k,$$

and such equality implies that A is a normal operator, with polar decompositions $A = U_0B_0 = B_0U_0$, where

$$B_0 = 2\pi \sum_{-\infty}^{\infty} |k|\langle \cdot, e_k\rangle e_k, \quad U_0 = i\sum_{-\infty}^{\infty} \mathrm{sign} k\langle \cdot, e_k\rangle e_k.$$

Of course it contradicts our previus considerations. What is wrong? We have choosen a basis which consists from periodic functons, as a derivative of a periodic function is again periodic we have silently restricted the domain of A to periodic functions only. Thus the operator A with the above expansion this time has a domain $\{x \in H; x' \in H, x(0) = x(1)\}$. A similar efect happens, when one tries to use sine basis (s_k) for approximation operator A. This shows that one has to be carefull while approximating unbounded operators.

Stone–von Neumann operator calculus

Operator calculus enables us to define $f(A)$, where f is a complex valued functions defined on a subset of the complex plane, and A an operator. If f is a polynomial $f(A)$ expands in powers of A in the same way as the polynomial.

In the case when A is a diagonalizable operator, i.e. $A = \sum \lambda_j\langle \cdot, e_k\rangle e_k$ we set $f(A) = \sum f(\lambda_j)\langle \cdot, e_k\rangle e_k$ and this definiton is consistent with the definition for polynomials.

Defining functions of selfadjoint operators is nearly the same task as that for diagonal ones. Let μ be a nonnegative Borel mesure defined on Borel subsets of real line. Let $H = L^2(\mu)$, and A be an operator defined by $Ax(t) = tx(t)$. This operator is selfadjoint, its spectrum coincides with the support of μ, and for any Borel function f the operator $f(A)$ is defined by $f(A)x(t) = f(t)x(t)$.

Operators of this kind are blocks from which any selfadjoint operator is composed. Namely, if A is a selfadjoint operator acting in a Hilbert space H then there exists a family $\{\mu_\alpha\}_\alpha$ of nonnegative Borel measures and a unitary operator $U : H \to \oplus_\alpha L^2(\mu_\alpha)$ such that $A = U^* \oplus_\alpha A_\alpha U$, where $A_\alpha x(t) = tx(t)$.

It is well known that spectral measure is a useful tool for studying selfadjoint operators. For each Borel subset $\Omega \subset R$ is an orthogonal projection acting in H. Spectral measure has properties similar to measure, $E(\Omega)$ is an orthogonal projection in H, $E(\Omega_1)E(\Omega_2) =$

$E(\Omega_1 \cap \Omega_2)$, $E(\emptyset) = 0$, $E(R) = I$. Moreover $E(\Omega)A = AE(\Omega)$, and the operator A and any Borel measureable function f of this opeator may be expressed as

$$A = \int t dE, \quad f(A) = \int f(t) dE.$$

if ess sup $|f| = \sup_{t \geq 0}\{t : E(\{x : |f(x)| \geq t\}) \neq 0\}$ is bounded, then this quantity equals $\|f(A)\|$, if it is unbounded the operator $f(A)$ is unbounded.

With the representation of A as the direct sum of operatores we can write

$$E(\Omega)x = U^* \oplus_\alpha \chi(\Omega) x_\alpha U,$$

where $\chi(\Omega)$ is the charecteristic function of the set Ω.

Note also that $\sigma(A) = \text{supp } E = \overline{\bigcup_\alpha \text{supp } \mu_\alpha}$

While investigating the proofs of the result of the paper, we can see that the most essentiall parts are those which refer to properties of a selfadjoint operator. We shall give the proofs in the case when $H = L^2(\mu)$, and A is defined by $Ax(t) = tx(t)$. We shall refer to this case as the *model case*. For some results we will present also a proof in the general case. All the others may be modified similarily.

We shall use result numbering as in the original paper, changing sometimes notations.

Index function definition A positive function $\psi : (0, \infty) \to (0, \infty)$ is an *index function* if it is increasing (non-decreasing) and continuous with $\lim_{t \to 0} \psi(t) = 0$.

Theorem 1. Let A be a nonnegative selfadjoint operator acting in H with $\ker A = \{0\}$. then

(a) For every $x \in H$ and $\varepsilon > 0$ there exists a bounded index function ψ such that the general source condition

$$x = \psi(A)w \text{ with } w \in H \text{ and } \|w\| \leq (1 + \varepsilon)\|x\|$$

is satisfied, and hence $x \in \text{ran}\psi(A)$.

(b) If $x \in \text{ran}\psi(A)$ for some unbounded index function ψ, then $x \in \text{ran}\psi_0(A)$ for every bounded index function ψ_0 which coincides with ψ on $(0, t_0]$ for some $t_0 > 0$.

Comment 1. Why there is $\|w\| \leq (1 + \varepsilon)\|x\|$ above? – substituting ψ with $\psi_\varepsilon = \frac{1+\varepsilon}{\varepsilon}\psi_\varepsilon$ and w by $w_\varepsilon = \frac{\varepsilon}{1+\varepsilon}w$ we get estimation $\|w\| \leq \varepsilon\|x\|$, while theorem suggests that the constant 1 and any smaller one cannot be achived. However I suppose thay have some good reason for it, for example the family of index function $\psi_\gamma(t) = \gamma\psi(t)$ $\gamma > 0$ should be represented by this one, that $\psi_\gamma(\bar{\alpha}) = 1$.

Proof of Th. 1 part (a) – model case version We assume $H = L^2(\mu)$, $Ax(t) = tx(t)$ and $\|x\| = 1$. We have $\|x\|^2 = \int_0^\infty |x(t)|^2 d\mu = 1$, therefore for any $\alpha \in (0, 1)$ there exists decreasing and converging to 0 sequence of numbers $\{\tau_n\}_{n=0}^\infty$ such that

$$\int_{(0,\tau_n)} |x(t)|^2 d\mu \leq \varepsilon\alpha^n, \quad \text{for } n = 0, 1, \ldots.$$

Define with $\beta > 1$ and such that $\alpha\beta^2 < 1$

$$\psi_0(t) = \begin{cases} 1 \text{ for } t \geq \tau_0 \\ \beta^{-n} \text{ for } t \in [\tau_n, \tau_{n-1}), \quad n = 1, 2, \dots \end{cases}. \tag{1}$$

Then

$$\int_{[\tau_n,\tau_{n-1})} |\psi_0^{-1}(t)x(t)|^2 d\mu \leq \varepsilon\beta^{2n}\alpha^{n-1}$$

and

$$\begin{aligned} \|\psi_0^{-1}(A)x\|^2 &= \int_{(0,\infty)} |\psi_0^{-1}(t)x(t)|^2 d\mu \\ &= \int_{[\tau_0,\infty)} |x(t)|^2 d\mu + \sum_{n-1}^{\infty} \int_{[\tau_n,\tau_{n-1})} |\psi_0^{-1}(t)x(t)|^2 d\mu \\ &\leq 1 + \frac{\varepsilon}{\alpha} \sum_{n=1}^{\infty} (\alpha\beta^2)^n = 1 + \varepsilon \frac{\beta^2}{1-\alpha\beta^2}. \end{aligned}$$

Thus with $\alpha = \frac{1}{4}$, $\beta^2 = \frac{4}{3}$ (then $\frac{\beta^2}{1-\alpha\beta^2} = 2$) we have $\|\psi_0^{-1}(A)x\| \leq \sqrt{1+2\varepsilon} < 1 + \varepsilon$ and therefore $w = \psi_0^{-1}(A)x$ satisfies the thesis (part (a)).

If we require ψ to be a continuous function we may define it as a continuous piecewise linear function, linear in intervals $[\tau_n, \tau_{n-1}]$ and such that $\psi(\tau_n) = \psi_0(\tau_n)$. Then $\psi_0(t) \geq \psi(t)$ and $\|\psi^{-1}(A)x\|^2 = \int |\psi^{-1}(t)x(t)|^2 \leq \int |\psi_0^{-1}(t)x(t)|^2 = \|\psi_0^{-1}(A)x\|^2$ and the thesis is satisfied for ψ.

Proof of Th. 1 part (a) – general version Let E be spectral measure for operator A, $\varepsilon > 0$ and $\alpha = \frac{1}{4}$. We can find decreasing and converging to 0 sequence of numbers $\{\tau_n\}_{n=0}^{\infty}$ such that $\|E((0,\tau_n))x\|^2 < \varepsilon\alpha^n$. With ψ_0 defined by (1) and $\beta^2 = \frac{4}{3}$ we have

$$\|\psi_0(\tau_n)^{-1}E([\tau_n,\tau_{n-1})x\|^2 \leq \varepsilon\alpha^{n-1}\beta^{2n}$$

Because

$$\sum_{n=1}^{\infty} \psi_0^{-1}(\tau_n)^{-1}E([\tau_n,\tau_{n-1}) + E((\tau_0,\infty)) = \psi_0^{-1}(A)$$

we have

$$\|\psi_0^{-1}(A)x\|^2 = \sum_{n=1}^{\infty} \|\psi_0^{-1}(\tau_n)^{-1}E([\tau_n,\tau_{n-1})x\|^2 + \|E((\tau_0,\infty))x\|^2 < 1 + 2\varepsilon < (1+\varepsilon)^2$$

Thus $w = \psi_0^{-1}(A)x$ satisfies part (a) of the thesis.

Proof of part (b). Assume $H = L^2(\mu)$ and action of A on a function is its multiplication by the argument. Then

$$\begin{aligned} \|\psi_0 f\|^2 &= \left(\int_{(0,t_0)} + \int_{[t_0,\infty)} \right) |\psi_0(t)f(t)|^2 \\ &\leq \int_{(0,t_0)} |\psi(t)f(t)|^2 + \sup_t \psi_0^2(t) \int_{[t_0,\infty)} |f(t)|^2 \leq \|\psi f\|^2 + \sup_t \psi_0^2(t)\|f\|^2. \end{aligned}$$

In the general case, with each nonzero $x \in H$ we may associate Borel measure on the line by $\mu(\Omega) = \|E(\Omega)x\|^2$. For any Borel measureable funciton ψ we then have

$$\|\psi(A)x\|^2 = \| \int \psi(t)xdE\|^2 = \int \|\psi(t)xdE\|^2 = \int |\psi(t)|^2\|xdE\|^2 = \int |\psi(t)|^2 d\mu.$$

The proof is analogous to the proof for $H = L^2(\mu)$ with $f = f(t) = 1$.

Regularization definition. Family $\{g_\alpha\}_{0<\alpha<\bar{\alpha}}$ of bounded Borel functions $g_\alpha : R^+ \to R^+$ is regularization if they are piecewise continuous in α and

a) $r_\alpha(t) = 1 - tg_\alpha(t) \to 0$ as $\alpha \to 0$,
b) $|r_\alpha(t)| = |1 - tg_\alpha(t)| < \gamma_1$, for all $\alpha \in (0, \bar{\alpha}]$, $t > 0$,
c) $\sqrt{t}|g_\alpha(t)| < \frac{\gamma_*}{\sqrt{\alpha}}$ for all $t > 0$.

Approximate solution of equation $Ax = y$ is defined as

$$x_\alpha^\delta = A^* g_\alpha(AA^*)y^\delta$$

where $\|y^\delta - y\| \leq \delta$.

Source condition for the solution $x^\dagger$ of the equation $Ax = y$ is $x^\dagger = \psi(A^*A)w$.

Definition

$$W = \{g : \|g\|_W = \sup_{s \in R^+} \sqrt{s}|g(s)| < \infty\}$$

Proposition 1. If $g \in W$ then $A^*g(AA^*) = g(A^*A)A^*$ and $\|A^*g(AA^*)\| \leq \|g\|_W$.

In the space $H = L^2(\mu)$ proposition takes the form

$$\sup_{t>0} |tg(t^2)| \leq \|g\|_W$$

and becomes trivial.

Corollary 2. With $r_\alpha(t) = 1 - tg_\alpha(t)$

$$\|r_\alpha(A^*A)\| \leq \gamma_1, \quad \|A^*g_\alpha(AA^*)\| \leq \frac{\gamma_*}{\sqrt{\alpha}}.$$

Proof. In our model case the thesis reads as

$$\sup_{t>0} |1 - t^2 g_\alpha(t^2)| \leq \gamma_1, \quad \sup_{t>0} |tg_\alpha(t^2)| \leq \frac{\gamma_*}{\sqrt{\alpha}},$$

so there is nothing to prove, as this is equivalent with definitions.

In the general case let $A = BU$ be polar decomposition of A and E be spectral measure for $B \geq 0$, then $A^*A = U^*B^2U$, $AA^* = B^2$ and

$$r_\alpha(A^*A) = U^* \int r_\alpha(t^2)dEU, \quad A^*g_\alpha(AA^*) = U^* \int tg(t^2)dE.$$

Thesis follows form the fact that $\|r_\alpha(A^*A)\| = \text{ess sup}\ _{t\in\sigma(A^*A)}|r_\alpha(t)| \le \gamma_1$ and similarily for $\|A^* g_\alpha(AA^*)\|$.

Lemma 6. With $x_\alpha = A^* g_\alpha(AA^*)Ax^\dagger = A^* g_\alpha(AA^*)y$ we have

$$\|x_\alpha - x_\alpha^\delta\| \le \gamma_* \frac{\delta}{\sqrt{\alpha}}.$$

Proof. In the model case $x_\alpha(t) = tg_\alpha(t^2)y(t)$. Therefore

$$|x_\alpha(t) - x_\alpha^\delta(t)| = |tg_\alpha(t^2)(y(t) - y^\delta(t))| \le \sup_{t>0} |tg_\alpha(t^2)||y(t) - y^\delta(t))| \le \frac{\gamma_*}{\sqrt{\alpha}}|y(t) - y^\delta(t)|$$

and this implies the thesis.

In the general case let $BU = A$ be polar decomposition of A, and E be spectral measure for B. Then $x_\alpha = U^* B g_\alpha(B^2)y$, $x_\alpha^\delta = U^* B g_\alpha(B^2)y^\delta$ and

$$x_\alpha - x_\alpha^\delta = U^* B g_\alpha(B^2)(y - y^\delta)$$

and because

$$Bg_\alpha(B^2) = \int tg_\alpha(t^2)dE$$

where E is the spectral measure of B, we have

$$\|Bg_\alpha(B^2)\| \le \sup_{t>0} t|g_\alpha(t^2)| \le \frac{\gamma_*}{\sqrt{\alpha}}$$

and finally

$$\|x_\alpha - x_\alpha^\delta\| = \|U^* B g_\alpha(B^2)y\| \le \|U^*\|\|\|Bg_\alpha(B^2)\|\|\|y - y^\delta\| \le \frac{\gamma_*}{\sqrt{\alpha}}\|y - y^\delta\|.$$

Lemma 7. If the solution $x^\dagger$ satisfies source condition $x^\dagger = \psi(A^*A)w$ then

$$\|x^\dagger - x_\alpha\| \le \|w\| \sup_{s\in\sigma(A^*A)} |r_\alpha(s)|\psi(s).$$

Proof. In the model case

$$x^\dagger(t) - x_\alpha(t) = \psi(t^2)w(t) - t^2 g_\alpha(t^2)\psi(t^2)w(t) = \psi(t^2)(1 - t^2 g_\alpha(t^2))w(t).$$

Hence

$$\|x^\dagger - x_\alpha\| \le \sup_{t\in \text{supp}\ \mu} |\psi(t^2)r_\alpha(t^2)| \cdot \|w\|.$$

Now the thesis follows form the fact that $\sigma(A^2) = \{t^2 : t \in \sigma(A)\}$ and in the model case $\sigma(A) = \text{supp}\ \mu$.

In the general case with the notation used in proof of Lemma 6 we have $A^*A = U^*BBU$ and therefore

$$x^\dagger = \psi(A^*A)w = U^*\psi(B^2)Uw,$$

,

$$\begin{aligned} x_\alpha =& U^*Bg_\alpha(B^2)y = U^*Bg_\alpha(B^2)Ax^\dagger \\ =& U^*Bg_\alpha(B^2)BUU^*\psi(B^2)Uw = U^*B^2g_\alpha(B^2)\psi(B^2)Uw \end{aligned}$$

Thus

$$x^\dagger - x_\alpha = U^*(I - B^2g_\alpha(B^2))\psi(B^2)Uw. \tag{2}$$

Note that

$$(I - B^2g_\alpha(B^2))\psi(B^2) = \int (1 - t^2g_\alpha(t^2))\psi(t^2)dE$$

$$\|(I - B^2g_\alpha(B^2))\psi(B^2)\| \le \sup_{t\in\sigma(B)} |1 - t^2g_\alpha(t^2)|\psi(t^2) = \sup_{t\in\sigma(A^*A)} |r(t)|\psi(t), \tag{3}$$

because $\sigma(A^*A)$ may differ from $\sigma(AA^*) = \sigma(B^2)$ only by 0, and by the spectral mapping theorem $\sigma(B^2) = \{t^2; t \in \sigma(B)\}$.

Because $\|U\| = \|U^*\| = 1$ (2) and (3) imply the thesis.

From Lemmata 6 and 7 we get final estimation

$$\|x^\dagger - x_\alpha^\delta\| \le \|x^\dagger - x_\alpha\| + \|x_\alpha - x_\alpha^\delta\| \le \|w\| \sup_s |r_\alpha(s)|\psi(s) + \gamma_* \frac{\delta}{\sqrt{\alpha}}. \tag{4}$$

Bias convergence. Using the notation of Lemma 6 we have

$$\begin{aligned} x^\dagger - x_\alpha =& x^\dagger - A^*g_\alpha(AA^*)Ax^\dagger = U^*Ux^\dagger - U^*Bg_\alpha(B^2)BUx^\dagger \\ =& U^*(I - B^2g_\alpha(B^2))Ux^\dagger = U^*r_\alpha(B^2)Ux^\dagger \end{aligned}$$

and

$$\|x^\dagger - x_\alpha\|^2 = \int r_\alpha^2(t^2)\|dEUx^\dagger\|^2.$$

($\|dEUx^\dagger\|^2 = d\mu$ where the measure μ is defined by $\mu(\Omega) = \|E(\Omega)Ux^\dagger\|^2$.) The convergence $\|x^\dagger - x_\alpha\| \to 0$ follows form definition of regularization (parts a) and b)) and the Lebesgue's dominated convergence theorem.

Qualification

Definition 2. Qualification definition An index function φ is a qualification of the regularisation g_α if there are constants $\gamma = \gamma_\varphi < \infty$, $\bar{\alpha}_\varphi$ such that

$$\sup_{s\in\sigma(A^*A)} |r_\alpha(s)|\varphi(s) \le \gamma\varphi(\alpha), \quad 0 < \alpha \le \bar{\alpha}_\varphi. \tag{5}$$

Usually we do not know $\sigma(A^*A)$ which appears in (5), all we know is that $\sigma(A^*A) \subset [0, \infty)$, moreover qualification function is not defined in 0, therefore (5) should be read as

$$\sup_{s>0} |r_\alpha(s)|\varphi(s) \le \gamma\varphi(\alpha), \quad 0 < \alpha \le \bar{\alpha}. \tag{5'}$$

Proposition 2. (reformulated) Let g_α be a regularization with some known qualification φ. If ψ is an index function such that
there exists $s_0 > 0$ such that the function $s \to \psi(s)/\varphi(s)$, $0 < s \le s_0$ is non-increasing,

$$\psi(s) \le C\varphi(s) \quad \text{for } s > s_0 \tag{GC}$$

then ψ is a qualification of g_α.

Proof. If $s \le \alpha$ then $\psi(s) \le \psi(\alpha)$ and by Regularization definition (b)

$$|r_\alpha(s)|\psi(s) \le \gamma_1 \psi(\alpha) \quad \text{for } s \le \alpha. \tag{$a1$}$$

We have

$$|r_\alpha(s)|\psi(s) = |r_\alpha(s)|\varphi(s)\frac{\psi(s)}{\varphi(s)} \le \gamma_\varphi \varphi(\alpha)\frac{\psi(s)}{\varphi(s)} \tag{$a2$}$$

If $\alpha \le s \le s_0$ then

$$\frac{\psi(s)}{\varphi(s)} \le \frac{\psi(\alpha)}{\varphi(\alpha)}.$$

This and (a2) show that

$$|r_\alpha(s)|\psi(s) \le \gamma_\varphi \varphi(\alpha)\frac{\psi(\alpha)}{\varphi(\alpha)} = \gamma_\varphi \psi(\alpha) \quad \text{if } \alpha \le s \le s_0 \tag{$a3$}$$

We write (a2) in the form

$$|r_\alpha(s)|\psi(s) \le \gamma_\varphi \psi(\alpha)\frac{\varphi(\alpha)}{\psi(\alpha)}\frac{\psi(s)}{\varphi(s)}.$$

If $\alpha \le s_0$ then

$$\frac{\psi(s_0)}{\varphi(s_0)} \le \frac{\psi(\alpha)}{\varphi(\alpha)} \quad \text{or equivalently } \frac{\varphi(\alpha)}{\psi(\alpha)} \le \frac{\varphi(s_0)}{\psi(s_0)}$$

Therefore

$$|r_\alpha(s)|\psi(s) \le \gamma_\varphi \psi(\alpha)\frac{\varphi(s_0)}{\psi(s_0)}\frac{\psi(s)}{\varphi(s)} \quad \text{if } \alpha \le s_0$$

If $s \ge s_0$ then $\psi(s) \le C\varphi(s)$ and

$$|r_\alpha(s)|\psi(s) \le C\gamma_\varphi \frac{\varphi(s_0)}{\psi(s_0)}\psi(\alpha) \quad \text{if } \alpha \le s_0 \text{ and } s \ge s_0. \tag{$a4$}$$

The inequalities (a1), (a2) and (a4) show that ψ is a qualification for g_α with constants $\bar{\alpha}_\psi = \min\{\bar{\alpha}, s_0\}$ $\gamma_\psi = \min\{C\gamma_\varphi \frac{\varphi(s_0)}{\psi(s_0)}, \gamma_\varphi, \gamma_1\}$.

Remark. It is easy to show, that if condition (GC) holds for some s_0 then it holds for any $s_0 > 0$, the constant C may change only. However s_0 appears also in the assumption on the monotonicity of $\frac{\psi(s)}{\varphi(s)}$. Thus we cannot ignore the constant $\bar{\alpha}_\psi$ as the authors of the paper have done.

Proposition 3. (reformulated) Let g_α be a regularization with some known qualification φ. If ψ is an index function such that
there exists $s_0 > 0$ such that the function $s \to \psi(s)/\varphi(s)$, $0 < s \leq s_0$ is non-decreasing, and (GC) holds. then

$$|r_\alpha(s)|\psi(s) \leq C\varphi(\alpha) \quad \text{for } \alpha \in (0, \bar{\alpha}), s > 0.$$

Proof.(omited by th authors) We have

$$\frac{\psi(s)}{\varphi(s)} \leq \frac{\psi(s_0)}{\varphi(s_0)} \quad \text{for } s \leq s_0,$$

hence from (a2) it follows that

$$|r_\alpha(s)|\psi(s) \leq \gamma_\varphi \varphi(\alpha)\frac{\psi(s)}{\varphi(s)} \leq \gamma_\varphi \varphi(\alpha)\frac{\psi(s_0)}{\varphi(s_0)} \quad \text{for } s \leq s_0. \tag{$a5$}$$

On the other hand

$$\psi(s) \leq C\varphi(s) \quad \text{for } s > s_0,$$

then again form (a2) we have

$$|r_\alpha(s)|\psi(s) \leq \gamma_\varphi \varphi(\alpha)\frac{\psi(s)}{\varphi(s)} \leq C\gamma_\varphi \varphi(\alpha) \quad \text{for } s > s_0. \tag{$a6$}$$

The thesis follows form (a5) and (a6) with C replaced by $\gamma_\varphi \max\{C, \frac{\psi(s_0)}{\varphi(s_0)}\}$.

Comment 2. In the original formulation of the above proposition authors claim that $1 \leq C \leq \infty$. There is no reason for it – if ψ satisfies the thesis with C then $\frac{1}{2C}\varphi$ satisfies the assumptions and the thesie with $C = \frac{1}{2}$. See Comment 1.

Lemma 7 and Propositions 2 and 3 lead to bias estimation.

Proposition 4. Let g_α be a regularization with qualification φ and $x^\dagger = \psi(A^*A)w$ a source condition with index function ψ, which satisfies (GC).

a) If the function $\frac{\psi(s)}{\varphi(s)}$ is non-increasing in $(0, s_0]$ then for some $C, \bar{\alpha} > 0$

$$\|x^\dagger - x_\alpha\| \leq C\psi(\alpha)\|w\|, \quad \alpha \in (0, \bar{\alpha}]$$

b) If the function $\frac{\psi(s)}{\varphi(s)}$ is non-decreasing in $(0, s_0]$ then for some $C, \bar{\alpha} > 0$

$$\|x^\dagger - x_\alpha\| \leq C\varphi(\alpha)\|w\|, \quad \alpha \in (0, \bar{\alpha}]$$

Remark Note that if we set

$$\begin{array}{ll} \varphi_0(s)=\varphi(s), \quad \psi_0(s)=\psi(s) & \text{for } s\in(0,s_0] \\ \varphi_0(s)=\varphi(s_0), \quad \psi_0(s)=\psi(s_0) & \text{for } s\in(s_0,\infty) \end{array}$$

then φ_0 is also the qualification for g_α (by Proposition 2) and the functions φ_0, ψ_0 satisfy the same assumptions as the functions φ, ψ in Propositions 2-4, therefore also the same claims for these functions hold.

Convergence rates.

With assumptions of Proposition 4 and Lemma 6 we get error estimates if the form

$$\|x^\dagger - x_\alpha^\delta\| \le C\psi(\alpha)\|w\| + \gamma_* \frac{\delta}{\sqrt{\alpha}}, \quad \alpha\in(0,\bar{\alpha}]$$

$$\|x^\dagger - x_\alpha^\delta\| \le C\varphi(\alpha)\|w\| + \gamma_* \frac{\delta}{\sqrt{\alpha}}, \quad \alpha\in(0,\bar{\alpha}]$$

If $\psi(s) = s^p$ for each fixed δ we may find $\alpha = \alpha(\delta)$ which minimizes the right hand side in these equalities. With such ψ or φ we get estimation of the form

$$\|x^\dagger - x_\alpha^\delta\| \le C\left(\alpha^p + \frac{c\delta}{\sqrt{\alpha}}\right).$$

The minimum is attained for $\alpha = \left(\frac{c\delta}{2p}\right)^{\frac{2}{2p+1}}$ and equals $c\delta^{\frac{2p}{2p+1}}$ with some new constant c. With this choice of α we get

$$\|x^\dagger - x_\alpha^\delta\| \le C\delta^{\frac{2p}{2p+1}}.$$

Splitting operator. Using polar decomosition and spectral measure we may split operator $A = A_b \oplus A_u$. Let $A = BU$ be the polar decomposition of A, $A : D(A) \to H_2$, $D(A) \subset H_1$, where H_1, H_2 are Hilbert spaces. $B : D(B) \to H_2$, $D(B) \subset H_2$ is a positive selfadjoint operator, and $U : H_1 \to H_2$ is an isometry.

With E - the spectral measure of B we set

$$\begin{aligned} H_{2,b} = E([0,s_0])H_2, \quad & H_{2,u} = E((s_0,\infty))H_2, \\ H_{1,b} = U^{-1}H_{2,b}, \quad & H_{1,u} = U^{-1}H_{2,u}. \end{aligned}$$

Now

$$\begin{aligned} A_b = A|_{H_{1,b}} :& H_{1,b} \to H_{2,b}, \\ A_u = A|_{H_{1,u}} :& D(A_u) \to H_{2,u}, D(A_u) = D(A)\cap H_{1,u}. \end{aligned}$$

In the model case $H_{1,b} = H_{2,b} = L^2_\mu([0,s_0])$, $H_{1,u} = H_{2,u} = L^2_\mu((s_0,\infty))$

A_b is a bounded operator $\|A_b\| \le s_0$, A_u may be unbounded, however it has a bounded inverse, because for $x \in D(A_u) \subset H_{1,u}$ we have

$$\|A_u x\| = \|BUx\| \ge s_0\|Ux\| = s_0\|x\|.$$

With this splitting regularization splits also

$$x_\alpha^\delta = A^* g_\alpha(AA^*) y^\delta = A_b^* g_\alpha(B^2) y_b^\delta \oplus A_u^* g_\alpha(B^2) y_u^\delta,$$

where $y_b^\delta = U^{-1}E([0, s_0])Uy^\delta$, $y_u^\delta = U^{-1}E((s_0, \infty))Uy^\delta$, and therefore $\|y_u^\delta\|^2 + \|y_b^\delta\|^2 = \|y^\delta\|^2$.

Regularization theory for bounded operators is known it suffices to check, how it may be applied for unbounded operators with bounded inverse.

If φ is a qualification for $\{g_\alpha\}$ then for any $s_0 > 0$

$$|r_\alpha(\xi)|\varphi(s_0) \le |r_\alpha(\xi)|\varphi(\xi) \le \sup_{s>0} |r_\alpha(s)|\varphi(s) \le \gamma\varphi(\alpha), \quad 0 < \alpha \le \bar{\alpha}, \quad \xi > s_0.$$

Then we can estimate some part of bias.

In the model case we have

$$E([s_0, \infty))(x^\dagger - x_\alpha) = (I - A_u^2 g_\alpha(A_u^2))E([s_0, \infty))x^\dagger$$

and

$$\|E([s_0, \infty))(x^\dagger - x_\alpha)\|^2 = \int_{[s_0,\infty)} |r_\alpha(t^2))x^\dagger(t)|^2 \le \frac{\gamma^2\varphi^2(\alpha)}{\varphi^2(s_0^2)} \|E([s_0, \infty))x^\dagger\|^2$$

In the general case we have

$$U^*E([s_0, \infty))U(x^\dagger - x_\alpha) = U^*(I - B^2 g_\alpha(B^2))E([s_0, \infty))Ux^\dagger$$

and therefore

$$\|U^*E([s_0, \infty))U(x^\dagger - x_\alpha)\|^2 = \int_{[s_0,\infty)} r_\alpha^2(t^2)\|dEUx^\dagger\|^2 \le \frac{\gamma^2\varphi^2(\alpha)}{\varphi^2(s_0^2)} \|E([s_0, \infty))Ux^\dagger\|^2.$$

Hence

$$\|U^*E([s_0, \infty))U(x^\dagger - x_\alpha)\| \le \frac{\gamma\varphi(\alpha)}{\varphi(s_0^2)} \|x^\dagger\|. \tag{6}$$

Part c) of regularization definition is mainly applicable to operators for which their positive part in polar decompsition is not strictly bounded by 0 from below. It is not the case for A_u.

$$\|U^*E([s_0, \infty))U(x_\alpha - x_\alpha^\delta)\| \le \sup_{t \ge s_0} t|g_\alpha(t^2)\|y - y^\delta\| \le \delta \sup_{t \ge s_0} t|g_\alpha(t^2).$$

For Tikhonov relularization $g_\alpha(t) = \frac{1}{t+\alpha}$ and

$$\sup_{t \ge s_0} t|g_\alpha(t^2) = \frac{s_0}{s_0^2 + \alpha} \le s_0^{-1} \quad \text{for } \alpha \le s_0^2$$

because the derivative or $\frac{t}{t^2+\alpha}$ is negative for $\alpha \le s_0^2$. Thus the bound does not depend on α.

References.

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1. T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag 1976.