Approximation of mappings with derivatives of low rank

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My talk is based on two recent joint papers with Paweł Goldstein.

Jacek Gałęski in 2017, in the context of his research in geometric measure theory, formulated the following conjecture:

Conjecture. Let $1 \leq m < n$ be integers and let $\Omega \subset \mathbb{R}^n$ be open. If $f \in C^1(\Omega, \mathbb{R}^n)$ satisfies $\operatorname{rank} Df \leq m$ everywhere in Ω, then f can be uniformly approximated by smooth mappings $g \in C^\infty(\Omega, \mathbb{R}^n)$ such that $\operatorname{rank} Dg \leq m$ everywhere in Ω.

One can also modify the conjecture and ask about a local approximation: smooth approximation in a neighborhood of any point. These are very natural problems with possible applications to PDEs and Calculus of Variations. However, the problems are difficult, because standard approximation techniques like the one based on convolution do not preserve the rank of the derivative. It is a highly nonlinear constraint, difficult to deal with.

In 2018 Goldstein and Hajłasz obtained infinitely many counterexamples to this conjecture. Here is one:

Example. There is $f \in C^1(\mathbb{R}^5, \mathbb{R}^5)$ with $\operatorname{rank} Df \leq 3$ that cannot be locally and uniformly approximated by mappings $g \in C^2(\mathbb{R}^5, \mathbb{R}^5)$ satisfying $\operatorname{rank} Dg \leq 3$.

This example is a special case of a much more general result and the construction heavily depends on algebraic topology including the homotopy groups of spheres and the Freudenthal suspension theorem.

More recently Goldstein and Hajłasz proved the conjecture in the positive in the case when $m = 1$. The proof is based this time on methods of analysis on metric spaces and in particular on factorization of a mapping through metric trees.

The method of factorization through metric trees used in the proof of the conjecture when $m = 1$ is very different and completely unrelated to the methods of algebraic topology used in the construction of counterexamples. However, quite surprisingly, both techniques have originally been used by Wenger and Young as tools for study of Lipschitz homotopy groups of the Heisenberg group, a problem that seems completely unrelated to problems discussed in this talk.