Causal perturbation theory

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1 Introduction

Causal perturbation theory is based on ideas of Stückelberg and Bogoliubov [2] which were rigorously worked out in the seminal paper of Epstein and Glaser (EG) [9]. It was further developed mainly by R. Stora (e.g., [15, 16, 20]) and the groups of G. Scharf [18] and K. Fredenhagen [3, 4, 6, 7]. Causal perturbation theory is a rigorous perturbative approach to Quantum Field Theory (QFT) – winning by its *conceptual clarity*. The latter relies on the following properties:

- The time-ordered product (T-product), which is the main building stone of a perturbative QFT, is defined by *axioms*, the most important being Causality.
- The interaction is adiabatically switched off. By this, the infrared (IR) problem is separated from the ultraviolet (UV) problem. The adiabatic limit (i.e., the limit which removes this unphysical switching of the interaction – this is the IR problem) is performed only at the end of the construction; typically it exists only for observable quantities, e.g., inclusive cross sections, and not for the individual S-matrix elements. However, local, algebraic properties of the observables can be obtained without performing the adiabatic limit, see Sect. 4.5. After performing the adiabtic limit, the results agree with what comes out from more conventional versions of perturbative QFT, e.g., BPHZ-renormalization or dimensional regularization.
- The T-product $T = (T_n)_{n\in\mathbb{N}}$ is constructed in position space, by induction on the number n of factors. Due to this, renormalization (i.e., the UV-problem) is the mathematically well-defined problem of extending inductively known distributions from $\mathcal{D}'(M^n \setminus \Delta_n)$ to $\mathcal{D}'(M^n)$, where M is the space-time manifold and Δ_n is the thin diagonal in M^n, see (4.15). As long as one does not consider the adiabatic limit, in each step, all quantities are mathematically well-defined.
- The observables are constructed as *formal* power series in the coupling constant and in $\hbar$ – questions concerning the convergence of this series are not touched.
- The EG-construction yields *all* solutions of the axioms. By the Main Theorem (Thm. 4.9) the set of solutions is the orbit of the Stückelberg–Petermann renormalization group (Def. 4.8) when acting on a particular solution (any solution may be chosen as starting point).

Further advantages of the EG-construction of the T-product are:

- Since it proceeds in position space, it is well suited for perturbative QFT on a globally hyperbolic, curved space-time manifold M [4, 11] – see the next article in this encyclopedia. For simplicity, in this article, we choose M to be the d-dimensional Minkowski space. EG-renormalization has been worked out also in Euclidean space [14].
- Overlapping divergences (which caused a lot of troubles in the history of pertrubative renormalization) do not appear, due to the inductive procedure in the construction of the T-product.
- It applies also to nonrenormalizable interactions, e.g. perturbative quantum gravity (see e.g. [5, 18]): In each order in the coupling constant it yields a well-defined result.

In most formulations of perturbative QFT (also in the work of EG [9]) the free quantum field is a Fock space operator. That is, it is an “on-shell field”, since it obeys the free field equation. In this article, (classical and quantum) fields are functionals on the classical configuration space, which is

$C^\infty(M,\mathbb{R})$ in case of a real scalar field. That is, our fields are "off-shell", since they are not restricted by any field equation. The algebra of classical fields is given by the pointwise product of functionals. Quantization of the free theory is obtained by *deformation* of this product, where the propagator of the resulting star product (i.e., the two-point function) contains the information that we are dealing with the free theory (see (3.1)). Working with off-shell fields is more flexible than the Fock space formalism; this is advantagous for various purposes – see [8, Preface].

Main references is the latter book, which for the most parts relies on [3,4,6,7]. In the following, solely references differing from the just mentioned ones are given.

2 Fields as functionals on the configuration space

Space of fields. For brevity we study the model of one real scalar field with mass $m \geq 0$. For the pertinent configuration space we choose $\mathcal{C} := \mathcal{C}^\infty(M,\mathbb{R})$. The partial derivative ∂^a (where[1] $a \in \mathbb{N}^d$) of the basic field $\varphi(x)$ is the functional

$$\partial^a\varphi(x) : \begin{cases} \mathcal{C} \longrightarrow \mathbb{R} \\ h \longmapsto \partial^a\varphi(x)[h] = \partial^a h(x)\ . \end{cases} \tag{2.1}$$

For simplicity, we study only fields which are polynomials in the basic field φ (and its derivatives).

Definition 2.1. The *space of (classical and quantum) fields* $\mathcal{F}$ is defined as the vector space of functionals $F \equiv F(\varphi)\colon \mathcal{C} \longrightarrow \mathbb{C}$ of the form

$$F(\varphi) = f_0 + \sum_{n=1}^{N} \int d^dx_1 \cdots d^dx_n\ \varphi(x_1)\cdots\varphi(x_n)\, f_n(x_1,\dots,x_n) =: \sum_{n=0}^{N} \langle f_n, \varphi^{\otimes n}\rangle \tag{2.2}$$

with $N < \infty$, where $F(\varphi)[h] := F(h)$. Here $f_0 \in \mathbb{C}$ is a constant and, for $n \geq 1$, f_n is a distribution (i.e., $f_n \in \mathcal{D}'(M^n,\mathbb{C})$) with compact support. In addition, each f_n is required to satisfy the wave front set property:

$$\mathrm{WF}(f_n) \subseteq \{\, (x_1,\dots,x_n;k_1,\dots,k_n) \,\big|\, (k_1,\dots,k_n) \notin \overline{V}_+^{\times n} \cup \overline{V}_-^{\times n} \,\}\ , \tag{2.3}$$

where $V_\pm$ denotes the forward/backward light cone. Convergence in $\mathcal{F}$ is understood in the pointwise sense: $\lim_{n\to\infty} F_n = F$ iff $\lim_{n\to\infty} F_n[h] = F[h]\ ,\ \forall h \in \mathcal{C}$.

The purpose of the wave front set condition is to ensure the existence of pointwise products of distributions which appear in our definition of the Poisson bracket (2.9) and, more generally, of the star product (see Sect. 3).

An important example is given by

$$f_n(x_1,\dots,x_n) := (-1)^{\sum_j |a_j|} \int dx\ g(x)\, \partial^{a_1}\delta(x_1-x)\cdots\partial^{a_n}\delta(x_n-x)\ ,\quad g \in \mathcal{D}(M,\mathbb{C})\ ,$$

and $f_k = 0$ for $k \neq n$, that is,

$$F(\varphi) = \int dx\, g(x)\, \partial^{a_1}\varphi(x)\cdots\partial^{a_n}\varphi(x) \in \mathcal{F}\ . \tag{2.4}$$

The *support* of $F \in \mathcal{F}$ is defined as

$$\operatorname{supp} F := \operatorname{supp} \frac{\delta F}{\delta\varphi(\cdot)}\ .$$

Algebra of classical fields. Introducing the pointwise product

$$F \cdot G \equiv FG : h \longmapsto F[h]G[h] \in \mathcal{F}, \tag{2.5}$$

and the "$*$-operation"

$$F = \sum_{n=0}^{N}\langle f_n, \varphi^{\otimes n}\rangle \longmapsto F^* = \sum_{n=0}^{N}\langle \overline{f_n}, \varphi^{\otimes n}\rangle \in \mathcal{F} \tag{2.6}$$

we obtain a *commutative $*$-algebra* – this is the algebra of classical fields.

[1] We use the French convention that $0 \in \mathbb{N}$.

Local fields. The example (2.4) is a local functional in the sense of the following definition:

Definition 2.2. The space $\mathcal{F}_{\text{loc}}$ of *local fields* is following subspace of $\mathcal{F}$: Let $\mathcal{P}$ be the space of polynomials in the variables $\{\partial^a\varphi \,|\, a \in \mathbb{N}^d\}$ with real coefficients ("field polynomials"); then

$$\mathcal{F}_{\text{loc}} := \Big\{ \sum_{i=1}^{K} A_i(g_i) := \sum_{i=1}^{K} \int dx\; A_i(x)\, g_i(x) \,\Big|\, A_i \in \mathcal{P},\; g_i \in \mathcal{D}(M,\mathbb{C}),\; K < \infty \Big\}. \tag{2.7}$$

Given $F = \sum_{i=1}^{K} A_i(g_i) \in \mathcal{F}_{\text{loc}}$, the pairs $(A_i, g_i)_{i=1}^{K}$ are not uniquely determined by F, since $\int dx\, \partial_\mu\big(A(x)g(x)\big) = 0$ for any $A \in \mathcal{P}$ and $g \in \mathcal{D}(M)$. This non-uniqueness can be removed in the following way:

Proposition 2.3. *There exists a subspace* $\mathcal{P}_{\text{bal}}$ *of* $\mathcal{P}$ *(the space of "*balanced fields*") with the following properties:*

(a) *Every* $0 \neq A \in \mathcal{P}$ *can* uniquely *be written as a finite sum of type*

$$A = \sum_{a\in\mathbb{N}^d} \partial^a B_a\;, \quad \textit{where} \quad B_a \in \mathcal{P}_{\text{bal}} \quad \textit{and} \quad B_a|_{\varphi=0} = 0 \;\; \forall a \neq 0\;. \tag{2.8}$$

(b) *For each* $F \in \mathcal{F}_{\text{loc}}$*, there exists a* unique $f \in \mathcal{D}(\mathbb{M}, \mathcal{P}_{\text{bal}})$ *such that*

$$F - F[0] = \int dx\, f(x) \quad \textit{and also} \quad f(x)|_{\varphi=0} = 0 \quad \forall x \in \mathbb{M}\;.$$

For example, $\varphi\partial^\mu\varphi$ cannot be a balanced field, since $\varphi\partial^\mu\varphi = \partial^\mu(\frac{1}{2}\varphi^2)$. In part (b), $F[0] \in \mathbb{C}$ must be excluded, since there are infinitely many $\tilde{f} \in \mathcal{D}(\mathbb{M},\mathbb{C})$ fulfilling $F(0) = \int dx\, \tilde{f}(x)$. It is an easy exercise to prove that part (a) implies part (b). Part (a) has been proved by giving an explicit construction of $\mathcal{P}_{\text{bal}}$.

Poisson bracket. Let $-\Delta_m^{\text{ret}} \in \mathcal{D}'(\mathbb{R}^d)$ be the retarded Green' function of the Klein–Gordon operator and let $\Delta_m(x) := \Delta_m^{\text{ret}}(x) - \Delta_m^{\text{ret}}(-x)$ be the commutator function.

Definition 2.4. The *Poisson bracket* of the free theory is the bilinear map $\mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}$ given by

$$\{F, G\} := \int dx\, dy\; \frac{\delta F}{\delta\varphi(x)}\, \Delta_m(x-y)\, \frac{\delta G}{\delta\varphi(y)}\;. \tag{2.9}$$

One proves: Since F and G satisfy the wave front set property (2.3) the pointwise product of distributions in (2.9) exists, and $\{F,G\}$ again satisfies this wave front set property, hence $\{F,G\} \in \mathcal{F}$. Obviously, it holds that $\{G,F\} = -\{F,G\}$; and one verifies that the bracket (2.9) fulfills the *Leibniz rule* and the *Jacobi identity*, hence, it is indeed a Poisson bracket.

3 Deformation quantization of the free theory

Deformation quantization is mainly due to [1]. Wheras that work deals with quantum mechanics (i.e., finite dimensional systems), we apply it here to QFT.

Definition and properties of the star product. Let $\mathcal{F}_\hbar$ be the space of formal polynomials in $\hbar$ with coefficients in $\mathcal{F}$. The star product $\star \equiv \star_\hbar : \mathcal{F}_\hbar \times \mathcal{F}_\hbar \longrightarrow \mathcal{F}_\hbar$ is a *deformation of the classical product* (2.5) with deformation parameter $\hbar$, which is required to be

(a) *bilinear* in its arguments;

(b) *associative*; and, for $F, G \sim \hbar^0$; should satisfy:

(c) $F \star_\hbar G \to F \cdot G$ (the classical product) as $\hbar \to 0$; and

(d) $(F \star_\hbar G - G \star_\hbar F)/i\hbar \to \{F, G\}$ (the Poisson bracket of the free theory) as $\hbar \to 0$.

Definition 3.1. Given a suitable two-point function $H \in \mathcal{D}'(\mathbb{R}^d)$, the star product is defined by

$$F \star_\hbar G := \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \int dx_1 \cdots dx_n \, dy_1 \cdots dy_n \, \frac{\delta^n F}{\delta\varphi(x_1) \cdots \delta\varphi(x_n)} \prod_{l=1}^{n} H(x_l - y_l) \, \frac{\delta^n G}{\delta\varphi(y_1) \cdots \delta\varphi(y_n)} \, . \quad (3.1)$$

Note that the sum over n is *finite* since $F, G \in \mathcal{F}_\hbar$ are polynomials in φ.

The two-point function $H \in \mathcal{D}'(\mathbb{R}^d)$ should satisfy the following properties:

(i) The wave front set of H should be such that the pointwise products of distributions appearing in (3.1) exist for all $F, G \in \mathcal{F}_\hbar$;

(ii) the above requirement (d) is satisfied iff the antisymmetric part of H is given by $\frac{1}{i}\big(H(z) - H(-z)\big) = \Delta_m(z)$; we also require

(iii) Lorentz invariance, $H(\Lambda z) = H(z)$ for all $\Lambda \in \mathcal{L}_+^\uparrow$,

(iv) that H is a solution of the free field equation, $(\square + m^2)H = 0$,

(v) and that $\overline{H(x)} = H(-x)$, which is equivalent to $(F \star G)^* = G^* \star F^*$.

Due to (ii) and (iv), $H \equiv H_m$ depends on the mass $m \geq 0$ appearing in the free field equation; hence, this holds also for the star product – sometimes we signify this by writing $\star_m$ (instead of $\star$ or $\star_\hbar$). The above requirements (a) and (c) are obviously satisfied and, with some effort, one can prove associativity [21].

The most obvious solution of the above requirements on H is the Wightman two-point function, $H_m = \Delta_m^+$. However, in even dimensions d, Δ_m^+ is not smooth in $m \geq 0$. The latter property can be reached using a Hadamard function instead, that is,

$$H_m(x) = H_m^\mu(x) = \Delta_m^+(x) - m^{d-2} \, f_d(m^2 x^2) \, \log(m^2/\mu^2) \, , \quad (3.2)$$

where $\mu > 0$ is a mass parameter and $f_d : \mathbb{R} \to \mathbb{R}$ is a certain analytic function (depending on the dimension d), hence $\mathrm{WF}(H_m^\mu) = \mathrm{WF}(\Delta_m^+)$. Since, in addition, $(\square_x + m^2) f_d(m^2 x^2) = 0$, H_m^μ solves also the above requirements (ii)-(v). Note that Δ_m^+ scales homogeneously, i.e., $\rho^{d-2} \Delta_{m/\rho}^+(\rho x) = \Delta_m^+(x)$, but H_m^μ scales only *almost homogeneously*, i.e., homogeneously up to logarithmic terms – see Def. 4.1.

States. By definition, a *state* ω on the algebra $(\mathcal{F}_\hbar, \star)$ is a functional $\omega : (\mathcal{F}_\hbar, \star) \longrightarrow \mathbb{C}$ which is *linear*, *real* (i.e., $\omega(F^*) = \overline{\omega(F)}$), *positive* (i.e., $\omega(F^* \star F) \geq 0$) and *normalized* (i.e., $\omega(1) = 1$). Note that ω itself may be a formal polynomial in $\hbar$; but, in $\omega(F)$ (with $F \in \mathcal{F}_\hbar$) the sum over the powers of $\hbar$ is an ordinary sum, in order that $\omega(F)$ is a complex number (depending on $\hbar$).

A simple, but important, example is the *vacuum state*:

$$\omega_0(F) := f_0 \quad \text{where} \quad F = f_0 + \sum_{n \geq 1} \langle f_n, \varphi^{\otimes n} \rangle \, . \quad (3.3)$$

For $H = \Delta_m^+$ one can prove that ω_0 is indeed positive, by using that $\int dx\, dy\, \overline{h(x)}\, \Delta_m^+(x-y)\, h(y) \geq 0$ for all $h \in \mathcal{D}(M)$; but the latter property may be violated for $H = H_m^\mu$.

On-shell fields. Introducing the space of solutions of the free field equation

$$\mathcal{C}_0 \equiv \mathcal{C}_0^{(m)} := \{\, h \in \mathcal{C} \,\big|\, (\square + m^2) h(x) = 0 \,\} \, ,$$

we define the space of *on-shell fields* to be

$$\mathcal{F}_0^{(m)} := \{\, F_0 := F\big|_{\mathcal{C}_0^{(m)}} \,\big|\, F \in \mathcal{F} \,\} \, .$$

This definition is motivated by the fact that $\varphi_0(x) := \varphi(x)\big|_{\mathcal{C}_0}$ satisfies the free field equation.

One verifies that the star product on $\mathcal{F}_\hbar$ induces a well-defined product on

$$\mathcal{F}_{0,\hbar}^{(m)} := \mathcal{F}_\hbar\big|_{\mathcal{C}_{S_0}^{(m)}} \quad \text{by setting} \quad F_0 \star G_0 := (F \star G)_0 \, .$$

Quantizing with Δ_m^+, on-shell fieds $F_0 \in \mathcal{F}_{0,\hbar}^{(m)}$ may be identified with linear operators on Fock space:

Theorem 3.2. *Let $\varphi^{\mathrm{op}}(x)$ be the free, real scalar field (for a given mass m) on the Fock space $\mathfrak{F}$. Then the map*

$$\Phi\colon \mathcal{F}^{(m)}_{0,\hbar} \longrightarrow \Phi(\mathcal{F}^{(m)}_{0,\hbar}) \subset \{\textit{linear operators on } \mathfrak{F}\} \tag{3.4}$$

given by

$$F_0 = \sum_{n=0}^{N} \langle f_n, \varphi_0^{\otimes n}\rangle \longmapsto \Phi(F_0) = \sum_{n=0}^{N} \int dx_1 \cdots dx_n \,{:}\varphi^{\mathrm{op}}(x_1)\cdots\varphi^{\mathrm{op}}(x_n){:}\, f_n(x_1,\ldots,x_n)$$

(where $:-:$ denotes normal ordering of Fock space operators) is an algebra isomorphism for the star product on the left and the operator product on the right (i.e., $\Phi(F_0 \star G_0) = \Phi(F_0)\,\Phi(G_0)$) and also for the classical product on the left and the normally ordered product on the right (i.e., $\Phi(F_0 \cdot G_0) = {:}\Phi(F_0)\,\Phi(G_0){:}$). In addition, Φ respects the $*$-operation:

$$\big\langle \psi_1, \Phi(F_0^*)\psi_2\big\rangle_{\mathfrak{F}} = \big\langle \Phi(F_0)\psi_1, \psi_2\big\rangle_{\mathfrak{F}} \qquad \forall F_0 \in \mathcal{F}^{(m)}_{0,\hbar} \tag{3.5}$$

and for all ψ_1, ψ_2 in the domain of $\Phi(F_0)$ or $\Phi(F_0^*)$, respectively.

4 Perturbative QFT

Let $L_{\mathrm{int}} = \sum_{k=1}^{\infty} L_k\, \kappa^k \in \mathcal{P}[\![\kappa]\!]$ be the interaction Lagrangian, where κ is the coupling constant. With that,

$$S \equiv S(g) := \int dx\, \sum_{k=1}^{\infty} \big(g(x)\,\kappa\big)^k L_k(x) \in \mathcal{F}_{\mathrm{loc}}[\![\kappa]\!]\ , \quad g \in \mathcal{D}(M)\ , \tag{4.1}$$

is the adiabatically switched off interaction. The main aim is to construct the pertinent scattering matrix (S-matrix), for which we make the ansatz

$$\mathbf{S}(S) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!\hbar^n}\, T_n(S^{\otimes n}) \in \mathcal{F}[\![\kappa]\!]\ , \tag{4.2}$$

which is a formal Laurent series in $\hbar$, where $T_n : (\mathcal{F}_{\mathrm{loc}})^{\otimes n} \to \mathcal{F}$ is the n-fold time-ordered product defined axiomatically in the following section.

4.1 Axioms for the T-product

In view of the inductive construction of $T = (T_n)$ we split the axioms into 'basic axioms' and 'renormalization conditions'. The first basic axiom is

(1) **Linearity:** We require that

$$T_n\colon \mathcal{F}_{\mathrm{loc}}^{\otimes n} \longrightarrow \mathcal{F} \quad \text{be linear.} \tag{4.3}$$

Note that here both the arguments and the values of T_n are *off-shell fields.*

Our construction of the T-product is an inductive construction of the $\mathcal{F}$-valued distributions $T_n\big(A_1(x_1),\ldots,A_n(x_n)\big) \in \mathcal{D}'(M^n,\mathcal{F})$ (for all $A_1,\ldots,A_n \in \mathcal{P}$), which should be connected to the maps $T_n : \mathcal{F}_{\mathrm{loc}}^{\otimes n} \to \mathcal{F}$ (4.3) by

$$\int dx_1\cdots dx_n\; T_n\big(A_1(x_1),\ldots,A_n(x_n)\big)\, g_1(x_1)\cdots g_n(x_n) = T_n\big(A_1(g_1)\otimes\cdots\otimes A_n(g_n)\big) \tag{4.4}$$

for all $g_1,\ldots,g_n \in \mathcal{D}(\mathbb{M})$. But there is a problem with this fomula (4.4): Since $\int dx\ \partial_x\big(g(x)\,A(x)\big) = 0$, (4.4) and Linearity of T_n imply

$$\int dx\,\Big((\partial g)(x)\,T_n\big(\ldots,A(x),\ldots\big) + g(x)\,T_n\big(\ldots,(\partial A)(x),\ldots\big)\Big) = 0\ ;$$

hence the *Action Ward Identity* (AWI) must hold, that is,

$$\textbf{AWI:}\quad \partial_{x_l} T_n\big(\ldots,A(x_l),\ldots\big) = T_n\big(\ldots,\partial_{x_l}A(x_l),\ldots\big), \quad \forall A \in \mathcal{P}\ ,\ 1 \le l \le n\ . \tag{4.5}$$

To define $T_n\big(A_1(x_1),\dots,A_n(x_n)\big)$ in terms of the map $T_n:\mathcal{F}_{\rm loc}^{\otimes n}\to\mathcal{F}$ in accordance with the AWI, we use Prop. 2.3: For balanced fields $B_1,\dots,B_n\in\mathcal{P}_{\rm bal}$ we define $T_n\big(B_1(x_1),\dots,B_n(x_n)\big)$ by the formula (4.4), and then, for arbitrary $A_1,\dots,A_n\in\mathcal{P}$, we define $T_n\big(A_1(x_1),\dots,A_n(x_n)\big)$ by first writing $A_i=\sum_{a_i}\partial^{a_i}B_{ia_i}$ where $B_{ia_i}\in\mathcal{P}_{\rm bal}$ and setting

$$T_n\big(A_1(x_1),\dots,A_n(x_n)\big):=\sum_{a_1,\dots,a_n}\partial^{a_1}_{x_1}\cdots\partial^{a_n}_{x_n}\,T_n\big(B_{1a_1}(x_1),\dots,B_{na_n}(x_n)\big)\ . \tag{4.6}$$

One easily verifies that with this definition the AWI (4.5) holds for arbitrary $A\in\mathcal{P}$ and that also the relation (4.4) holds for all $A_1,\dots,A_n\in\mathcal{P}$.

The further basic axioms are:

(2) **Initial condition:** $T_1(F)=F$ for any $F\in\mathcal{F}_{\rm loc}$;

(3) **Symmetry:** $T_n(F_{\pi(1)}\otimes\cdots\otimes F_{\pi(n)})=T_n(F_1\otimes\cdots\otimes F_n)\quad\forall\,F_1,\dots,F_n\in\mathcal{F}_{\rm loc}$ and for all permutations π;

(4) **Causality:** For all $A_1,\dots,A_n\in\mathcal{P}$, T_n fulfills the causal factorization:

$$T_n\big(A_1(x_1),\dots,A_n(x_n)\big)=T_k\big(A_1(x_1),\dots,A_k(x_k)\big)\star_m T_{n-k}\big(A_{k+1}(x_{k+1}),\dots,A_n(x_n)\big)$$

whenever $\{x_1,\dots,x_k\}\cap\big(\{x_{k+1},\dots,x_n\}+\overline{V}_-\big)=\emptyset$;

Assuming validity of axiom (3), one proves that axiom (4) is equivalent to the following causality relation for the S-matrix:

$$\mathbf{S}(H+G+F)=\mathbf{S}(H+G)\star\mathbf{S}(G)^{\star-1}\star\mathbf{S}(G+F)\quad\text{if}\quad\operatorname{supp}H\cap(\operatorname{supp}F+\overline{V}_-)=\emptyset\ , \tag{4.7}$$

where $(-)^{\star-1}$ means the inverse w.r.t. the star product.

Due to the axioms (1) (Linearity) and (3) (Symmetry) it holds that

$$\mathbf{S}^{(n)}:=\mathbf{S}^{(n)}(0)=\tfrac{i^n}{n!}\,T_n\quad(n\text{th derivative of }\mathbf{S}(F)\text{ at }F=0)\text{, i.e., }\mathbf{S}^{(n)}(F^{\otimes n})=\tfrac{d^n}{d\lambda^n}\big|_{\lambda=0}\mathbf{S}(\lambda F).$$

We turn to the renormalization conditions:

(5) **Field independence:** $\delta T_n/\delta\varphi=0$.

Performing a (finite) Taylor expansion of $T_n\big(A_1(x_1),\dots,A_n(x_n)\big)$ in φ with respect to $\varphi=0$, one shows that Field Independence is equivalent to the validity of the *causal Wick expansion*: For *monomials* $A_1,\dots,A_n\in\mathcal{P}$ it holds that

$$T_n\big(A_1(x_1),\dots,A_n(x_n)\big)=\sum_{\underline{A}_l\subseteq A_l}\omega_0\Big(T_n\big(\underline{A}_1(x_1),\dots,\underline{A}_n(x_n)\big)\Big)\,\overline{A}_1(x_1)\cdots\overline{A}_n(x_n), \tag{4.8}$$

where each *submonomial* $\underline{A}$ of a given monomial A and its *complementary submonomial* $\overline{A}$ are defined by

$$\underline{A}:=\frac{\partial^k A}{\partial(\partial^{a_1}\varphi)\cdots\partial(\partial^{a_k}\varphi)}\neq 0,\qquad\overline{A}:=C_{a_1\dots a_k}\,\partial^{a_1}\varphi\cdots\partial^{a_k}\varphi \tag{4.9}$$

(no sum over $a_1,\dots,a_k$ in the formula for $\overline{A}$), with $C_{a_1\dots a_k}$ being a certain combinatorial factor. The range of the sum $\sum_{\underline{A}\subseteq A}$ are all $k\in\mathbb{N}$ and $a_1,\dots,a_k\in\mathbb{N}^d$ which yield a $\underline{A}\neq 0$. (For $k=0$ we have $\underline{A}=A$ and $\overline{A}=1$.) The main message of the causal Wick expansion is that $T_n\big(A_1(x_1),\dots\big)\in\mathcal{D}'(M^n,\mathcal{F})$ is uniquely determined by the $\mathbb{C}$-valued distributions $\omega_0\big(T_n\big(\underline{A}_1(x_1),\dots\big)\big)$.

(6) **Unitarity and field parity:** In order that $\mathbf{S}(S)$ is unitary for real interactions (i.e., $S=S^*$), we require

$$\mathbf{S}(F)^*=\mathbf{S}(F^*)^{\star-1}\quad\forall F\in\mathcal{F}_{\rm loc}. \tag{4.10}$$

Field parity is the condition

$$\alpha\circ T_n=T_n\circ\alpha^{\otimes n},\quad\text{where }\alpha:\mathcal{F}\to\mathcal{F}\text{ is defined by}\quad(\alpha F)[h]:=F[-h]\quad\forall h\in\mathcal{F}. \tag{4.11}$$

(7) **Poincaré covariance:** $\beta_{\Lambda,a} \circ T_n = T_n \circ \beta_{\Lambda,a}^{\otimes n} \quad \forall (\Lambda, a) \in \mathcal{P}_+^\uparrow$,
where $\beta_{\Lambda,a} : \mathcal{F} \to \mathcal{F}$ is defined by

$$\beta_{\Lambda,a} \sum_n \langle f_n, \varphi^{\otimes n} \rangle := \sum_n \langle f_n(x_1, \dots, x_n), \varphi(\Lambda x_1 + a) \otimes \cdots \otimes \varphi(\Lambda x_n + a) \rangle.$$

Considering only Translation covariance (i.e., $\Lambda = 1$), we conclude that the $\mathbb{C}$-valued distributions

$$t_n(A_1, \dots, A_n)(x_1 - x_n, \dots, x_{n-1} - x_n) := \omega_0\Big(T_n\big(A_1(x_1), \dots, A_n(x_n)\big)\Big) \tag{4.12}$$

depend *only* on the relative coordinates, since $\omega_0 \circ \beta_{\Lambda,a} = \omega_0$. By using the causal Wick expansion (4.8), one easily verifies that this property (4.12) is in fact *equivalent* to Translation covariance.

(8) **Off-shell field equation:**

$$\begin{aligned} T_n\big(\varphi(g) \otimes F_1 \otimes \cdots \otimes F_{n-1}\big) &= \varphi(g)\; T_{n-1}\big(F_1 \otimes \cdots \otimes F_{n-1}\big) \\ &+ \hbar \int dx\, dy\; g(x)\, H_m^F(x-y) \sum_{k=1}^{n-1} T_{n-1}\Big(F_1 \otimes \cdots \otimes \frac{\delta F_k}{\delta \varphi(y)} \otimes \cdots \otimes F_{n-1}\Big)\;, \end{aligned} \tag{4.13}$$

where $g \in \mathcal{D}(\mathbb{M})$ and H_m^F is the Feynman propagator belonging to the two-point function H_m, that is, $H_m^F(x) := \theta(x^0)\, H_m(x) + \theta(-x^0)\, H(-x)$.

(9) **Smoothness in the mass $m \geq 0$:** By the basic axioms, $T \equiv T^{(m)}$ depends on the mass m of the free field equation via the star product $\star_m$ appearing in the Causality axiom. We require that the distributions

$$t_n^{(m)}(A_1, \dots, A_n) \quad \text{(4.12) depend smoothly on } m \geq 0, \text{ for all } A_1, \dots, A_n \in \mathcal{P} \text{ and all } n.$$

This axiom excludes quantization with Δ_m^+; hence, in the following we quantize with a Hadamard function H_m^μ

The next axiom deals with the scaling behaviour of the T-product. For this and also in view of the inductive construction of the T-product we introduce some notions:

Definition 4.1. Let $f \in \mathcal{D}'(\mathbb{R}^k)$ or $f \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$.

(a) We say that f scales *almost homogeneously* with *degree* $D \in \mathbb{C}$ and *power* $N \in \mathbb{N}$ iff

$$\Big(\mathbb{E}_k + D\Big)^{N+1} f(y_1, \dots, y_k) = 0 \quad \text{and} \quad \Big(\mathbb{E}_k + D\Big)^{N} f(y_1, \dots, y_k) \neq 0\;, \tag{4.14}$$

where $\mathbb{E}_k := \sum_{r=1}^k y_r\, \partial/\partial y_r$ is the Euler operator. When $N = 0$, we say there is *homogeneous* scaling of degree D.

(b) For $f(y) = f^{(m)}(y)$ (where $y \in \mathbb{R}^k$) being differentiable in the mass $m \geq 0$, we say that $f^{(m)}(y)$ scales *almost homogeneously under* $(y, m) \mapsto (\rho y, m/\rho)$ with *degree* $D \in \mathbb{C}$ and *power* $N \in \mathbb{N}$ iff the relations (4.14) hold for $(\mathbb{E} - m\, \partial/\partial m)$ in place of $\mathbb{E}$.

(c) The *scaling degree* (with respect to the origin) of f is given by

$$\mathrm{sd}(f) := \inf\{\, r \in \mathbb{R} \;\big|\; \lim_{\rho \downarrow 0} \rho^r\, f(\rho y) = 0 \,\}\;,$$

where $\inf \emptyset := \infty$ and $\inf \mathbb{R} := -\infty$.

For example, $\partial^a \delta_{(k)} \in \mathcal{D}'(\mathbb{R}^k)$ scales homogeneously with degree $D = k + |a|$. Obviously, a distribution f scaling almost homogeneously with degree $D \in \mathbb{C}$ (and arbitrary power N) has scaling degree $\mathrm{sd}(f) = \mathrm{Re}\, D$.

We also need the "mass dimension" of a *monomial* $A \in \mathcal{P}$.

Definition 4.2. The *mass dimension* of $\partial^a\varphi \in \mathcal{P}$ is defined by

$$\dim \partial^a\varphi := \frac{d-2}{2} + |a| \quad \text{for} \quad a \in \mathbb{N}^d \ .$$

For *monomials* $A_1, A_2 \in \mathcal{P}$, we agree that $\dim(A_1A_2) := \dim A_1 + \dim A_2$.

Denoting by $\mathcal{P}_j$ the vector space spanned by all monomials $A \in \mathcal{P}$ with $\dim A = j$ we introduce the *set* of "homogeneous" polynomials: $\mathcal{P}_{\mathrm{hom}} := \bigcup_{j\in\mathbb{N}/2} \mathcal{P}_j$.

(10) **Scaling:** For all *monomials* $A_1, \dots, A_n \in \mathcal{P}$ and all $n \geq 2$ we require that

$t_n^{(m)}(A_1, \dots, A_n)(y)$ scales almost homogeneously under $(y, m) \mapsto (\rho y, m/\rho)$ with degree $D = \sum_{l=1}^n \dim A_l$ and an arbitrary power $N < \infty$ (where $y \in \mathbb{R}^{d(n-1)}$).

(11) **$\hbar$-dependence:** For all *monomials* $A_1, \dots, A_n \in \mathcal{P}$ fulfilling $A_j \sim \hbar^0\ \forall j$ and all $n \geq 2$, we require

$$t_n(A_1, \dots, A_n) \sim \hbar^{\sum_{j=1}^n |A_j|/2} \ ,$$

where $|A|$ is the degree of the monomial A, i.e., $A(x)[\lambda h] = \lambda^{|A|}\, A(x)[h]$ for all $h \in \mathcal{C}$, $\lambda > 0$.

We point out that in axiom (10) the degree D fulfills $D \in \mathbb{N}$ (also in odd dimensions d), and in axiom (11) the power of $\hbar$ satisfies $\sum_{j=1}^n |A_j|/2 \in \mathbb{N}$. Both statements rely on the observation that Field parity (4.11) and $\omega_0 \circ \alpha = \omega_0$ imply that

$$t_n(A_1, \dots, A_n) = 0 \quad \text{if} \quad \sum_{j=1}^n |A_j| \quad \text{is odd;}$$

and, for the statement about the degree, we also use that for $A = c \prod_{j=1}^J \partial^{a_j}\varphi$ ($c \in \mathbb{R}$) it holds that $\dim A = J \cdot \dim\varphi + \sum_{j=1}^J |a_j| = |A| \cdot \frac{d-2}{2} + \sum_{j=1}^J |a_j|$.

4.2 Inductive construction of the T-product

The T-product $T = (T_n)$ is constructed by induction on n, starting with axiom (2) (Initial condition). Turning to the inductive step $(n-1) \to n$ we introduce the thin diagonal in M^n:

$$\Delta_n := \{\, (x_1, \dots, x_n) \in M^n \,\big|\, x_1 = \ldots = x_n \,\} \ . \tag{4.15}$$

Inductive step, off the thin diagonal Δ_n. The axioms (4) (Causality) and (3) (Symmetry) imply the following: For each point $x \in M^n \setminus \Delta_n$ there exist a neighbouhood $U_x \subset M^n$ of x, a $k \in \{1, \dots, n-1\}$ and a permutation π such that

$$\begin{aligned} T_n\big(A_1(x_1), \dots, A_n(x_n)\big) = {} & T_k\big(A_{\pi(1)}(x_{\pi(1)}), \dots, A_{\pi(k)}(x_{\pi(k)})\big) \\ & \star\, T_{n-k}\big(A_{\pi(k+1)}(x_{\pi(k+1)}), \dots, A_{\pi(n)}(x_{\pi(n)})\big) \quad \text{on } \mathcal{D}(U_x) \end{aligned}$$

for all $A_1, \dots, A_n \in \mathcal{P}$; that is, $T_n\big(A_1(x_1), \dots\big)|_{\mathcal{D}(U_x)}$ is uniquely determined in terms of the inductively known $(T_k)_{1\leq k<n}$. This observation gives part (a) of the following Theorem.

Theorem 4.3. *Let $(T_k)_{1\leq k<n}$ be constructed.*

(a) Uniqueness: If there exists some map $T_n : \mathcal{P}^{\otimes n} \to \mathcal{D}'(M^n, \mathcal{F})$ fulfilling the basic axioms, then its image is uniquely determined on $\mathcal{D}(M^n \setminus \Delta_n)$.

(b) Existence: There exists a map $T_n^0 : \mathcal{P}^{\otimes n} \to \mathcal{D}'(M^n \setminus \Delta_n, \mathcal{F})$ satisfying all axioms (1)-(11).

The proof of part (b) is constructive, using a partition of unity subordinate to an open cover of $M^n \setminus \Delta_n$ [4, 20].

Alternatively, the use of a partition of unity in the inductive construction of T_n^0 can be avoided by working with the distribution splitting method of Epstein and Glaser [9, 18], on the price of a more complicated combinatorics.[2]

[2] Note that for the construction of T_n^0 the distribution splitting can be done by multiplication with a Heaviside-function. Hence, the distribution splitting problem in the Epstein–Glaser construction is an equivalent reformulation of the extension problem $T_n^0 \to T_n$ treated in the next paragraph.

Extension to the thin diagonal Δ_n. The extension of T_n^0 (taking values in $\mathcal{D}'(M^n \setminus \Delta_n, \mathcal{F})$) to a well-defined T_n (taking values in $\mathcal{D}'(M^n, \mathcal{F})$) is non-unique and it corresponds to what is called "renormalization" in conventional approaches. By part (a) of Thm. 4.3, the renormalization conditions are not used for the construction of T_n^0, but they give guidance on how to do this extension and reduce the non-uniqueness drastically.

Since we are constructing $T_n : \mathcal{P}^{\otimes n} \to \mathcal{D}'(M^n, \mathcal{F})$ (instead of $T_n : \mathcal{F}_{\mathrm{loc}}^{\otimes n} \to \mathcal{F}$), the AWI plays the role of an additional renormalization condition. We fulfil it by first constructing $T_n\big(B_1(x_1), \ldots, B_n(x_n)\big)$ for all $B_1, \ldots, B_n \in \mathcal{P}_{\mathrm{bal}}$ and then we define $T_n\big(A_1(x_1), \ldots, A_n(x_n)\big)$ for arbitrary $A_1, \ldots, A_n \in \mathcal{P}$ by (4.6).

By Linearity (axiom (1)), the causal Wick expansion (axiom (5)) and Translation covariance (axiom (7)), the problem is simplified to the extension of

$$t^0(y) \equiv t_n^{(m),0}(B_1, \ldots, B_n)(y) := \omega_0\Big(T_n^{(m),0}\big(B_1(x_1), \ldots, B_n(x_n)\big)\Big) \in \mathcal{D}'(\mathbb{R}^{d(n-1)} \setminus \{0\}, \mathbb{C}),$$

where $y := (x_1 - x_n, \ldots, x_{n-1} - x_n)$ and $B_1, \ldots, B_n \in \mathcal{P}_{\mathrm{bal}} \cap \mathcal{P}_{\mathrm{hom}}$, to a distribution $t \in \mathcal{D}'(\mathbb{R}^{d(n-1)}, \mathbb{C})$. In view of axiom (10) (Scaling) we require here and in the following that $B_j \in \mathcal{P}_{\mathrm{bal}} \cap \mathcal{P}_{\mathrm{hom}}$; this is sufficient since there exists a basis of $\mathcal{P}_{\mathrm{bal}}$ lying in $\mathcal{P}_{\mathrm{hom}}$.

Obviously, any extension $t \in \mathcal{D}'(\mathbb{R}^k)$ of a given $t^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ obeys $\mathrm{sd}(t) \geq \mathrm{sd}(t^0)$. Looking for extensions which do not increase the scaling degree, existence and uniqueness of this problem is answered by the following [4]:

Theorem 4.4. *Let $t^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$. Then:*

(a) *If $\mathrm{sd}(t^0) < k$, there is a* unique *extension (called "direct extension") $t \in \mathcal{D}'(\mathbb{R}^k)$ fulfilling the condition $\mathrm{sd}(t) = \mathrm{sd}(t^0)$.*

(b) *If $k \leq \mathrm{sd}(t^0) < \infty$, there are* several *extensions $t \in \mathcal{D}'(\mathbb{R}^k)$ satisfying the condition $\mathrm{sd}(t) = \mathrm{sd}(t^0)$. In this case, given a particular solution t_0, the general solution is of the form*

$$t = t_0 + \sum_{|a| \leq \mathrm{sd}(t^0) - k} C_a\, \partial^a \delta_{(k)} \quad \textit{with} \quad C_a \in \mathbb{C}. \tag{4.16}$$

The proof is constructive, the idea of the construction is given in Sect. 4.3.

Turning to the maintenance of the axioms (9) (Smoothness in $m \geq 0$) and (10) (Scaling) in the extension $t^0 \equiv t_n^{(m),0}(B_1, \ldots, B_n) \to t$ (where $B_1, \ldots, B_n \in \mathcal{P}_{\mathrm{bal}} \cap \mathcal{P}_{\mathrm{hom}}$), we set $D := \sum_{j=1}^n \dim B_j \in \mathbb{N}$ and we first point out that in the case $m = 0$ we may apply the following [11, 13, 15]:

Proposition 4.5. *Let $t^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ scale almost homogeneously with degree $D \in \mathbb{C}$ and power $N_0 \in \mathbb{N}$. Then there exists an extension $t \in \mathcal{D}'(\mathbb{R}^k)$ which scales also almost homogeneously with degree D and power $N \geq N_0$:*

(i) *If $D \notin \mathbb{N} + k$, then t is unique and $N = N_0$;*

(ii) *if $D \in \mathbb{N} + k$, then t is non-unique and $N = N_0$ or $N = N_0 + 1$. In this case, given a particular solution t_0, the general solution is of the form*

$$t = t_0 + \sum_{|a| = D - k} C_a\, \partial^a \delta_{(k)} \quad \textit{with arbitrary} \quad C_a \in \mathbb{C}. \tag{4.17}$$

For the subcase $\mathrm{Re}\, D\ (= \mathrm{sd}(t^0)) < k$ of case (i)*, the unique t agrees with the direct extension of t^0 of part (a) of Thm. 4.4.*

For $m > 0$, Smoothness in $m \geq 0$ of $t^{(m),0} \in \mathcal{D}'(\mathbb{R}^{d(n-1)} \setminus \{0\}, \mathbb{C})$ ensures the existence of the Taylor expansion

$$t^{(m),0}(y) = \sum_{l=0}^{D - d(n-1)} \frac{m^l}{l!}\, u_l^0(y) + m^{D - d(n-1) + 1}\, t_{\mathrm{red}}^{(m),0}(y) \quad \text{with} \quad u_l^0(y) := \frac{\partial^l t^{(m),0}(y)}{\partial m^l}\bigg|_{m=0}. \tag{4.18}$$

Almost homogeneous scaling under $(y, m) \mapsto (\rho y, m/\rho)$ of $t^{(m),0}$ with degree D implies almost homogeneous scaling under $y \mapsto \rho y$ of u_l^0 with degree $D - l \in \mathbb{N} + d(n-1)$, hence we may apply part (ii) of Prop. 4.5 for the extension $u_l^0 \to u_l \in \mathcal{D}'(\mathbb{R}^{d(n-1)}, \mathbb{C})$. Considering $t_{\mathrm{red}}^{(m),0}$, one shows that the validity of the axioms (9) and (10) for $t^{(m),0}$ implies the validity of (9) for $t_{\mathrm{red}}^{(m),0}$ and that $\mathrm{sd}(t_{\mathrm{red}}^{(m),0}) < d(n-1)$. Hence, Thm. 4.4(a) (i.e., the direct extension) provides a unique extension $t_{\mathrm{red}}^{(m)}$ with $\mathrm{sd}(t_{\mathrm{red}}^{(m)}) = \mathrm{sd}(t_{\mathrm{red}}^{(m),0})$. The latter maintains smoothness in $m \geq 0$ and one verifies that it also maintains almost homogeneous scaling under $(y, m) \mapsto (\rho y, m/\rho)$ with degree $d(n-1) - 1$.

The construction given so far guarantees that the resulting T_n satisfies the renormalization conditions AWI, Field independence, Translation covariance, Smoothness in $m \geq 0$ and Scaling. How to maintain the further renormalization conditions in the extension $t_n^0 \to t_n$?

- For axiom (10) ($\hbar$-dependence) this is reached by doing the extension in each order of $\hbar$ individually.
- Turning to the symmetries required in axiom (6) (Unitarity and Field parity) and (7) (Lorentz covariance) we quote the following general result: *Given a t_n^0 scaling almost homogeneously and being symmetric w.r.t. a certain group $\mathcal{G}$, there exists an extension t_n which scales also almost homogeneously and is also $\mathcal{G}$-symmetric, if all finite-dimensional representations of $\mathcal{G}$ are completely reducible.* This assumption is satisfied for Unitarity and Field parity (in both cases $\mathcal{G}$ is the group $(\{-1, 1\}, \cdot)$ and also for the Lorentz group $\mathcal{L}_+^\uparrow$.

  The *construction* of a $\mathcal{G}$-symmetric t_n is an easy task for the Star-structure and the Field parity (one starts with an extension satisfying all other renormalization conditions and symmetrizes it w.r.t. $\mathcal{G}$), but for Lorentz covariance this requires some effort. (See also [16, 18]).
- The Off-shell field equation (axiom (8)) is satisfied by defining $t_n(\varphi, B_1, \ldots, B_{n-1})$ (with $B_1, \ldots, B_{n-1} \in \mathcal{P}_{\mathrm{bal}} \cap \mathcal{P}_{\mathrm{hom}}$) in terms of t_{n-1} by the vacuum expectation value of the relation (4.13) (note, that the first term on the r.h.s. of (4.13) does not contribute). The so defined $t_n(\varphi, B_1, \ldots, B_{n-1})$ is an extension of $t_n^0(\varphi, B_1, \ldots, B_{n-1})$ (because the latter fulfills (4.13)) and one verifies easily that it satisfies all other renormalization conditions.

4.3 Techniques to renormalize

In this section we sketch main techniques to compute the extension $\mathcal{D}'(\mathbb{R}^k \setminus \{0\}) \ni t^0 \to t \in \mathcal{D}'(\mathbb{R}^k)$.

Direct extension and W-extension. In the proof of Thm. 4.4 the extension t is constructed as follows [4, 9]:

- For $\mathrm{sd}(t^0) < k$ the (unique) "direct extension" is obtained by

  $$\langle t, h \rangle := \lim_{\rho \to \infty} \langle t^0, \chi_\rho h \rangle \qquad \forall h \in \mathcal{D}(\mathbb{R}^k) , \tag{4.19}$$

  where $\chi \in C^\infty(\mathbb{R}^k)$ is such that $0 \leq \chi(x) \leq 1$, $\chi(x) = 0$ for $|x| \leq 1$ and $\chi(x) = 1$ for $|x| \geq 2$ and we use the notation $\chi_\rho(x) := \chi(\rho x)$. Since $\chi_\rho(x) h(x) \in \mathcal{D}(\mathbb{R}^k \setminus \{0\})$ for any $\rho > 0$, the expression $\langle t^0, \chi_\rho h \rangle$ exists. One proves that the limit (4.19) exists and defines a distribution $t \in \mathcal{D}'(\mathbb{R}^k)$. Since for any $h_1 \in \mathcal{D}(\mathbb{R}^k \setminus \{0\})$ it holds that $\chi_\rho h_1 = h_1$ for ρ sufficiently large, this t is indeed an extension of t^0. In addition one shows that $\mathrm{sd}(t) = \mathrm{sd}(t^0)$.

  For practical computations the formula (4.19) means that the direct extension t is given by the same formula as t^0.
- For $\mathrm{sd}(t^0) \geq k$: Let $\omega := \mathrm{sd}(t^0) - k$ be the *singular order* of t^0. Introducing the subspace of test functions

  $$\mathcal{D}_\omega := \mathcal{D}_\omega(\mathbb{R}^k) := \{\, h \in \mathcal{D}(\mathbb{R}^k) \,\big|\, \partial^a h(0) = 0 \text{ for } |a| \leq \omega \,\} , \tag{4.20}$$

  one proves that t^0 *has a unique extension* t_ω *to* $\mathcal{D}'_\omega$ *satisfying* $\mathrm{sd}(t_\omega) = \mathrm{sd}(t^0)$ – roughly speaking, the direct extension applies also in this case:

  $$\langle t_\omega, h \rangle = \lim_{\rho \to \infty} \langle t^0, \chi_\rho h \rangle . \tag{4.21}$$

Each projector $W\colon \mathcal{D}(\mathbb{R}^k) \longrightarrow \mathcal{D}_\omega$ defines an extension $t^W \in \mathcal{D}'(\mathbb{R}^k)$ (called "W-extension") by

$$\langle t^W, h\rangle := \langle t_\omega, Wh\rangle\,, \tag{4.22}$$

Since $Wh = h$ for $h \in \mathcal{D}(\mathbb{R}^k \setminus \{0\})$, the relations

$$\langle t^W, h\rangle = \langle t_\omega, Wh\rangle = \langle t_\omega, h\rangle = \langle t^0, h\rangle$$

show that t^W is indeed an extension of t^0. More elaborate is the proof of $\mathrm{sd}(t^W) = \mathrm{sd}(t^0)$.

Any set of functions $w_a \in \mathcal{D}(\mathbb{R}^k)$ (where $a \in \mathbb{N}^k$ with $|a| \le \omega$) satisfying

$$\partial^b w_a(0) = \delta^b_a \quad \forall b \in \mathbb{N}^k\ ,\ |b| \le \omega \tag{4.23}$$

defines such a projector W by

$$Wh(x) := h(x) - \sum_{|a|\le\omega} \partial^a h(0)\, w_a(x). \tag{4.24}$$

One can prove that every extension t having $\mathrm{sd}(t) = \mathrm{sd}(t^0)$ is a W-extension (4.22) with the projector W given in terms of a family of functions (w_a) (4.23) by (4.24).

For practical computations, the W-extension has essential disadvantages: Firstly, for t^W (given by (4.22) and (4.24)) Lorentz covariance is at least not manifest, since there does not exist any Lorentz covariant $w_a \in \mathcal{D}(\mathbb{R}^k)$. Secondly, to compute t^W explicitly, one needs explicit formulas for the functions (w_a) which makes the computation unhandy.

Due to this and due to the Taylor expansion in $m \ge 0$ of t^0 (4.18), in praxis one is left with the following problem: *Given a distribution $t^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ which scales almost homogeneously with degree $D \in k+\mathbb{N}$, find an extension $t \in \mathcal{D}'(\mathbb{R}^k)$ which scales also almost homogeneously with degree D.* We are going to sketch two techniques solving this problem.

Differential renormalization. The idea is to trace back the case $D \ge k$ to the simple case $D < k$, in which the solution is unique and obtained by the direct extension (Prop. 4.5), in the following way: Write t^0 as a derivative of a distribution $f^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ which scales almost homogeneously with degree $D - l < k$, where $l \in \mathbb{N} \setminus \{0\}$; more precisely

$$t^0 = \mathfrak{D} f^0 \quad \text{with} \quad \mathfrak{D} = \sum_{|a|=l} C_a \partial^a\ ,\ \ C_a \in \mathbb{C}\ , \quad \text{and} \quad (\mathbb{E}_k + D - l)^N f^0 = 0 \tag{4.25}$$

for $N \in \mathbb{N}$ sufficiently large. Let $f \in \mathcal{D}'(\mathbb{R}^k)$ be the direct extension of f^0; it scales also almost homogeneously with the same degree $D - l$ and the same power. Then

$$t := \mathfrak{D} f \tag{4.26}$$

exists in $\mathcal{D}'(\mathbb{R}^k)$ and is an extension of t^0, because for $h \in \mathcal{D}(\mathbb{R}^k \setminus \{0\})$ we get

$$\langle t, h\rangle = \langle \mathfrak{D} f, h\rangle = (-1)^l\, \langle f, \mathfrak{D} h\rangle = (-1)^l\, \langle f^0, \mathfrak{D} h\rangle = \langle \mathfrak{D} f^0, h\rangle = \langle t^0, h\rangle\ , \tag{4.27}$$

and one easily verifies that $(\mathbb{E}_k + D)^N\, t = 0$

In praxis, the difficult step is to find a distribution $f^0 \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ satisfying the conditions (4.25). A general method to solve this problem is not known; however, differential renormalization has been successfully applied to a wealth of concrete examples, see e.g. [10, 17].

Analytic renormalization. [3] Roughly, the idea is to solve the above given extension problem as follows: One introduces a ζ-dependent "regularized" distribution $t^{\zeta\,0} \in \mathcal{D}'(\mathbb{R}^k \setminus \{0\})$ (where $\zeta \in \mathbb{C} \setminus \{0\}$ and $|\zeta|$ sufficiently small), such that $\lim_{\zeta\to 0} t^{\zeta\,0} = t^0$ and that $t^{\zeta\,0}$ scales almost homogeneously with a *non-integer degree D_ζ*. From Prop. 4.5 we know that $t^{\zeta\,0}$ has a unique extension $t^\zeta \in \mathcal{D}'(\mathbb{R}^k)$ which

[3] Analytic renormalization was first applied to x-space Epstein–Glaser renormalization by Hollands [12]. Essentially, we follow [7].

scales almost homogeneously with the same degree D_ζ. The explicit computation of t^ζ is generically much simpler than the computation of a solution of the original extension task, mostly this can be done by means of differential renormalization – this is the gain of the regularization. For the resulting extension t^ζ one then removes the regularization, i.e., one performs the limit $\zeta \to 0$. In order that this limit exists in $\mathcal{D}'(\mathbb{R}^k)$, one has to subtract a suitable local term.

To explain this in detail, first note that almost homogeneous scaling of t^0 implies that the above introduced unique extension t_ω to $\mathcal{D}'_\omega(\mathbb{R}^k)$ (4.20) (where $\omega := D-k$) scales also almost homogeneously.

Definition 4.6 (Analytic regularization)**.** With the given assumptions (see the above formulated problem) and notations, a family of distributions $\{t^\zeta\}_{\zeta\in\Omega\setminus\{0\}}$, $t^\zeta \in \mathcal{D}'(\mathbb{R}^k)$, with $\Omega \subseteq \mathbb{C}$ a neighbourhood of the origin, is called a *regularization* of t^0, if

$$\lim_{\zeta\to 0}\langle t^\zeta, h\rangle = \langle t_\omega, h\rangle \qquad \forall h \in \mathcal{D}_\omega(\mathbb{R}^k)\ , \tag{4.28}$$

and if t^ζ scales almost homogeneously with degree $D_\zeta = D + D_1\,\zeta$ for some constant $D_1 \in \mathbb{C}\setminus\{0\}$. The regularization $\{t^\zeta\}$ is called *analytic*, if for all $h \in \mathcal{D}(\mathbb{R}^k)$ the map

$$\Omega\setminus\{0\} \ni \zeta \longmapsto \langle t^\zeta, h\rangle \tag{4.29}$$

is analytic with a pole of finite order at the origin.

Let $\{t^\zeta\}$ be an analytic regularization of t^0 and let t be an almost homogeneous extension of t^0. As mentioned above, t can be written as a W-extension, that is, there exist functions (w_a) (4.23) such that

$$\langle t,h\rangle = \langle t^W,h\rangle = \langle t_\omega, Wh\rangle = \lim_{\zeta\to 0}\langle t^\zeta, Wh\rangle = \lim_{\zeta\to 0}\Big(\langle t^\zeta,h\rangle - \sum_{|a|\le\omega}\langle t^\zeta, w_a\rangle\,\partial^a h(0)\Big)\ , \tag{4.30}$$

by using (4.22), (4.28) and finally (4.24). In general, the limit of the individual terms on the right-hand side might not exist. However, each term can be expanded in a Laurent series around $\zeta = 0$, and since the overall limit is finite, the principal parts (pp) of these Laurent series must cancel out:

$$\big\langle \mathrm{pp}(t^\zeta), h\big\rangle := \mathrm{pp}\big(\langle t^\zeta,h\rangle\big) = \sum_{|a|\le\omega}\mathrm{pp}\big(\langle t^\zeta, w_a\rangle\big)\ \partial^a h(0)\ ,\quad \forall h\in\mathcal{D}(\mathbb{R}^k)\ . \tag{4.31}$$

We conclude that

$$\mathrm{pp}\big(t^\zeta(x)\big) = \sum_{|a|\le D-k} C_a(\zeta)\,\partial^a\delta(x)\ ,\quad \text{where}\quad C_a(\zeta) = (-1)^{|a|}\,\mathrm{pp}\big(\langle t^\zeta, w_a\rangle\big)\ . \tag{4.32}$$

Proposition 4.7 (Minimal subtraction)**.** (a) *The sum in* (4.32) *runs only over* $|a| = D-k$*, that is, the principal part* $\mathrm{pp}(t^\zeta)$ *is a local distribution which scales homogeneously with degree* D*.*

(b) *The regular part* $\mathrm{rp}(t^\zeta) := t^\zeta - \mathrm{pp}(t^\zeta)$ *defines by*

$$\langle t^{\mathrm{MS}}, h\rangle := \lim_{\zeta\to 0}\ \mathrm{rp}\big(\langle t^\zeta,h\rangle\big)\ ,\quad \forall h\in\mathcal{D}(\mathbb{R}^k)\ , \tag{4.33}$$

a distinguished extension of t^0 *which scales almost homogeneously with degree* D *("minimal subtraction").*

That t^{MS} is an extension of t^0 with $\mathrm{sd}(t^{\mathrm{MS}}) = \mathrm{sd}(t^0)$, can be seen as follows: We compare t^{MS} with the initial extension $t = t^W$. Using (4.30), (4.31) and the definition of t^{MS} (4.33), we obtain

$$\begin{aligned}\langle t,h\rangle &= \lim_{\zeta\to 0}\Bigg(\langle t^\zeta,h\rangle - \sum_{|a|\le\omega}\Big(\mathrm{pp}\big(\langle t^\zeta,w_a\rangle\big) + \mathrm{rp}\big(\langle t^\zeta,w_a\rangle\big)\Big)\ \partial^a h(0)\Bigg)\\ &= \langle t^{\mathrm{MS}},h\rangle - \sum_{|a|\le\omega}\langle t^{\mathrm{MS}},w_a\rangle\ \partial^a h(0)\ ,\quad h\in\mathcal{D}(\mathbb{R}^k)\ .\end{aligned} \tag{4.34}$$

Hence, t^{MS} differs from t by a term of the form $t^{\mathrm{MS}} - t = \sum_{|a|\le\omega} b_a\,\partial^a\delta\ ,\ b_a\in\mathbb{C}$. More involved is the proof of the statements that $\mathrm{pp}(t^\zeta)$ and t^{MS} scale homogenously or almost homogenously, respectively, with degree D.

4.4 Stückelberg–Petermann renormalization group and Main Theorem

The Main Theorem is essentially the following statement: A change $\mathbf{S} \mapsto \hat{\mathbf{S}}$ of the renormalization prescription can equivalently be described by a renormalization of the interaction $F \mapsto Z(F)$:

$$\hat{\mathbf{S}}(F) = \mathbf{S}\big(Z(F)\big) \quad \forall F \in \mathcal{F}_{\mathrm{loc}} \ . \tag{4.35}$$

The definition of the Stückelberg–Petermann renormalization group (RG) is such that the set of all maps $Z\colon \mathcal{F}_{\mathrm{loc}} \longrightarrow \mathcal{F}_{\mathrm{loc}}$ appearing in this relation, when $(\mathbf{S}, \hat{\mathbf{S}})$ runs through all admissible pairs of S-matrices, is precisely the Stückelberg–Petermann RG. Consequently, the definition of the Stückelberg–Petermann RG depends on the set of renormalization conditions for the T-product. For brevity, we use here only Field Independence and Translation covariance.

Definition 4.8. The *Stückelberg–Petermann RG* is the set $\mathcal{R}$ of all maps $Z\colon \mathcal{F}_{\mathrm{loc}}[\![\hbar,\kappa]\!] \longrightarrow \mathcal{F}_{\mathrm{loc}}[\![\hbar,\kappa]\!]$ satisfying the following properties:

(1) *Analyticity*: Z is analytic in the sense that

$$Z(F) = \sum_{n=0}^{\infty} \frac{1}{n!}\, Z^{(n)}(F^{\otimes n}) \ , \quad \text{where} \quad Z^{(n)} := Z^{(n)}(0) \ : \ \mathcal{F}_{\mathrm{loc}}[\![\hbar,\kappa]\!]^{\otimes n} \longrightarrow \mathcal{F}_{\mathrm{loc}}[\![\hbar,\kappa]\!]$$

is the nth derivative of $Z(F)$ at $F = 0$ (i.e., $Z^{(n)}$ is linear, symmetrical in all factors and can be computed by $Z^{(n)}(F^{\otimes n}) = \frac{d^n}{d\lambda^n}\big|_{\lambda=0} Z(\lambda F)$).

(2) *Lowest orders*: $Z^{(0)} \equiv Z(0) = 0, \ \ Z^{(1)} = \mathrm{Id}$.

(3) *Locality, Translation covariance*: For all monomials $A_1, \ldots \ldots, A_n \in \mathcal{P}$ the VEV

$$z^{(n)}(A_1,\ldots,A_n)(x_1 - x_n,\ldots) := \omega_0\Big(Z^{(n)}\big(A_1(x_1),\ldots,A_n(x_n)\big)\Big) \tag{4.36}$$

depends only on the relative coordinates and is of the form

$$z^{(n)}(A_1,\ldots,A_n)(x_1 - x_n,\ldots) = \mathfrak{S}_n \sum_{a\in\mathbb{N}^{d(n-1)}} C^a(A_1,\ldots,A_n)\, \partial^a\delta(x_1 - x_n,\ldots,x_{n-1} - x_n)\,,$$

with constant coefficients $C^a(A_1,\ldots,A_n)$ and $\mathfrak{S}_n$ denotes symmetrization in $(A_1,x_1),\ldots,(A_n,x_n)$.

(4) *Field independence*: $\delta Z/\delta\varphi = 0$.

The distributions $Z^{(n)}\big(A_1(x_1),\ldots,A_n(x_n)\big) \in \mathcal{D}'(\mathbb{M}^n,\mathcal{F}_{\mathrm{loc}})$, $A_1,\ldots,A_n \in \mathcal{P}$ appearing in (4.36) are defined analogously to $T_n\big(A_1(x_1),\ldots,A_n(x_n)\big)$ (see (4.6)), hence, they also satisfy the AWI.

Similarly to the T-product, the Field independence property (4) for Z is equivalent to the validity of the (causal) Wick expansion (4.8) for $Z^{(n)}$, $\forall n \geq 2$.

In the framework of causal perturbation theory, a first version of the Main theorem was given by Popineau and Stora [16]; we present here a more elaborated version.

Theorem 4.9 (Main Theorem)**.** (a) *Given two S-matrices $\mathbf{S}$ and $\widehat{\mathbf{S}}$ both fulfilling the axioms, there exists an analytic map $Z\colon \mathcal{F}_{\mathrm{loc}}[\![\hbar,\kappa]\!] \longrightarrow \mathcal{F}_{\mathrm{loc}}[\![\hbar,\kappa]\!]$ (i.e., Z satisfies the property* (1) *of Definition* 4.8)*, which is uniquely determined by*

$$\widehat{\mathbf{S}} = \mathbf{S} \circ Z \ . \tag{4.37}$$

This Z is an element of the Stückelberg–Petermann RG $\mathcal{R}$.

(b) *Conversely, given an S-matrix $\mathbf{S}$ fulfilling the axioms for the T-product and an arbitrary $Z \in \mathcal{R}$, the composition $\widehat{\mathbf{S}} := \mathbf{S} \circ Z$ also satisfies these axioms.*

A corollary of this Theorem states that $(\mathcal{R}, \circ)$ is indeed a group.

4.5 Interacting fields and the algebraic adiabatic limit

Interacting fields are obtained from the S-matrix by Bogoliubov's definition [2]: for $F \in \mathcal{F}_{\rm loc}$ the formal power series

$$F_S := \frac{\hbar}{i}\,\frac{d}{d\lambda}\Big|_{\lambda=0} \mathbf{S}(S)^{\star-1} \star \mathbf{S}(S+\lambda F) \in \mathcal{F}[\![\hbar,\kappa]\!] \tag{4.38}$$

is the interacting field belonging to the free field F (i.e., $F_S|_{\kappa=0} = F$) and to the interaction S. By using only the basic axioms for T, one proves that the so defined interacting fields satisfy Causality, that is,

$$F_{S+G} = F_S \quad \text{if} \quad \operatorname{supp} G \cap (\operatorname{supp} F + \overline{V}_-) = \emptyset, \tag{4.39}$$

and the Glaser-Lehmann-Zimmermann (GLZ) relation (which plays an important role in Steinmann's inductive construction of the perturbative interacting fields [19])

$$\frac{1}{i\hbar}\big[G_S\,,\,F_S\big]_\star = \frac{d}{d\lambda}\Big|_{\lambda=0}\Big(F_{S+\lambda G} - G_{S+\lambda F}\Big)\;, \tag{4.40}$$

(where $[\,\cdot\,,\,\cdot\,]_\star$ denotes the commutator w.r.t. the star product). Combining these two properties we get spacelike commutativity:

$$[G_S,\,F_S]_\star = 0 \quad \text{if} \quad (x-y)^2 < 0 \quad \text{for all} \quad (x,y) \in \operatorname{supp} G \times \operatorname{supp} F.$$

The validity of the renormalization conditions for T implies corresponding properties for the interacting fields, e.g., unitarity (axiom (6)) implies $(F_S)^* = (F^*)_{S^*}$ and, as the name says, axiom (8) implies

$$(\square + m^2)\varphi(x)_S = (\square + m^2)\varphi(x) + \Big(\frac{\delta S}{\delta\varphi(x)}\Big)_S\;.$$

In addition, axiom 11 ($\hbar$-dependence) implies that, for $S,\,F \sim \hbar^0$, the interacting field F_S is a formal power series in $\hbar$; hence, its limit $\hbar \to 0$ exists and gives the pertinent (perturbative) *classical* interacting field – however, the limit $\lim_{\hbar\to 0} \mathbf{S}(S)$ does not exist.

Algebraic adiabatic limit. To obtain scattering amplitudes contributing to inclusive cross sections, one has to perform the (weak) adiabatic limit $\lim_{g\to 1} \omega_0\big(F \star \mathbf{S}(S(g)) \star G\big)$ for appropriate $G, F \in \mathcal{F}$ describing the in- and out-state, respectively. In contrast, from the interacting fields one can extract observable quantities without performing this limit; to wit, the *local, algebraic* structure of these fields does not depend on the adiabatic switching of the interaction.

To explain this, let $\mathcal{O}$ be an open double cone (i.e., $\mathcal{O} = (x+V_+)\cap(y+V_-)$ for some pair $(x,y) \in M^2$ fulfilling $y \in (x+V_+)$) and let $\mathcal{F}_{\rm loc}(\mathcal{O}) := \{\, F \in \mathcal{F}_{\rm loc} \,\big|\, \operatorname{supp} F \subset \mathcal{O}\,\}$. We introduce the algebra of interacting fields localized in $\mathcal{O}$:

$$\mathcal{A}_{L_{\rm int}}(\mathcal{O}) := \bigvee\nolimits_\star \{\, F_{S(g)} \,\big|\, F \in \mathcal{F}_{\rm loc}(\mathcal{O})\,\} \quad \text{with} \quad g \in \mathcal{G}(\mathcal{O}) := \{\, g \in \mathcal{D}(\mathbb{M},\mathbb{R}) \,\big|\, g|_{\overline{\mathcal{O}}} = 1\,\}, \tag{4.41}$$

where $\bigvee_\star$ means the algebra, under the $\star$-product, generated by members of the indicated set and $S(g)$ is obtained from $L_{\rm int}$ by (4.1). A main problem is that $F_{S(g)}$ depends on the restriction of g to $\mathcal{O} + \overline{V}_-$ (by causality, see (4.39)); but the algebra $\mathcal{A}_{L_{\rm int}}(\mathcal{O})$ should rather be independent of g. This is indeed the case [4].

Theorem 4.10. *As an abstract algebra* (4.41) *is independent of the choice of* $g \in \mathcal{G}(\mathcal{O})$. *Concretely, for any* $g_1, g_2 \in \mathcal{G}(\mathcal{O})$, *there is a unitary*[4] *element* $U_{g_1,g_2} \in \mathcal{F}[\![\kappa]\!]$ *such that*

$$U_{g_1,g_2} \star F_{S(g_1)} \star (U_{g_1,g_2})^{\star-1} = F_{S(g_2)}\,, \quad \textit{for all} \quad F \in \mathcal{F}_{\rm loc}(\mathcal{O})\,. \tag{4.42}$$

The proof uses only causality and unitarity of the S-matrix, i.e., (4.7) and (4.10).

[4]To say that U is *unitary* means that $U^* \star U = U \star U^* = (1,0,0,\dots)$ in $\mathcal{F}[\![\kappa]\!]$.

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