Decomposition theorems for vector valued Hardy Martingales

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Topics

1. Basic Examples

2. Maximal Functions

3. Davis Decomposition

4. Martingale Transforms and Consequences

5. Davis Garsia Inequalities

The main sources

A. Pelczynski, Banach Spaces of analytic functions and absolutely summing operators, (1977)

J. Bourgain. *Embedding* L^1 *to* L^1/H^1, TAMS 278 (1983).

PFXM. *A decomposition for Hardy Martingales,* Indiana Univ. Math. J. 61 (2013) 1801–1816

PFXM. *A decomposition for Hardy Martingales II*, Math. Proc. Cambr. Philos. Soc. 157 (2014) 189–207

PFXM. *A decomposition for Hardy Martingales III,* Preprint.

Complex analytic Hardy Spaces

$f \in L^p(\mathbb{T}, X)$, $\mathbb{T} = \{e^{i\theta} : |\theta| \le \pi\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

The harmonic extension of f to the unit disk

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - |z|^2}{|z - e^{i\alpha}|^2} f(e^{i\alpha}) d\alpha, \qquad z \in \mathbb{D}.$$

Define $f \in H^p(\mathbb{T}, X)$ if $f \in L^p(\mathbb{T}, X)$ and the **harmonic extension of f is analytic** in $\mathbb{D}$.

Hardy Martingales $H^1(\mathbb{T}^{\mathbb{N}}, X)$

$\mathbb{T}^{\mathbb{N}}$ the infinite torus-product with Haar measure $d\mathbb{P}$.

$F_k : \mathbb{T}^{\mathbb{N}} \to \mathbb{C}$ is $\mathcal{F}_k$ measurable iff

$$F_k(x) = F_k(x_1, \ldots, x_k), \quad x = (x_i)_{i=1}^{\infty}$$

Conditional expectation $\mathbb{E}_k F = \mathbb{E}(F|\mathcal{F}_k)$ is integration,

$$\mathbb{E}_k F(x) = \int_{\mathbb{T}^{\mathbb{N}}} F(x_1, \ldots, x_k, w) d\mathbb{P}(w).$$

An $(\mathcal{F}_k)$ martingale $F = (F_k)$ is a **Hardy martingale** if

$$y \to F_k(x_1, \ldots, x_{k-1}, y) \quad \in H^1(\mathbb{T}, X).$$

Martingale differences $\Delta F_k = F_k - F_{k-1}$.

Example: Maurey's embedding.

Fix $\epsilon > 0, w = (w_k) \in \mathbb{T}^{\mathbb{N}}$. Put $\varphi_1(w) = \epsilon w_1$, and

$$\varphi_n(w) = \varphi_{n-1}(w) + \epsilon(1 - |\varphi_{n-1}(w)|)^2 w_n.$$

Then $\lim |\varphi_n| = 1$ and $\varphi = \lim \varphi_n$ is uniformly distributed over $\mathbb{T}$.

For any $f \in H^1(\mathbb{T}, X)$

$$F_n(w) = f(\varphi_n(w)), \qquad w \in \mathbb{T}^{\mathbb{N}}$$

is an integrable Hardy martingale with **uniformly small** increments

$$\sup_{n\in\mathbb{N}} \mathbb{E}(\|F_n\|_X) = \int_{\mathbb{T}} \|f\|_X dm \quad \text{and} \quad \|\Delta F_n\|_X \leq 2\epsilon \int_{\mathbb{T}} \|f\|_X dm.$$

Pointwise estimates for ΔF_n.

Fix $w \in \mathbb{T}^{\mathbb{N}}, \quad n \in \mathbb{N}, \quad z = \varphi_n(w), \quad u = \varphi_{n-1}(w)$

$$\Delta F_n(w) = f(\varphi_n(w)) - f(\varphi_{n-1}(w)).$$

Cauchy integral formula

$$f(z) - f(u) = \int_{\mathbb{T}} \left\{ \frac{\zeta}{\zeta - z} - \frac{\zeta}{\zeta - u} \right\} f(\zeta) dm(\zeta).$$

Triangle inequality

$$\|f(z) - f(u)\|_X \leq \frac{|z-u|}{(1-|u|)(1-|z|)} \int_{\mathbb{T}} \|f\|_X dm$$

Example: Rudin Shapiro Martingales

Fix a complex sequence (c_n) with $\sum_{k=1}^{\infty} |c_k|^2 \leq 1$.

Define recursively: $F_1 = G_1 = 1$ and for $w = (w_n) \in \mathbb{T}^{\mathbb{N}}$

$$F_{m+1}(w) = F_m(w) + \overline{G_m}(w) c_{m+1} w_{m+1},$$

$$G_{m+1}(w) = G_m(w) - \overline{F_m}(w) c_{m+1} w_{m+1}.$$

Pythagoras for (F_m, G_m) and $(\overline{G}_m, -\overline{F}_m)$ gives

$|F_{m+1}(w)|^2 + |G_{m+1}(w)|^2 = (1 + |c_{m+1}|^2)(|F_m(w)|^2 + |G_m(w)|^2).$

and repeat

$$|F_{m+1}(w)|^2 + |G_{m+1}(w)|^2 = \prod_{k=1}^{m+1} (1 + |c_k|^2) 2.$$

Rudin Shapiro Martingales II

$F = (F_n)$ a uniformly bounded Hardy martingale

$$F_n(w) = \sum_{m=1}^{n} \overline{G_m}(w) c_{m+1} w_{m+1}$$

for which the martingale differences reproduce the (c_m).

$$\mathbb{E}_w(\overline{w_m}(F_n(w) - F_{n-1}(w)) = c_{m+1}\mathbb{E}_w\overline{G_m}(w) = c_{m+1}.$$

Rudin Shapiro martingales gives the cotype 2 estimate for L^1/H^1

$$\mathbb{E}_w\| \sum_{m=1}^{n} w_m x_m \|_{L^1/H^1} \geq c \left(\sum \|x_m\|^2_{L^1/H^1} \right)^{1/2}.$$

when the x_m have **well separated** Fourier spectrum.

The Origins I

A. Pelczynski posed **famous problems** in "Banach Spaces of analytic functions and absolutely summing operators, (1977)."

Does H^1 have an unconditional basis?

Does there exist a subspace of L^1/H^1 isomorphic to L^1?

Does L^1/H^1 have cotype 2?

Are the spaces $A(D^n)$ and $A(D^m)$ not isomorphic when $n \neq m$?

The Origins II

Hardy martingales gave rise to the operators by which **Maurey** proved that H^1 has an unconditional basis;

and to the isomorphic invariants by which **Bourgain** proved the dimension conjecture, that L^1/H^1 has cotype 2 and that L^1 embeds into L^1/H^1.

Pisier's L^1/H^1 valued Riesz products form a Hardy martingale that is strongly intertwined with Bourgain's solutions and played an important role for the work of **Garling, Tomczak-Jaegermann, W. Davis** on Hardy martingale cotype and complex uniformly convex renormings of Banach spaces.

Garling's Maximal Functions estimate I .

For any X valued Hardy martingale $F = (F_k)$

$$\mathbb{E}(\sup_{k\in\mathbb{N}} \|F_k\|) \leq e \sup_{k\in\mathbb{N}} \mathbb{E}(\|F_k\|).$$

For any $0 < \alpha \leq 1$, $(\|F_{k-1}\|_X^\alpha)$ is a non- negative sub-martingale

$$\|F_{k-1}\|_X^\alpha \leq \mathbb{E}_{k-1}(\|F_k\|_X^\alpha).$$

Brownian Motion

Let Ω denote the Wiener space $\{z_t : t > 0\}$ denotes complex Brownian Motion started at $0 \in \mathbb{D}$, and define

$$\tau = \inf\{t > 0 : |z_t| > 1\}.$$

For $f \in H^1(\mathbb{T}, X)$, $0 < \alpha < 1$ and $0 < t < \tau$,

$$\|f(z_t)\|_X^\alpha \le \mathbb{E}(\|f(z_\tau)\|_X^\alpha | \mathcal{F}_t),$$

and

$$\mathbb{E}(\sup_{t<\tau} \|f(z_t)\|_X) \le e \sup_{t<\tau} \mathbb{E}(\|f(z_t)\|_X),$$

where the integration is over the Wiener space Ω.

Garling's Maximal Functions estimate II .

$$\Sigma = \mathbb{T}^{k-1} \times \Omega, \quad x \in \mathbb{T}^{k-1}, \quad \omega \in \Omega.$$

For any X valued Hardy martingale $F = (F_k)$, the maximal function

$$F_k^*(x, \omega) = \max \left\{ \max_{m \le k-1} \|F_m(x)\|_X, \sup_{t < \tau} \|F_k(x, z_t(\omega))\|_X \right\}$$

satisfies

$$\mathbb{E}_\Sigma(F_k^*) \le e^2 \mathbb{E}(\|F_k\|_X).$$

Davies Decomposition I.

Let $F=(F_k)_{k=1}^n$ be an X valued Hardy martingale.

With the **maximal function estimates,** the standard B. Davies decomposition and **Doob's projection** we obtain a splitting of F into **Hardy martingales**

$$F=G+B$$

satisfying

$$\|\Delta G_k\|_X \leq \max_{m\leq k-1} \|F_m\|_X,$$

and

$$\mathbb{E}(\sum_{k=1}^n \|\Delta B_k\|_X) \leq C\mathbb{E}(\|F\|_X).$$

Sketch of Proof. Fix $x \in \mathbb{T}^{k-1}, v \in \mathbb{T}$. Define

$$f(v) = \Delta F_k(x, v), \qquad \lambda = \max_{m \le k-1} \|F_m(x)\|_X.$$

and

$$\rho = \inf\{t < \tau : \|f(z_t)\|_X > 2\lambda\}, \quad R_k = f(z_\rho), \quad S_k = f(z_\rho) - f(z_\tau).$$

• $F_k^*(x,\omega) \le 4(F_k^*(x,\omega) - F_{k-1}^*(x,\omega)), \quad \omega \in A = \{\rho < \tau\}.$

• $\|S_k\|_X \le 2F_k^* \le 8(F_k^* - F_{k-1}^*), \quad \sum_{k=1}^n \|S_k\|_X \le 8F_n^*.$

• By choice of the stopping time ρ, $\|R_k\| \le 2\lambda$.

Doob's projection generates the analytic functions

$$\Delta B_k = \mathbb{E}(S_k | z_\tau = z), \quad \Delta G_k = \mathbb{E}(R_k | z_\tau = z), \qquad z \in \mathbb{T}.$$

Improved Davies Decomposition (PFXM) A Hardy martingale $F = (F_k)$ can be decomposed into Hardy martingales as $F = G + B$ such that

$$\|\Delta G_k\|_X \leq C\|F_{k-1}\|_X,$$

and

$$\mathbb{E}(\sum_{k=1}^{\infty} \|\Delta B_k\|_X) \leq C\mathbb{E}(\|F\|_X).$$

Lemma

If $h \in H_0^1(\mathbb{T}, X)$, $z \in X$ there exists $g \in H_0^\infty(\mathbb{T}, X)$ with

$$\|g(\zeta)\|_X \leq C_0\|z\|_X, \qquad \zeta \in \mathbb{T}$$

and

$$\|z\|_X + \frac{1}{8}\int_{\mathbb{T}} \|h - g\|_X dm \leq \int_{\mathbb{T}} \|z + h\|_X dm.$$

Sketch of Proof. Fix $x \in \mathbb{T}^{k-1}$. Put

$$h(y) = \Delta F_k(x, y) \quad \text{and} \quad z = F_{k-1}(x).$$

Lemma yields a bounded analytic g with

$$\|z\|_X + 1/8 \int_{\mathbb{T}} \|h - g\|_X dm \leq \int_{\mathbb{T}} \|z + h\|_X dm; \quad \|g(\zeta)\|_X \leq C_0 \|z\|_X.$$

Define

$$\Delta G_k(x, y) = g(y), \quad \Delta B_k(x, y) = h(y) - g(y).$$

Then

$$\|F_{k-1}\|_X + 1/8 \mathbb{E}_{k-1}(\|\Delta B_k\|_X) \leq \mathbb{E}_{k-1}(\|F_k\|_X).$$

Integrate and take the sum,

$$\sum \mathbb{E}(\|\Delta B_k\|_X) \leq 4 \sup \mathbb{E}(\|F_k\|_X).$$

The strong Davis decomposition yields vector valued Davis and Garsia Inequalities. At this point we need to make an assumption on the Banach space X:

Let $q \geq 2$. A Banach space X satisfies the hypothesis $\mathcal{H}(q)$, if for each $M \geq 1$ there exists $\delta = \delta(M) > 0$ such that for any $x \in X$ with $\|x\| = 1$ and $g \in H_0^\infty(\mathbb{T}, X)$ with $\|g\|_\infty \leq M$,

$$\int_{\mathbb{T}} \|z + g\|_X dm \geq (1 + \delta \int_{\mathbb{T}} \|g\|_X^q dm)^{1/q}. \qquad (1)$$

Condition (1) is required for uniformly bounded analytic functions g, and $\delta = \delta(M) > 0$ is allowed to depend on the uniform estimates $\|g\|_\infty \leq M$. When $X = \mathbb{C}$, the hypothesis "$\mathcal{H}(q)$" hold true with $q = 2$.

Satz 1 *Let $q \geq 2$. Let X be a Banach satisfying $\mathcal{H}(q)$. Any X valued Hardy martingale $F = (F_k)$ can be decomposed into the sum of X valued Hardy martingales $F = G + B$ such that*

$$\mathbb{E}(\sum_{k=1}^{\infty} \mathbb{E}_{k-1}(\|\Delta G_k\|_X^q))^{1/q} + \mathbb{E}(\sum_{k=1}^{\infty} \|\Delta B_k\|_X) \leq A_q \mathbb{E}(\|F\|_X).$$

Satz 2 *Let $q \geq 2$. Let X be a Banach satisfying $\mathcal{H}(q)$. There exists $M > 0$ $\delta_q > 0$ such that for any $h \in H_0^1(\mathbb{T}, X)$ and $z \in X$ there exists $g \in H_0^\infty(\mathbb{T}, X)$ satisfying*

$$\|g(\zeta)\|_X \leq M\|z\|_X, \qquad \zeta \in \mathbb{T}, \tag{2}$$

and

$$\int_{\mathbb{T}} \|z+h\|_X dm \geq \left(\|z\|_X^q + \delta_q \int_{\mathbb{T}} \|g\|_X^q dm\right)^{1/q} + \frac{1}{16}\int_{\mathbb{T}} \|h-g\|_X dm. \tag{3}$$

The strong Davis decomposition and hypothesis "$\mathcal{H}(q)$" gives a decomposition into Hardy martingales as $F = G + B$ such that

$$\mathbb{E}(\sum_{k=1}^{\infty}(\mathbb{E}_{k-1}\|\Delta G_k\|_X^q))^{1/q}+\mathbb{E}(\sum_{k=1}^{\infty}\|\Delta B_k\|_X)\leq A_q\mathbb{E}(\|F\|_X).$$

If we replace hypothesis "$\mathcal{H}(q)$" by the weaker hypothesis

$$\int_{\mathbb{T}}\|z+g\|_X dm \geq (1+\delta(\int_{\mathbb{T}}\|g\|_X dm)^q)^{1/q}, \qquad (4)$$

then we are able to prove that the strong Davis decomposition yields

$$\mathbb{E}(\sum_{k=1}^{\infty}(\mathbb{E}_{k-1}\|\Delta G_k\|_X)^q)^{1/q}+\mathbb{E}(\sum_{k=1}^{\infty}\|\Delta B_k\|_X)\leq A_q\mathbb{E}(\|F\|_X).$$

We note that for scalar valued analytic functions, when $X = \mathbb{C}$, the hypothesis "$\mathcal{H}(q)$" hold true with $q = 2$.

Recall the Iteration Lemma: If

$$\mathbb{E}(M_{k-1}^2 + v_k^2)^{1/2} + \mathbb{E}w_k \le \mathbb{E}M_k \qquad \text{for } 1 \le k \le n, \tag{5}$$

then

$$\mathbb{E}(\sum_{k=1}^n v_k^2)^{1/2} + \mathbb{E}\sum_{k=1}^n w_k \le 2\sqrt{\mathbb{E}M_n \mathbb{E}\max_{k\le n} M_k} \tag{6}$$

(All random variables are non-negative, integrable)

For $0 \le s \le 1$, and $A, B \ge 0$,

$$Bs \le s^2 A + (A^2 + B^2)^{1/2} - A. \tag{7}$$

Let $0 \leq \epsilon \leq 1$. Choose bounded functions $0 \leq s_k \leq \epsilon$ with $\sum_{k=1}^n s_k^2 \leq \epsilon^2$ to linearize the square function.

$$v_k s_k \leq s_k^2 M_{k-1} + (M_{k-1}^2 + v_k^2)^{1/2} - M_{k-1} \tag{8}$$

Integrate

$$\mathbb{E}(v_k s_k) \leq \mathbb{E}(s_k^2 M_{k-1}) + \mathbb{E}(M_{k-1}^2 + v_k^2)^{1/2} - \mathbb{E}M_{k-1}.$$

Use hypothesis for $\mathbb{E}(M_{k-1}^2 + v_k^2)^{1/2}$.

$$\mathbb{E}(v_k s_k) \leq \mathbb{E}(s_k^2 M_{k-1}) + \mathbb{E}M_k - \mathbb{E}M_{k-1} - \mathbb{E}w_k.$$

Sum over $k \leq n$

$$\mathbb{E}(\sum_{k=1}^{n} v_k s_k) + \sum_{k=1}^{n} \mathbb{E} w_k \leq \mathbb{E} M_n + \mathbb{E}(\sum_{k=1}^{n} s_k^2 M_{k-1}) \\ \leq \mathbb{E} M_n + \epsilon^2 \mathbb{E} \max_{k \leq n} M_{k-1} \tag{9}$$

Since $\sum_{k=1}^{n} s_k^2 \leq \epsilon^2$,

$$\epsilon \mathbb{E}(\sum_{k=1}^{n} v_k^2)^{1/2} + \sum_{k=1}^{n} \mathbb{E} w_k \leq \mathbb{E} M_n + \epsilon^2 \mathbb{E} \max_{k \leq n} M_{k-1}.$$

Divide by $0 < \epsilon \leq 1$, with

$$\epsilon^2 = (\mathbb{E} M_n)(\mathbb{E} \max_{k \leq n} M_k)^{-1}.$$