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To the memory of Jerzy Urbanowicz

Elements in the Milnor group

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In this lecture we present some results of Urbanowicz given in his paper [U] concerning elements in the Milnor group $K_2(\Lambda)$ of a ring Λ with 1. Basic properties of this group are given in the books by Milnor [M] and Steinberg [St].

Main result presented here (Theorem 6.1) concerns presentation of elements of order n in $K_2(\Lambda)$, where Λ is a field, as products of transfers of cyclotomic elements in the Milnor group of some extensions of Λ.

1. An example.

Example 1. The symmetric group S_3 can be defined by generators $\sigma = (1,2)$ and $\tau = (1,2,3)$, and relations $\sigma^2 = \tau^3 = \sigma\tau\sigma\tau = 1$.

Suppose that we know only the relations $\sigma^2 = \tau^3 = 1$, and we are looking for further relations defining the group S_3.

Equivalently, we consider the group

$$\mathrm{St} = \langle x, y \ : \ x^2 = y^3 = 1 \rangle,$$

and canonical surjective homomorphism

$$\varphi \ : \ \mathrm{St} \to S_3, \qquad \varphi(x) = \sigma, \ \varphi(y) = \tau.$$

Then the nontrivial elements of the group $K_2 := \ker\varphi$ give relations in S_3 independent of $\sigma^2 = \tau^3 = 1$.

Thus we have an exact sequence

$$1 \longrightarrow K_2 \longrightarrow \mathrm{St} \xrightarrow{\ \varphi\ } S_3 \longrightarrow 1.$$

We shall describe the group K_2 and find its generators.

The group St is the free product of the cyclic groups $\langle x \rangle$ of order 2, and $\langle y \rangle$ of order 3. Hence every element of St can be written uniquely in the form

$$xy^{\alpha_1}xy^{\alpha_2}... \qquad \text{or} \qquad y^{\alpha_1}xy^{\alpha_2}x...,$$

where every α_j equals 1 or 2.

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Proposition 1. *The group K_2 is the free group with two generators $u = xyxy$ and $v = yxyx$.*

Proof. We verify directly that the elements

$$x, y, xy, xy^2, xyx, yxy, yxy^2$$

do not belong to $\ker\varphi$. Then the same property have their inverses:

$$x, y^2, y^2x, yx, xy^2x, y^2xy^2, yxy^2.$$

Next we verify that elements $u = xyxy$ and $v = yxyx$ belong to $\ker\varphi$. Then also $u^{-1} = y^2xy^2x$ and $v^{-1} = xy^2xy^2$ belong to $\ker\varphi$.

We shall prove inductively (with respect to the number of factors x or y of an element) that every element of $K_2 = \ker\varphi$ belongs to the group $\langle u, v\rangle$ generated by u and v.

Let $w \in K_2,\ w \neq 1,$ Assume that

$$w = xy^{\alpha_1}xy^{\alpha_2}x....$$

Then

$$w = (xy^{\alpha_1}xy^{\alpha_1})(y^{\alpha_2-\alpha_1}x...),$$

and the first factor in brackets equals u or v^{-1} and the second factor is shorter than w. Hence, by the inductive assumption, the second factor belongs to the group $\langle u, v\rangle$. Then $w \in \langle u, v\rangle$.

If $w = y^{\alpha_1}xy^{\alpha_2}xy^{\alpha_3}...$, then we consider two cases:

1^0 $\alpha_1 = \alpha_2$. Then

$$w = (y^{\alpha_1}xy^{\alpha_1}x)(y^{\alpha_3}...),$$

and the first factor in brackets equals v or u^{-1}, and the second factor is shorter than w. Similarly as above we conclude that $w \in \langle u, v\rangle$.

2^0 $\alpha_1 \neq \alpha_2$. then $\alpha_2 \equiv 2\alpha_1 \pmod 3$. Hence

$$w = (y^{\alpha_1}xy^{\alpha_1}x)(xy^{\alpha_1}xy^{\alpha_3}...).$$

The first factor in brackets equals v or u^{-1}, and the second factor is shorter that w.

Therefore $w \in \langle u, v\rangle$, by the same argument as above.

Thus we have proved that $K_2 = \langle u, v\rangle$.

To prove that u, v are free generators of K_2 it is sufficient to prove that every element $w \in K_2$ can be writen uniquely as the product of factors u, v, u^{-1}, v^{-1} (with no cancellation between them).

Let us observe that in all possible products of two factors:

$$uu,\ uv,\ uv^{-1},\ vu,\ vv,\ vu^{-1},\ u^{-1}u^{-1},\ u^{-1}v,\ u^{-1}v^{-1},\ v^{-1}u,\ v^{-1}u^{-1},\ v^{-1}v^{-1}$$

at most one cancellation holds, namely if the first factor ends with x, and the second begins with x. Then there is no further cancellation.

Therefore after all cancellations the first three terms of the product of any elements u, v, u^{-1}, v^{-1} are the same as the first three terms of the first factor of the product.

Since all the elements

$$u = xyxy,\ v = yxyx,\ u^{-1} = y^2xy^2x,\ v^{-1} = xy^2xy^2$$

have three first terms different, we conclude that the first factor of an element $w \in K_2$ is determined uniquely. Then we proceed inductively, and conclude that the representation of w as the product of elements u, v, u^{-1} and v^{-1} is unique. □

Let us observe that $v = yxyx = x(xyxy)x^{-1} = xux^{-1}$. Thus the subgroup K_2 with two generators u, v considered as a normal divisor of St has only one generator, e.g. u. Elements in K_2, which are conjugate in St give the same relation in S_3.

Consequently the complete set of relations in S_3 is $\sigma^2 = \tau^3 = \sigma\tau\sigma\tau = 1$.

This example can be generalized as follows. We have a group G with a set of generators $A = (g_j)_{j\in J}$ and some set R of relations satisfied by these generators. It may happen that the set R does not define the group G, some further relations are necessary.

We are going to extend the set R to get a set of relations defining the group G.

We consider the group St with generators $(x_j)_{j\in J}$, defined by relations R with every g_j replaced by x_j. There is a canonical surjective homomorphism $\varphi : \mathrm{St} \to G$, satisfying $\varphi(x_j) = g_j$ for $j \in J$. Denote by K_2 the kernel of φ. Let R' be a set of generators of the group K_2, as a normal divisor of the group St. Then replacing every x_j by g_j in every relation in R' we get the set of relations $R \cup R'$ which defines the group G.

In general it is a nontrivial problem to determine a set of generators of the group K_2 in this situation.

2. The group of elementary matrices.

Let Λ be an associative ring with 1, but not necessarily commutative. For $n \geq 2$ and $1 \leq i, j \leq n,\ i \neq j$, let e_{ij} be the $n \times n$ matrix with 1 on the place (i, j) and 0 on all other places. For $\lambda \in \Lambda$ let $e_{ij}(\lambda) = I + \lambda e_{ij}$, where I is the $n \times n$ unit matrix. The group $E_n(\Lambda)$ generated by all matrices $e_{ij}(\lambda)$ we call the group of elementary matrices, it is a subgroup of $\mathrm{SL}_n(\Lambda)$.

It can be verified that the following relations hold, where $\lambda, \mu \in \Lambda$:

$$\begin{aligned} &e_{ij}(\lambda)\, e_{ij}(\mu) = e_{ij}(\lambda + \mu), \\ &[e_{ij}(\lambda), e_{kl}(\mu)] = \begin{cases} e_{il}(\lambda\mu), & \text{if} \quad j = k,\ i \neq l, \\ 1, & \text{if} \quad j \neq k,\ i \neq l. \end{cases} \end{aligned} \tag{1}$$

Problem. Do the relations (1) define the group $E_n(\Lambda)$, or some further relations are necessary ?

The answer depends on the ring Λ. E.g. it is positive for $\Lambda = \mathbb{Z}[i]$, and negative for $\Lambda = \mathbb{Z}$.

To investigate this problem we proceed similarly as in Example 1. First we define the (Steinberg) group $\mathrm{St}_n(\Lambda)$ by generators $x_{ij}(\lambda),\ 1 \le i,j \le n,\ i \ne j,\ \lambda \in \Lambda$, and relations analogous to (1) with $\lambda, \mu \in \Lambda$:

$$\begin{aligned} &x_{ij}(\lambda)\, x_{ij}(\mu) = x_{ij}(\lambda+\mu), \\ &[x_{ij}(\lambda), x_{kl}(\mu)] = \begin{cases} x_{il}(\lambda\mu), & \text{if} \quad j = k,\ i \ne l, \\ 1, & \text{if} \quad j \ne k,\ i \ne l. \end{cases} \end{aligned} \tag{2}$$

Then we consider the canonical surjective homomorphism

$$\varphi_n \ :\ \mathrm{St}_n(\Lambda) \to E_n(\Lambda), \qquad \varphi_n(x_{ij}(\lambda)) = e_{ij}(\lambda).$$

The generators of the kernel $K_{2,n}(\Lambda) := \ker(\varphi_n)$ give missing relations for the group $E_n(\Lambda)$.

For technical reasons it is more convenient not to fix the value of n, but consider $n \to \infty$. We have canonical homomorphisms

$$E_n(\Lambda) \to E_{n+1}(\Lambda), \quad e_{ij}(\lambda) \mapsto e_{ij}(\lambda) \quad \text{for} \quad 1 \le i,j \le n, i \ne j, \lambda \in \Lambda,$$

and

$$\mathrm{St}_n(\Lambda) \to \mathrm{St}_{n+1}(\Lambda), \quad x_{ij}(\lambda) \mapsto x_{ij}(\lambda) \quad \text{for} \quad 1 \le i,j \le n, i \ne j, \lambda \in \Lambda,$$

Then we get a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_{2,n}(\Lambda) & \longrightarrow & \mathrm{St}_n(\Lambda) & \xrightarrow{\varphi_n} & E_n(\Lambda) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K_{2,n+1}(\Lambda) & \longrightarrow & \mathrm{St}_{n+1}(\Lambda) & \xrightarrow{\varphi_{n+1}} & E_{n+1}(\Lambda) & \longrightarrow & 1 \end{array}$$

Taking direct limits

$$K_2(\Lambda) = \varinjlim K_{2,n}(\Lambda), \quad \mathrm{St}\,(\Lambda) = \varinjlim \mathrm{St}_n(\Lambda), \quad E(\Lambda) = \varinjlim \mathrm{St}_n(\Lambda),$$

we get the exact sequence

$$1 \longrightarrow K_2(\Lambda) \longrightarrow \mathrm{St}\,(\Lambda) \xrightarrow{\varphi} E(\Lambda) \longrightarrow 1.$$

The group $E(\Lambda)$ is generated by elements $e_{ij}(\lambda)$, $i \ne j$, $i,j \in \mathbb{N}$, $\lambda \in \Lambda$ which satisfy relations (1) for all values $n \in \mathbb{N}$. Then generators of $K_2(\Lambda)$ give relations which should be added to relations (1) to get the set of relations defining the group $E(\Lambda)$.

It follows that for a given ring Λ it is important to determine the group $K_2(\Lambda)$, or at least to find some of its nontrivial elements.

3. Some elements of $K_2(\Lambda)$.

Example 2. Let us consider the following element of $E_2(\Lambda)$.

$$a = e_{12}(1)e_{21}(-1)e_{12}(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is the matrix of rotation by $\pi/2$. Therefore $a^4 = 1$. It follows that

$$b := \Big(x_{12}(1)x_{21}(-1)x_{12}(1)\Big)^4 \in K_2(\Lambda).$$

3.1. Steinberg symbols.

Generalizing Example 2 we define for an invertible element $u \in \Lambda^*$ an element in $\mathrm{St}\,(\Lambda)$ as follows:

Put $w_{ij}(u) := x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$, and $h_{ij}(u) := w_{ij}(u)w_{ij}(-1)$. Next for $u, v \in \Lambda^*$ satisfying $uv = vu$ we define the Steinberg symbol

$$\{u, v\}_{ij} := h_{ij}(uv)h_{ij}(u)^{-1}h_{ij}(v)^{-1}.$$

The corresponding matrices are (we assume for simplicity that $i = 1, j = 2$) : $\varphi(w_{12}(u)) = \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}$ and $\varphi(h_{12}(u)) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$. Hence

$$\varphi(\{u, v\}_{12}) = \begin{pmatrix} uv & 0 \\ 0 & (uv)^{-1} \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}^{-1} \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}^{-1} = I,$$

since u and v commute.

Therefore $\{u, v\}_{ij} \in K_2(\Lambda)$. It can be verified that $\{u, v\}_{ij}$ does not depend on i, j, so we omit the indices i, j. We call $\{u, v\}$ the Steinberg symbol. It satisfies the following relations:

$$\begin{aligned} \{v, u\} &= \{u, v\}^{-1}, \\ \{u_1u_2, v\} &= \{u_1, v\}\{u_2, v\}, \\ \{u, v_1v_2\} &= \{u, v_1\}\{u, v_2\}, \\ \{u, v\} &= 1 \quad \text{if} \quad u + v = 0 \text{ or } 1. \end{aligned}$$

Theorem 3.1 (H. Matsumoto). *If Λ is a field then $K_2(\Lambda)$ is generated by Steinberg symbols.*

More precisely, $K_2(\Lambda) = (\Lambda^* \otimes \Lambda^*)/I$, where I is the subgroup of $\Lambda^* \otimes \Lambda^*$ generated by all elements of the form $u \otimes v$, with $u + v = 1$.

For the ring of algebraic integers $\mathcal{O}_F$ of a real quadratic number field $F = \mathbb{Q}(\sqrt{d})$, $d > 0$ squarefree, the group $K_2(\mathcal{O}_F)$ is generated by symbols iff $d = 2, 5$ or 13.

Therefore the Steinberg symbols do not suffice to describe all elements of $K_2(\mathcal{O}_F)$, in general.

For these three fields the group $K_2(\mathcal{O}_F) = \mathbb{Z}/2 \times \mathbb{Z}/2$ is generated by symbols $\{-1,-1\}$ and $\{-1,\varepsilon\}$, where ε is the fundamental unit of $\mathcal{O}_F$.

3.2. Dennis-Stein symbols.

Now let $a, b \in \Lambda$ satisfy $u := 1 - ab \in \Lambda^*$. Define the following element in $\mathrm{St}\,(\Lambda)$:

$$\langle a, b\rangle_{ij} := x_{ji}(-b/u)x_{ij}(-a)x_{ji}(b)x_{ij}(a/u)h_{ij}(u)^{-1}.$$

Then (taking $i = 1, j = 2$) :

$$\begin{aligned}
&\varphi(x_{ji}(-b/u)x_{ij}(-a)x_{ji}(b)x_{ij}(a/u)) \\
&\qquad = \begin{pmatrix} 1 & 0 \\ -b/u & 1 \end{pmatrix}\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} 1 & a/u \\ 0 & 1 \end{pmatrix} \\
&\qquad = \begin{pmatrix} 1 & 0 \\ -b/u & 1 \end{pmatrix}\begin{pmatrix} u & -a \\ b & 1 \end{pmatrix}\begin{pmatrix} 1 & a/u \\ 0 & 1 \end{pmatrix} \\
&\qquad = \begin{pmatrix} u & -a \\ 0 & 1+ab/u \end{pmatrix}\begin{pmatrix} 1 & a/u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1/u \end{pmatrix}
\end{aligned}$$

and

$$\varphi(h_{12}(u)) = \begin{pmatrix} u & 0 \\ 0 & 1/u \end{pmatrix}.$$

Consequently $\langle a, b\rangle_{ij} \in K_2(\Lambda)$.

It can be proved that $\langle a, b\rangle_{ij}$ does not depend on i, j, therefore we omit these indices.

We call $\langle a, b\rangle$ the Dennis-Stein symbol. It has the following properties:

$$\begin{aligned}
\langle a, b\rangle &= \langle b, a\rangle^{-1} \\
\langle a, b\rangle\langle a, c\rangle &= \langle a, b + c - abc\rangle \\
\langle a, bc\rangle &= \langle ab, c\rangle\langle ac, b\rangle \\
\langle a, b\rangle &= \{a, 1 - ab\} \quad \text{if} \quad a \in \Lambda^*
\end{aligned}$$

Examples (see [G]).

Let $F = \mathbb{Q}(\sqrt{d})$, $\mathcal{O}_F$ is the ring of algebraic integers in F.

For $d = 3$ we have $K_2(\mathcal{O}_F) = \mathbb{Z}/2 \times \mathbb{Z}/2$ is generated by symbols $\{-1,-1\}$ and $\langle 1 + \sqrt{3}, 1 - \sqrt{3}\rangle$.

For $d = 6$ we have $K_2(\mathcal{O}_F) = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/3$ is generated by symbols $\{-1,-1\}$, $\langle 2 + \sqrt{6}, 2 - \sqrt{6}\rangle$ and $\langle 2 + \sqrt{6}, 2\rangle$ (of order 3).

For $d = 17$ we have $K_2(\mathcal{O}_F) = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ is generated by $\{-1,-1\}$, $\{-1,\varepsilon\}$ and $\langle (3 + \sqrt{17})/2, (3 - \sqrt{17})/2\rangle$.

3.3. Cyclotomic elements.

Let

$$\Phi_n(x) = \prod_{\substack{j=1 \\ (j,n)=1}}^{n} (1 - \zeta_n^j x)$$

be the n-th cyclotomic polynomial. Equivalently,

$$\Phi_1(x) = 1 - x, \qquad \Phi_n(x) = \prod_{\substack{j=1 \\ (j,n)=1}}^{n} (x - \zeta_n^j) \quad \text{for} \quad n \geq 2.$$

Then

$$1 - x^n = \prod_{d \mid n} \Phi_d(x). \tag{3}$$

For $a \in \Lambda^*$, such that $\Phi_n(a) \in \Lambda^*$, we define the cyclotomic element

$$c_n(a) := \{a, \Phi_n(a)\} \in K_2(\Lambda).$$

Lemma 3.1. $c_n(a)^n = 1$.

Proof. We proceed by induction on n. For $n = 1$ we have $\Phi_1(x) = 1 - x$, hence $c_1(a) = \{a, \Phi_1(a)\} = \{a, 1 - a\} = 1$, provided $a \neq 1$.

By (3) we have

$$\begin{aligned} 1 = \{a^n, 1 - a^n\} &= \prod_{d|n} \{a^n, \Phi_d(a)\} = \prod_{d|n} \{a^d, \Phi_d(a)\}^{n/d} \\ &= \{a, \Phi_n(a)\}^n \cdot \prod_{\substack{d|n \\ d<n}} \left(\{a, \Phi_d(a)\}^d\right)^{n/d}. \end{aligned} \tag{4}$$

By the inductive assumption, $\{a, \Phi_d(a)\}^d = 1$ for $d < n$. Then from (4) we conclude that $\{a, \Phi_n(a)\}^n = 1$. □

Urbanowicz proved (see [U]) that if $n = 3$ and Λ is a field of characteristic $\neq 3$, then every element of order 3 in $K_2(\Lambda)$ is a cyclotomic element $c_3(a) = \{a, a^2 + a + 1\}$ for some $a \in \Lambda$ such that $a^2 + a + 1 \neq 0$.

Problem. Is every element of order n in $K_2(\Lambda)$ a cyclotomic element $c_n(a)$ for some a in Λ^*, or it is the product of cyclotomic elements ?

Some partial results for Λ a field and $n = 2, 4, 6$ and 12 are given in [B82]. For $n > 3$ the answer is negative for many rings Λ, see e.g. [B07], [CXQ], [Guo], [Q94], [Q99], [Q07], [X02], [X07], [XQ01a], [XQ01b], [XQ02], [XQ03], [XM], [XW], [ZL].

3.4. Transfer symbols.

Urbanowicz defined in [U] some elements $\langle U, V \rangle \in K_2(\Lambda)$ called the transfer symbols, where U, V are invertible matrices over Λ satisfying $UV = VU$. They generalize the Steinberg symbols and cyclotomic elements. In particular $\langle U, \Phi_n(U) \rangle^n = 1$ holds, see below.

4. The change of rings.

The following general theorem holds

Theorem 4.1. *Let Γ, Λ be rings with 1. Then for every homomorphism $\Psi : E(\Gamma) \to E(\Lambda)$ there is a unique homomorphism $\Psi_S : \mathrm{St}\,(\Gamma) \to \mathrm{St}\,(\Lambda)$ such that the following diagram commutes:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(\Gamma) & \longrightarrow & \mathrm{St}\,(\Gamma) & \longrightarrow & E(\Gamma) & \longrightarrow & 1 \\ & & \big\downarrow \Psi'_S & & \big\downarrow \Psi_S & & \big\downarrow \Psi & & \\ 1 & \longrightarrow & K_2(\Lambda) & \longrightarrow & \mathrm{St}\,(\Lambda) & \longrightarrow & E(\Lambda) & \longrightarrow & 1 \end{array} \tag{5}$$

where $\Psi'_S = \Psi_S | K_2(\Gamma)$.

We postpone the proof to Section 7 (Theorem 7.3). We apply this theorem to some special rings Γ and Λ, and give direct proofs in these particular cases.

Let $r \geq 1$ and let $\Gamma = M_r(\Lambda)$ be the ring of $r \times r$ matrices over Λ. We define a homomorphism $\Psi : E(M_r(\Lambda)) \to E(\Lambda)$ as follows.

For every $n \geq 1$ there is a canonical isomorphism of rings

$$\Psi : M_n(M_r(\Lambda)) \to M_{nr}(\Lambda).$$

Namely, it is sufficient in every matrix $A \in M_n(M_r(\Lambda))$, $A = (A_{ij})_{1 \leq i,j \leq n}$, where $A_{ij} \in M_r(\Lambda)$, to replace every matrix A_{ij} by the table of its elements. Then we get a matrix in $M_{nr}(\Lambda)$.

This mapping is 1–1 and it is a ring homomorphism.

Lemma 4.1. *The mapping Ψ defined above maps $E(M_r(\Lambda))$ into $E(\Lambda)$.*

Proof. For a matrix $A = (a_{st})_{1 \leq s,t \leq r} \in M_r(\Lambda)$ let

$$e_{ij}(A) = \begin{pmatrix} I & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \\ .. & .. & \cdots & I & \cdots & A & \cdots & 0 \\ \\ .. & .. & \cdots & 0 & \cdots & I & \cdots & 0 \\ \\ .. & .. & \cdots & 0 & \cdots & 0 & \cdots & I \end{pmatrix}_{n \times n} \in M_n(M_r(\Lambda))$$

be a generator of the group $E_n(M_r(\Lambda))$, where $1 \leq i, j \leq n$, $i \neq j$. Then

$$\Psi(e_{ij}(A)) = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & & \cdots & & \cdots \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & & \cdots & & \cdots \\ & & & & & & a_{11} & a_{12} & \cdots & a_{1r} & \\ \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & & & & \cdots \\ & & & & & & a_{r1} & a_{r2} & \cdots & a_{rr} & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & & 1 \end{pmatrix}_{nr \times nr}$$

It is sufficient to prove that $\Psi(e_{ij}(A)) \in E_{nr}(\Lambda)$.

In the matrix $\Psi(e_{ij}(A))$ the element $a_{st},\ 1 \le s,t \le r$ is on the place $((i-1)r+s,(j-1)r+t)$, which we denote simpler by $v_{ij}+(s,t)$, where v_{ij} is the vector $r(i-1,j-1)$. Namely, the matrix A is shifted from the basic position by the vector v_{ij}.

Since the matrix $v_{ij}+A$ is above (or below) the diagonal of the matrix $e_{ij}(A)$, the first coordinates of all the points $v_{ij}+(s,t), 1 \le s,t \le r$, are greater (respectively, smaller) than the second coordinates of these points.

Consequently,

$$e_{v_{ij}+(s,t)} \cdot e_{v_{ij}+(s',t')} = 0 \quad \text{for} \quad 1 \le s,t,s',t' \le r.$$

Therefore the matrices

$$e := e_{v_{ij}+(s,t)}(\lambda) = I + \lambda e_{v_{ij}+(s,t)} \quad \text{and} \quad e' := e_{v_{ij}+(s',t')}(\lambda') = I + \lambda' e_{v_{ij}+(s',t')}$$

commute. More precisely,

$$e \cdot e' = I + \lambda e_{v_{ij}+(s,t)} + \lambda' e_{v_{ij}+(s',t')}. \tag{6}$$

Hence, by (6),

$$\Psi(e_{ij}(A)) = I + \sum_{1\le s,t\le r} a_{st} e_{v_{ij}+(s,t)} = \prod_{1\le s,t\le r} e_{v_{ij}+(s,t)}(a_{st}) \in E_{nr}(\Lambda)$$

and the factors in the last product commute.

Thus we have proved that $\Psi(e_{ij}(A)) \in E(\Lambda)$. □

Theorem 4.2. *There is a unique homomorphism $\Psi_S : \mathrm{St}\,(M_r(\Lambda)) \to \mathrm{St}\,(\Lambda)$ such that the following diagram commutes*

$$\begin{array}{ccccc} \mathrm{St}\,(M_r(\Lambda)) & \longrightarrow & E(M_r(\Lambda)) & \longrightarrow & 1 \\ \downarrow{\scriptstyle \Psi_S} & & \downarrow{\scriptstyle \Psi} & & \\ \mathrm{St}\,(\Lambda) & \longrightarrow & E(\Lambda) & \longrightarrow & 1 \end{array}$$

where Ψ is the homomorphism defined above.

Proof. We apply the notation used in the proof of the last Lemma. Define the mapping Ψ_S on generators $x_{ij}(A)$, where $i \neq j$, $A = (a_{st})_{1\le s,t\le r} \in M_r(A)$, of the group $\mathrm{St}\,(M_r(\Lambda))$ by the formula

$$\Psi_S(x_{ij}(A)) := \prod_{1\le s,t\le r} x_{v_{ij}+(s,t)}(a_{st}). \tag{7}$$

Similarly as in the proof of the Lemma above it can be observed that the factors in the product (7) commute.

To prove that the mapping Ψ_S is well defined, we should verify that it preserves the relations (2) defining the Steinberg group $\mathrm{St}\,(M_r(\Lambda))$.

We consider the first relation. Let $A, B \in M_r(\Lambda)$, where $A = (a_{st})_{1\le s,t\le r}$, $B = (b_{st})_{1\le s,t\le r}$. In $\mathrm{St}\,(M_r(\Lambda))$ we have

$$x_{ij}(A)x_{ij}(B) = x_{ij}(A+B), \quad \text{where} \quad i \neq j.$$

Now

$$\Psi_S(x_{ij}(A)) \cdot \Psi_S(x_{ij}(B)) = \prod_{1\le s,t\le r} x_{v_{ij}+(s,t)}(a_{st}) \cdot \prod_{1\le s',t'\le r} x_{v_{ij}+(s',t')}(b_{s't'}) \quad (8).$$

From $i \neq j$ it follows that the first coordinates of all vectors $v_{ij} + (s,t)$ are different from the second coordinates of these vectors. Therefore all factors in both products on the r.h.s. of (8) commute. We can rearrange them as follows

$$\prod_{1\le s,t\le r} x_{v_{ij}+(s,t)}(a_{st})x_{v_{ij}+(s,t)}(b_{st}) = \prod_{1\le s,t\le r} x_{v_{ij}+(s,t)}(a_{st}+b_{st})) = \Psi_S(x_{ij}(A+B)).$$

The proof of the second relation in (2) is similar. □

Thus we get a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(M_r(\Lambda)) & \longrightarrow & \mathrm{St}\,(M_r(\Lambda)) & \longrightarrow & E(M_r(\Lambda)) & \longrightarrow & 1 \\ & & \downarrow \Psi'_S & & \downarrow \Psi_S & & \downarrow \Psi & & \\ 1 & \longrightarrow & K_2(\Lambda) & \longrightarrow & \mathrm{St}\,(\Lambda) & \longrightarrow & E(\Lambda) & \longrightarrow & 1 \end{array} \quad (9)$$

It can be applied to find some element in $K_2(\Lambda)$. E.g. if U, V are commuting matrices in $\mathrm{GL}_r(\Lambda)$ then the image by Ψ'_S of the Steinberg symbol $\{U, V\}$ in $K_2(M_r(\Lambda))$ belongs to $K_2(\Lambda)$.

It is the transfer symbol $\langle U, V\rangle$ defined by Urbanowicz.

5. The transfer mapping.

If $m : \Gamma \to M_r(\Lambda)$ is a homomorphism of rings then there are canonical homomorphisms (where $\gamma \in \Gamma,\ i \neq j$) :

$$\Phi :\ E(\Gamma) \to E(M_r(\Lambda)), \qquad e_{ij}(\gamma) \mapsto e_{ij}(m(\gamma)),$$

$$\Phi_S :\ \mathrm{St}\,(\Gamma) \to \mathrm{St}\,(M_r(\Lambda)), \qquad x_{ij}(\gamma) \mapsto x_{ij}(m(\gamma)),$$

such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(\Gamma) & \longrightarrow & \mathrm{St}\,(\Gamma) & \longrightarrow & E(\Gamma) & \longrightarrow & 1 \\ & & \downarrow \Phi'_S & & \downarrow \Phi_S & & \downarrow \Phi & & \\ 1 & \longrightarrow & K_2(M_r(\Lambda)) & \longrightarrow & \mathrm{St}\,(M_r(\Lambda)) & \longrightarrow & E(M_r(\Lambda)) & \longrightarrow & 1 \end{array} \quad (10)$$

where $\Phi'_S = \Phi_S \,|\, K_2(\Gamma)$.

Now let Λ be a field and F an extension of Λ of degree r.

Lemma 5.1. *There is a subring of the ring $M_r(\Lambda)$ of matrices isomorphic with F.*

Proof. We fix a basis $b_1, \dots, b_r$ of F over Λ. Then to every $a \in F$ there corresponds the Λ-linear mapping $F \to F,\ b \mapsto ab$.

Denote by $m(a)$ the matrix corresponding to this linear map in the basis $b_1, \dots, b_r$. Let $m(a) = (a_{ij})_{1 \le i,j \le r}$, then

$$ab_j = \sum_{i=1}^{r} a_{ij} b_i, \quad \text{for} \quad j = 1, \dots, r.$$

The mapping $m : F \to M_r(\Lambda)$ is a homomorphism of rings which is a Λ-injection. An element $\lambda \in \Lambda$ is mapped by m on the scalar matrix λI_r.

Taking $\Gamma = F$ and $\Phi = m$ in the diagram (10), and joining it with the diagram (9) we obtain the commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K_2(F) & \longrightarrow & \mathrm{St}\,(F) & \longrightarrow & E(F) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle T'_S} & & \downarrow{\scriptstyle T_S} & & \downarrow{\scriptstyle T} & & \\
1 & \longrightarrow & K_2(\Lambda) & \longrightarrow & \mathrm{St}\,(\Lambda) & \longrightarrow & E(\Lambda) & \longrightarrow & 1
\end{array}$$

where $T = \Psi \circ m,\ T_S = \Psi_S \circ m_S,\ T'_S = \Psi'_S \circ m'_S$.

The mapping T'_S is called the transfer $\mathrm{Tr}_{F/\Lambda} : K_2(F) \to K_2(\Lambda)$. It does not depend on the basis $b_1, \dots, b_r$ chosen at the beginning.

Basic properties of the transfer are given in the paper by Rosset and Tate [RT].

In particular, there is given an algorithm which gives a representation of transfer of the Steinberg symbols in $K_2(F)$ as products of the Steinberg symbols in $K_2(\Lambda)$.

6. Main result.

Theorem 6.1. *Let $n > 2$ be prime to $\varphi(n)$ and to the characteristic of the field Λ. Assume that the n-th cyclotomic polynomial $\Phi_n(x)$ is irreducible in $\Lambda[x]$.*

Then every element $a \in K_2(\Lambda)$ satisfying $a^n = 1$ has the form

$$a = \prod_{j=1}^{t} Tr_{F_j/\Lambda}(c_n(a_j))$$

where elements a_j are algebraic over Λ, $F_j = \Lambda(a_j)$, and $c_n(a_j) = \{a_j, \Phi_n(a_j)\}$ are cyclotomic elements in $K_2(F_j)$.

Moreover

$$\sum_{j=1}^{t} (F_j : \Lambda) < (\Lambda(\zeta_n) : \Lambda) = \varphi(n). \tag{12}$$

Remarks.

1. From the assumptions it follows that n is odd.

2. If $n = 3$ then in (12) we have $\varphi(3) = 2$, hence $t = 1$ and $(F_1 : \Lambda) = 1$. Consequently a is a cyclotomic element in $K_2(\Lambda)$, $a = c_3(a_1) = \{a_1, a_1^2 + a_1 + 1\}$ for some $a_1 \in \Lambda$.

3. We do not claim that the fields F_j are subfields of $\Lambda(\zeta_n)$.

Proof. 1. We prove that a belongs to $\mathrm{Tr}_{F/\Lambda}(K_2(F))$.

Let $F = \Lambda(\zeta_n)$ and let $j : K_2(\Lambda) \to K_2(F)$ be the canonical homomorphism induced by the inclusion of Λ into F.

It is known that $(\mathrm{Tr} \circ j)(b) = b^{(F:\Lambda)}$ for $b \in K_2(\Lambda)$. Here $(F : \Lambda) = \varphi(n)$, since $\Phi_n(x)$ is irreducible in $\Lambda[x]$.

By assumption, $(n, \varphi(n)) = 1$, hence there exists $d \in \mathbb{N}$ such that $d\varphi(n) \equiv 1 \pmod{n}$. Therefore for the given element $a \in K_2(\Lambda)$ we have

$$\mathrm{Tr}_{F/\Lambda}(j(a^d)) = a^{d\varphi(n)} = a, \quad \text{since} \quad a^n = 1. \tag{13}$$

2. Now we define elements a_j.

From $a^n = 1$ it follows that $j(a^d) \in K_2(F)$ satisfies $(j(a^d))^n = 1$. Since $\zeta_n \in F$, by theorems of Tate and Suslin (see [T] and [S]) we have

$$j(a^d) = \{\zeta_n, b\}_F \quad \text{for some} \quad b \in F^*. \tag{14}$$

Then (13) and (14) imply that

$$a = \mathrm{Tr}_{F/\Lambda}\{\zeta_n, b\}_F. \tag{15}$$

Here $b = f(\zeta_n)$ for some polynomial $f(x) \in \Lambda[x]$ of degree less than $\varphi(n) = (F : \Lambda)$. The polynomial $f(x)$ is reducible in general. It can be written in the form

$$f(x) = c \prod_{j=1}^{t} f_j(x), \tag{16}$$

where $f_j(x) \in \Lambda[x]$ are monic and irreducible and $c \in \Lambda^*$. Choose a root a_j of $f_j(x)$, and let $F_j = \Lambda(a_j)$.

Then, by (15) and (16),

$$a = \mathrm{Tr}_{F/\Lambda}\{\zeta_n, b\}_F = \mathrm{Tr}_{F/\Lambda}\{\zeta_n, c\}_F \prod_{j=1}^{t} \mathrm{Tr}_{F/\Lambda}\{\zeta_n, f_j(\zeta_n)\}_F. \tag{17}$$

Since $c \in \Lambda^*$, we have

$$\mathrm{Tr}_{F/\Lambda}\{\zeta_n, c\}_F = \{N_{F/\Lambda}(\zeta_n), c\}_\Lambda = 1, \tag{18}$$

since $N_{F/\Lambda}(\zeta_n) = (-1)^{\deg \Phi_n(x)} \cdot \Phi_n(0) = 1$.

3. We apply the reciprocity law for K_2-transfer.

Reciprocity law. *Let $g, h \in \Lambda[x]$ be monic irreducible polynomials over a field Λ, and let $g(\beta) = h(\alpha) = 0$ for some α, β algebraic over Λ, where $\alpha\beta \neq 0$.*

Then

$$Tr_{\Lambda(\alpha)/\Lambda}\{\alpha, g(\alpha)\}_{\Lambda(\alpha)} = \{h(0), g(0)\}_\Lambda \cdot \{(-1)^{\deg h}, (-1)^{\deg g}\}_\Lambda \cdot Tr_{\Lambda(\beta)/\Lambda}\{\beta, h(\beta)\}_{\Lambda(\beta)} \tag{19}$$

Let us remark, that if $\deg g \leq \deg h$, and $h(x) = q(x)g(x) + r(x)$, with $\deg r(x) < \deg g(x)$, then $h(\beta) = r(\beta)$.

Substitute in (19): $\alpha = \zeta_n,\ \beta = a_j,\ g(x) = f_j(x),\ h(x) = \Phi_n(x)$. Since $\deg \Phi_n(x)$ is even, and $\Phi_n(0) = 1$, from (19) we get

$$\mathrm{Tr}_{F/\Lambda}\{\zeta_n, f_j(\zeta_n)\}_F = \mathrm{Tr}_{F_j/\Lambda}\{\alpha_j, \Phi_n(\alpha_j)\}_{F_j} \tag{20}$$

Hence, by (17) and (18),

$$\begin{aligned} a = \mathrm{Tr}_{F/\Lambda}\{\zeta_n, b\}_F &= \prod_{j=1}^{t} \mathrm{Tr}_{F/\Lambda}\{\zeta_n, f_j(\zeta_n)\}_F \\ &= \prod_{j=1}^{t} \mathrm{Tr}_{F_j/\Lambda}\{\alpha_j, \Phi_n(\alpha_j)\}_{F_j} \end{aligned}$$

Thus we have proved that a is the product of transfers of cyclotomic elements $c_n(\alpha_j) = \{\alpha_j, \Phi_n(\alpha_j)\}_{F_j}$.

4. We shall estimate the sum of degrees of the fields F_j.

Since $f_j \in \Lambda[x]$ are irreducible, and $f_j(a_j) = 0$, then $(F_j : \Lambda) = \deg f_j$. Therefore from (16) we get

$$\sum_{j=1}^{t}(F_j : \Lambda) = \sum_{j=1}^{t} \deg f_j(x) = \deg f(x) < \varphi(n). \qquad \square$$

7. Central extensions of groups.

Let us consider the exact sequence of groups:

$$1 \longrightarrow K \longrightarrow S \longrightarrow G \longrightarrow 1$$

Then we say that S is an extension of the group G by the group K. We say that the extension S is central if K is contained in the center $\mathfrak{z}(S)$ of the group S.

The central extension S is called universal if for every central extension $\widetilde{S}$

$$1 \longrightarrow \widetilde{K} \longrightarrow \widetilde{S} \longrightarrow G \longrightarrow 1$$

of the group G there exists a unique homomorphism $\theta : S \to \widetilde{S}$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & S & \longrightarrow & G & \longrightarrow & 1 \\ & & {\scriptstyle\theta'}\big\downarrow & & {\scriptstyle\theta}\big\downarrow & & \big\| & & \\ 1 & \longrightarrow & \widetilde{K} & \longrightarrow & \widetilde{S} & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

where $\theta' = \theta|K$.

Theorem 7.1. *There are given two central extensions*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & S & \xrightarrow{\varphi} & G & \longrightarrow & 1 \\ \\ 1 & \longrightarrow & K^* & \longrightarrow & S^* & \xrightarrow{\varphi^*} & G^* & \longrightarrow & 1 \end{array} \tag{21}$$

of groups G and G^*, respectively. Assume that the first one is universal.

Then for every homomorphism $\Psi : G \to G^*$ there exists a unique homomorphism $\Psi_S : S \to S^*$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & S & \xrightarrow{\varphi} & G & \longrightarrow & 1 \\ & & \Psi'_S\big\downarrow & & \Psi_S\big\downarrow & & \Psi\big\downarrow & & \\ 1 & \longrightarrow & K^* & \longrightarrow & S^* & \xrightarrow{\varphi^*} & G^* & \longrightarrow & 1 \end{array}$$

where $\Psi'_S = \Psi_S|K$.

Proof. First we consider the pullback $H \subseteq G \times S^*$:

$$\begin{array}{ccccc} H & \xrightarrow{\pi_1} & G & & \\ \pi_2\big\downarrow & & \Psi\big\downarrow & & \\ S^* & \xrightarrow{\varphi^*} & G^* & \longrightarrow & 1 \end{array} \tag{22}$$

where

$$H := \{(g, s^*) : g \in G, s^* \in S^*, \Psi(g) = \varphi(s^*)\}$$

and

$$\pi_1(g, s^*) = g, \quad \pi_2(g, s^*) = s^*, \quad \text{for} \quad (g, s^*) \in H.$$

The homomorphism π_1 is surjective, since for every $g \in G$ we can choose an $s^* \in S^*$ such that $\varphi^*(s^*) = \Psi(g)$, by the surjectivity of φ^*.

Next, $\ker \pi_1 \subseteq \mathfrak{z}(H)$. Namely, if $(g, s^*) \in \ker \pi_1$, then $g = \pi_1(g, s^*) = 1$. Consequently from $\varphi(g) = \varphi^*(s^*)$ we get $\varphi^*(s^*) = 1$, i.e. $s^* \in \ker \varphi^* = K^* \subseteq \mathfrak{z}(S^*)$.

Therefore $(g, s^*) = (1, s^*) \in \mathfrak{z}(H)$, hence $\ker \pi_1 = 1 \times K^* \subseteq \mathfrak{z}(H)$.

Thus we have proved that the extension H of the group G :

$$1 \longrightarrow \ker \pi_1 \longrightarrow H \xrightarrow{\pi_1} G \longrightarrow 1$$

is central.

By assumption the extension S of G is central universal. Therefore there is a unique homomorphism θ such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & S & \xrightarrow{\varphi} & G & \longrightarrow & 1 \\ & & \theta'\big\downarrow & & \theta\big\downarrow & & \big\| & & \\ 1 & \longrightarrow & \ker \pi_1 & \longrightarrow & H & \xrightarrow{\pi_1} & G & \longrightarrow & 1 \end{array}$$

where $\theta' = \theta|K$, is commutative.

From diagram (22) we get

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \ker\pi_1 & \longrightarrow & H & \xrightarrow{\pi_1} & G & \longrightarrow & 1 \\ & & {\scriptstyle \pi_2'}\big\downarrow & & {\scriptstyle \pi_2}\big\downarrow & & {\scriptstyle \Psi}\big\downarrow & & \\ 1 & \longrightarrow & K^* & \longrightarrow & S^* & \xrightarrow{\varphi^*} & G^* & \longrightarrow & 1 \end{array}$$

where $\pi_2' = \pi_2|\ker\pi_1$. Since $\ker\pi_1 = 1 \times K^*$, we get $\pi_2'(\ker\pi_1) \subseteq K^*$.

Joining last two diagrams we get a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & S & \xrightarrow{\varphi} & G & \longrightarrow & 1 \\ & & {\scriptstyle \pi_2'\theta'}\big\downarrow & & {\scriptstyle \pi_2\theta}\big\downarrow & & {\scriptstyle \Psi}\big\downarrow & & \\ 1 & \longrightarrow & K^* & \longrightarrow & S^* & \xrightarrow{\varphi^*} & G^* & \longrightarrow & 1 \end{array}$$

Thus it is sufficient to put $\Psi_S = \pi_2\theta$ to get the first part of the theorem.

To prove the uniqueness of Ψ_S assume that there is a homomorphism $\nu : S \to S^*$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & S & \xrightarrow{\varphi} & G & \longrightarrow & 1 \\ & & {\scriptstyle \nu'}\big\downarrow & & {\scriptstyle \nu}\big\downarrow & & {\scriptstyle \Psi}\big\downarrow & & \\ 1 & \longrightarrow & K^* & \longrightarrow & S^* & \xrightarrow{\varphi^*} & G^* & \longrightarrow & 1 \end{array} \tag{23}$$

where $\nu' = \nu|K$.

We define the homomorphism $\lambda : S \to H$, $\lambda(s) := (\varphi(s), \nu(s))$ for $s \in S$. Then $\nu = \pi_2 \circ \lambda$. The image of λ belongs to the pullback H, since $\Psi(\varphi(s)) = \varphi^*(\nu(s))$, by the commutativity of the diagram (23).

We have proved above that H is a central extension of G. By the universality of the central extension S of G, we conclude that the homomorphism λ satisfying the above conditions is unique. Hence $\lambda = \theta$.

Therefore $\nu = \pi_2 \circ \lambda = \pi_2 \circ \theta = \Psi_S$. □

Theorem 7.2. *For an arbitrary ring* Λ *we have* $K_2(\Lambda) = \mathfrak{z}(\mathrm{St}\,(\Lambda))$. *Moreover,* $\mathrm{St}\,(\Lambda)$ *is a universal central extension of* $E(\Lambda)$.

Proof. See [M]. □

Theorem 7.3. *Assume that for some rings* Γ *and* Λ *there is a homomorphism* $\Psi : E(\Gamma) \to E(\Lambda)$. *Then there is a unique homomorphism* $\Psi_S : \mathrm{St}\,(\Gamma) \to \mathrm{St}\,(\Lambda)$, *such that* $\Psi_S(K_2(\Gamma)) \subseteq K_2(\Lambda)$ *and the following diagram is commutative.*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(\Gamma) & \longrightarrow & \mathrm{St}\,(\Gamma) & \longrightarrow & E(\Gamma) & \longrightarrow & 1 \\ & & {\scriptstyle \Psi_S'}\big\downarrow & & {\scriptstyle \Psi_S}\big\downarrow & & {\scriptstyle \Psi}\big\downarrow & & \\ 1 & \longrightarrow & K_2(\Lambda) & \longrightarrow & \mathrm{St}\,(\Lambda) & \longrightarrow & E(\Lambda) & \longrightarrow & 1 \end{array}$$

Proof. The theorem follows immediately from Theorem 7.1 and Theorem 7.2. □

8. Problems.

1. Let Λ be a commutative ring, and H_r the subgroup of $K_2(\Lambda)$ generated by the transfer symbols $\langle A, B\rangle$, where $A, B \in M_s(\Lambda)^*$, $s \le r$, and $AB = BA$.

1.1) Does $K_2(\Lambda) = H_r$ hold for some $r \le \infty$?

1.2) Does $H_1 \ne H_2$ hold for some Λ?

1.3) Is $\langle A, B\rangle$ the product of Steinberg symbols in $K_2(\Lambda)$?

2. Let Λ be a commutative ring and $\Lambda[\zeta_n]$ be a free Λ-module of rank r.

2.1) Does every element in $(K_2(\Lambda))_n$ have the form $\langle U, A\rangle$ for some $A \in M_r(\Lambda)^*$, where $U = m(\zeta_n)$?

3. Assume that $\Phi_n(x)$ is reducible over the field F. Let $\phi(x)$ be its monic irreducible factor of degree r. Assume that n is prime to r and to the characteristic of F.

3.1) Is it true that $\langle A, \phi(A)\rangle^n = 1$ for every $A \in M_s(\Lambda)$, where $s < r$, such that $\phi(A) \in M_s(\Lambda)^*$?

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