

WORKSHOP : KONTSEVICH FORMALITY THEOREM AND DUFLO ISOMORPHISM

Introduction(15 min)(D. Calaque)

- presentation and goals of the workshop

Lecture 1 (45 min) : **Introduction to deformation quantization after Kontsevich I**

- Definition of a differential graded Lie algebra (DGLA)
- Definition of T_{poly} : vector fields, polyvector fields, Schouten bracket
- Poisson bracket \Leftrightarrow Maurer-Cartan element (MCE) in T_{poly}
- Definition of D_{poly} : differential operators, polydifferential operators, Gerstenhaber bracket, Hochschild coboundary operator
- Associative product \Leftrightarrow MCE in D_{poly}

Lecture 2 (45 min) : **Introduction to deformation quantization after Kontsevich II**

- Formal deformation \Rightarrow Poisson structure (semi-classical limit)
- Statement of the deformation quantization problem
- Statement of the Hochschild-Kostant-Rosenberg theorem
- Lie algebra morphisms between DGLAs induces a map between MC elements
- Consequence : if H-K-R was a Lie morphism, then we would obtain a solution to the deformation quantization problem
- Default of being a Lie morphism (example)

Lecture 3 (45 min) : **Introduction to the formality theorem**

- Reminder on cofree coalgebras
- Definition of L_∞ -algebras (example : DGLAs are L_∞ -algebras) and L_∞ -morphisms
- MCE elements are sent to MCE via a L_∞ -morphism
- Statement of the formality theorem (existence of a L_∞ -morphism)
- Consequences for deformation quantization

Lecture 4 (1h) : **Kontsevich's graphs and formality**

- Definition of an admissible graph
- Polydifferential operator associated to an admissible graph (examples)
- Sign ambiguity in the previous definition
- Construction of what-would-be-a-formality-map in terms of graphs (weights of graphs not yet specified)
- Quadratic identities on weights of graphs $\Rightarrow L_\infty$ -condition

Lecture 5 (1h30) : **Angle forms on compactified configuration spaces**

- Configuration spaces $C_{n,m}^+$ and their compactifications
- Combinatorics of the boundary of the compactified configuration spaces
- Angle forms on configuration spaces

Lecture 6 (45min) : **Proof of Kontsevich's formality theorem**

- Weights of graphs as integrals of angle forms

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- Sign ambiguity in the definition (cancel with the sign ambiguity of Lecture 4)
- Stokes theorem
- Vanishing lemmata
- Examples of two graphs and their weights
- Proof of the appropriate identities on weights (i.e proof of the L_∞ condition)

Lecture 7 (45min) : **Compatibility between cup products I**

- The homotopy equation
- Expression of the cup-products in terms of weights and graphs
- A path in $\tilde{C}_{2,0}$
- Expression of the homotopy in terms of weights and graphs

Lecture 8 (45min) : **Compatibility between cup products II**

- Stokes formula
- Proof of the homotopy equation

Lecture 9 (1h30) : **Duflo isomorphism**

- Statement (without proof!) of the Poincaré-Birkhoff-Witt theorem (isomorphism of vector spaces between $S(\mathfrak{g})$ and $U(\mathfrak{g})$)
- Statement of the Duflo isomorphism (isomorphism of algebras between $S(\mathfrak{g})^{\mathfrak{g}}$ and $U(\mathfrak{g})^{\mathfrak{g}}$)
- Proof of the statement via the compatibility between cup-products (after Kontsevich)

Lecture 10 (1h30) : **The combinatorial Kashiwara-Vergne conjecture and the Duflo isomorphism**

- Baker-Campbell-Hausdorff formula
- Presentation of the KV conjecture
- Statement of the combinatorial KV conjecture
- KV implies Duflo (algebraically)

Lecture 11 (1h30) : **Proof of the KV conjecture**

- Lie- and wheel-type graphs
- Reformulation of the KV conjecture via deformations (equivalence)
- A deformation of the Baker-Campbell-Hausdorff formula
- Alekseev-Meinrenken proof (sketch)

Lecture 12 (1h30) : **Introduction to Drinfeld associators**

- Universal Lie algebra t_n , relation with semi-simple Lie algebras
- Definition of associators
- Categorical interpretation
- Torsor structure
- Existence/ \mathbb{C} = existence/ \mathbb{Q}

Lecture 13 (1h30) : **KV associators, after Alekseev-Torossian** (D. Calaque)

- tder, sder, tr, kv_2 , kv_3 , etc ...
- Definition of KV-associators
- Relation with Drinfeld associators
- Relation with the KV problem

Lecture 14 (1h30 min) : **Existence of Drinfeld associators**

- Knizhnik-Zamolodchikov equations
- Construction of the KZ associator
- Proof that it satisfies the axioms of associators

Lecture 15 (45 min) : **Drinfeld double** (P. Hajac)

- The Drinfeld double and its quasi-triangular structure

Lecture 16 (45 min) : **Quantization of Lie bialgebras** (T. Maszczyk)

- Lie bialgebras, quantization problem
- Strategy of Etingof-Kazhdan proof

Lecture 17 : **Alekseev-Torossian explicit proof of the KV conjecture**

References :

- 1,... & 6 : Kontsevich, Deformation quantization of Poisson manifolds, *Lett. Math. Phys.* 66 (2003), no. 3, 157–216.
- 1,2 & 3 : Keller’s part in [CKTB] : Déformation, Quantification, Théorie de Lie, Panoramas et Synthèses 20 (2005).
- 5 : Bruguières’s appendix in [CKTB].
- 1,... & 5 : Cattaneo & Alberto, Formality and star products, Lecture notes taken by D. Indelicato, London Math. Soc. Lecture Note Ser., 323, Poisson geometry, deformation quantisation and group representations, 79–144, Cambridge Univ. Press, Cambridge, 2005.
- 6 : Torossian’s part in [CKTB].
- 6 : Arnal, Manchon & Masmoudi, Choix des signes pour la formalité de M. Kontsevich. (French) [Choice of signs for M. Kontsevich’s formality theorem] *Pacific J. Math.* 203 (2002), no. 1, 23–66.
- 7 & 8 : Manchon & Torossian, Cohomologie tangente et cup-produit pour la quantification de Kontsevich, *Annales Mathématiques Blaise Pascal*, Vol 10 (1), 2003, 75-106.
- 7, 8 & 9 : Calaque & Rossi, Lectures on Duflo isomorphisms in Lie algebras and geometry, *SIGMA Symmetry Integrability Geom. Methods Appl.* 4 (2008), Paper 060, <http://math.univ-lyon1.fr/~calaque/LectureNotes/LectETH.pdf>.
- 9 : Torossian, Cours de Master : Formule de Campbell-Haudorff, Isomorphisme de Duflo et Conjecture de Kashiwara-Vergne, <http://people.math.jussieu.fr/~torossian/publication/tunis.pdf>.
- 9 : Kontsevich, Deformation quantization of Poisson manifolds, *Lett. Math. Phys.* 66 (2003), no. 3, 157–216.
- 10 & 11 : Torossian, La conjecture de Kashiwara-Vergne, séminaire Bourbaki, juin 2007.
- 10 & 11 : Alekseev & Meinrenken, On the Kashiwara-Vergne conjecture, *Invent. Math.* 164 (2006), no. 3, 615–634.
- 11 : Torossian, Sur la conjecture combinatoire de Kashiwara-Vergne, *Ann. Math. Blaise Pascal* 10 (2003), no. 1, 75–106.
- 12 : Drinfeld, On quasi-triangular quasi-Hopf algebras and a group closely connected with $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, *Leningrad Math. J.*, vol. 2 no. 4, 829–860, (1991).
- 12 : Kassel, *Quantum groups*, New York NY : Springer, 1995.
- 13 : Alekseev & Torossian, The Kashiwara-Vergne and Drinfeld associators, preprint arXiv math0802.4300.
- 13 : Torossian, Kashiwara-Vergne problem and Drinfeld’s associators, <http://people.math.jussieu.fr/~torossian/exposeCIRM.pdf>
- 14, 15 & 16 : Drinfeld, On quasi-triangular quasi-Hopf algebras and a group closely connected with $\text{Gal}(\mathbb{Q}/\mathbb{Q})$, *Leningrad Math. J.*, vol. 2 no. 4, 829–860, (1991).
- 14, 15 & 16 : Drinfeld, Quantum groups, *Proceedings of the International Congress of Mathematicians*, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc.,

Providence, RI, 1987.

14, 15 & 16 : Etingof, Schiffman, Lectures on Quantum groups, Lectures in Mathematical Physics. International Press, Somerville, MA, 2002.

14, 15 & 16 : Etingof, Kazhdan, Quantization of Lie bialgebras I, Selecta Math. (N.S.) 2 (1996), no. 1, 1–41.