On and around the Drinfeld double

for Emily

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Yetter-Drinfeld modules

Let H be a Hopf algebra over a field k. Assume that its antipode S is bijective. Let M be a left module and a right comodule over H, i.e.,

$$H \otimes M \to M, \quad h \otimes m \mapsto h \triangleright m,$$
$$\Delta_M : M \to M \otimes H, \quad m \mapsto m^{(0)} \otimes m^{(1)}.$$

We say that M is a *Yetter-Drinfeld module* if

$$\Delta_M(h \triangleright m) = h^{(2)} \triangleright m^{(0)} \otimes h^{(3)} m^{(1)} S^{-1}(h^{(1)}).$$

The Yetter-Drinfeld modules form a monoidal category for the diagonal action and coaction. The category is braided by the formula:

$$\sigma : M \otimes N \to N \otimes M, \quad m \otimes n \mapsto m^{(1)} \triangleright n \otimes m^{(0)}.$$

Drinfeld (quantum) double

Let H be a finite-dimensional Hopf algebra, and $D(H) := (H^*)^{\mathrm{cop}} \otimes H$ the diagonal coalgebra. The formula

$$
\begin{aligned}
&(\varphi \otimes h)(\varphi' \otimes h') \\
&= \varphi'^{(1)}(S^{-1}(h^{(3)}))\varphi'^{(3)}(h^{(1)})\, \varphi\varphi'^{(2)} \otimes h^{(2)}h'
\end{aligned}
$$

turns $D(H)$ into a Hopf algebra. Yetter-Drinfeld modules over H are equivalent to modules over $D(H)$.

Stable anti-Yetter-Drinfeld modules

Let H be a Hopf algebra over a field k. Assume that its antipode S is bijective. Let M be a left module and a right comodule over H. We say that M is an *anti-Yetter-Drinfeld module* if

$$\Delta_M(h \triangleright m) = h^{(2)} \triangleright m^{(0)} \otimes h^{(3)} m^{(1)} S(h^{(1)}).$$

The tensor product of an anti-Yetter-Drinfeld module and a Yetter-Drinfeld module is an anti-Yetter-Drinfeld module. A module comodule M is called *stable* if

$$m^{(1)} \triangleright m^{(0)} = m.$$

The stable anti-Yetter-Drinfeld modules were discovered as coefficients of Hopf-cyclic cohomology.

Anti-Drinfeld double

Let H be a finite-dimensional Hopf algebra. The formula

$$(\varphi \otimes h)(\varphi' \otimes h') \\ = \varphi'^{(1)}(S^{-1}(h^{(3)}))\varphi'^{(3)}(S^2(h^{(1)}))\, \varphi\varphi'^{(2)} \otimes h^{(2)}h'$$

turns the vector space $A(H) := H^* \otimes H$ into an associative algebra with the unit $\varepsilon \otimes 1$.

If M is an anti-Yetter-Drinfeld module, it becomes a left $A(H)$-module by

$$(\varphi \otimes h) \triangleright m := \varphi((h \triangleright m)^{(1)})\,(h \triangleright m)^{(0)}.$$

Conversely, if M is a left $A(H)$-module, it becomes an anti-Yetter-Drinfeld module by

$$h \triangleright m := (\varepsilon \otimes h) \triangleright m, \quad \Delta_M(m) := \sum_{i=1}^{n} (h_i^* \otimes 1) \triangleright m \otimes h_i.$$

Here $\{h_1, \ldots, h_n\}$ is a basis of H and $\{h_1^*, \ldots, h_n^*\}$ its dual basis.

The formula $(\varphi \otimes h) \mapsto (\varphi^{(2)} \otimes h^{(1)}) \otimes (\varphi^{(1)} \otimes h^{(2)})$ makes $A(H)$ a right comodule algebra (Galois object) over the Drinfeld double $D(H)$.

Examples of (stable anti-) Yetter-Drinfeld modules

1. Yetter-Drinfeld:

$$M = H, \quad \Delta_M = \Delta, \quad h \triangleright k = h^{(2)} k S^{-1}(h^{(1)})$$

2. Stable anti-Yetter-Drinfeld:

$$M = H, \quad \Delta_M = \Delta, \quad h \triangleright k = h^{(2)} k S(h^{(1)})$$

3. Stable (anti-)Yetter-Drinfeld:

$$H = kG, \quad M = \bigoplus_{g \in G} M_g, \quad \Delta_M(m_g) = m_g \otimes g,$$

$$h \triangleright m_g \in M_{hgh^{-1}}, \quad g \triangleright m_g = m_g$$

4. Stable (anti-)Yetter-Drinfeld:

$$H = \mathcal{O}(SU(2)),\ M = \mathcal{O}(S^2),\ \Delta_M(m)(x,g) = m(xg),$$

$$U(1)\backslash SU(2) \cong S^2 \cong \{x \in SU(2) | \operatorname{Tr} x = 0\} \overset{j}{\hookrightarrow} SU(2),$$

$$(h \triangleright m)(x) = h(j(x))m(x), \quad j(xg) = g^{-1} j(x) g$$

Quasi-triangular structure

A *quasi-triangular* structure R on a bialgebra B is an invertible element in $B \otimes B$ s.t.

1. $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}, \quad (\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$,

2. $\Delta^{\mathsf{op}}(h)R = R\Delta(h)$.

For a quasi-triangular Hopf algebra H, we can define a ribbon element $r \in H$, which is fundamental for constructing invariants of framed links.

For $B = D(H)$, we take $R = \sum_i (1 \otimes h_i) \otimes (h_i^* \otimes 1)$. Kauffman and Radford prove that $(D(H), R)$ admits a ribbon element if and only if there exists a pair in involution (δ, g) s.t. $S^4(h) = g^2(\delta^2 \triangleright h \triangleleft \delta^{-2})g^{-2}$.

Modular pairs in involution

Let H be a Hopf algebra over a field k. Assume that its antipode S is bijective. Let M be a 1-dimensional vector space over k that is a left module via a character $\delta : H \to k$ and a right comodule via a group-like $g \in H$. We say that (δ, g) is a *(modular) pair in involution* if M is a (stable) anti-Yetter-Drinfeld module.

Modular pairs in involution provide a one-to-one correspondence between the Yetter-Drinfeld and anti-Yetter-Drinfeld modules. If $\dim H < \infty$, $D(H)$ and $A(H)$ are isomorphic as (comodule) algebras if and only if there exists a modular pair in involution.

Also, the stability of an anti-Yetter-Drinfeld module is equivalent to the trivial action of the Kauffman-Radford ribbon element in $D(H)$ on the corresponding Yetter-Drinfeld module.

Doubles for infinite-dimensional Hopf algebras

There exist corings whose corepresentation categories are equivalent to the category of Yetter-Drinfeld and the category of anti-Yetter-Drinfeld modules, respectively. The convolution duals of these corings are algebras generalizing the Drinfeld and anti-Drinfeld doubles. These are $\mathrm{End}(H)$ with a complicated multiplication structure derived from their action on (anti-)Yetter-Drinfeld modules:

$$f \triangleright m := f(m^{(1)}) \triangleright m^{(0)}.$$

The category of Yetter-Drinfeld and the category anti-Yetter-Drinfeld modules are full subcategories of the corresponding categories of modules over an appropriate $\mathrm{End}(H)$.