Knizhnik-Zamolodchikov and Drinfeld-Kohno

U. Krähmer

U Glasgow

Warsaw 8.4.2009

Connections and parallel transport

- Input: A smooth vector bundle $\pi : E \to X$, a point $x \in X$, and a connection $\nabla : \Gamma(E) \to \Omega^1(X, E) := \Omega^1(X) \otimes \Gamma(E)$ on E,

$$\nabla(f\psi) = \mathrm{d}f \otimes \psi + f\nabla(\psi), \quad f \in C^\infty(X), \psi \in \Gamma(E).$$

- Given $\xi \in E_x := \pi^{-1}(x)$ and a closed smooth curve

$$\gamma : [0,1] \to X, \quad \gamma(0) = \gamma(1) = x,$$

  there is a unique smooth curve $\psi : [0,1] \to E$ with

$$\psi(0) = \xi, \quad \pi(\psi(t)) = \gamma(t), \quad \nabla_{\dot{\gamma}(t)}\psi(t) = 0.$$

  The parallel transport along γ is the linear map

$$E_x \ni \xi = \psi(0) \mapsto \psi(1) \in E_x.$$

Holonomy, monodromy, and curvature

Definition

The group $\mathrm{Hol}(x) \subset GL(E_x)$ *of parallel transports along arbitrary curves* γ *is the holonomy of the connection in* x*. The monodromy of* ∇ *in* x *is the quotient group* $\mathrm{Hol}(x)/\mathrm{Hol}^0(x)$*, where* $\mathrm{Hol}^0(x)$ *is the normal subgroup arising from contractible* γ*.*

Definition

The curvature of ∇ *is the (*$C^\infty(X)$*-linear) map*

$$\nabla \circ \nabla : \Gamma(E) \to \Omega^2(X, E),$$

where we extend ∇ *to* $\Omega^1(X, E)$ *using the Leibniz rule,*

$$\nabla(\omega \otimes \psi) := \mathrm{d}\omega \otimes \psi - \omega \wedge \nabla(\psi), \quad \omega \in \Omega^1(X), \psi \in \Gamma(E).$$

Flat connections and monodromy representations

Definition

A connection is flat if $\nabla \circ \nabla = 0$*.*

Theorem

If ∇ *is flat, then the parallel transport along* γ *only depends on the class of* γ *in the fundamtental group* $\pi_1(X)$*. Thus one obtains*

$$\rho : \pi_1(X) \longrightarrow GL(E_x).$$

Definition

ρ *is the monodromy representation of* $\pi_1(X)$ *(defined by* E, x, ∇*).*

The braid group

Definition

Put

$$Y_n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, z_i \neq z_j \text{ for } i \neq j\}.$$

Then the configuration space of n points in the plane is

$$X_n := Y_n / S_n,$$

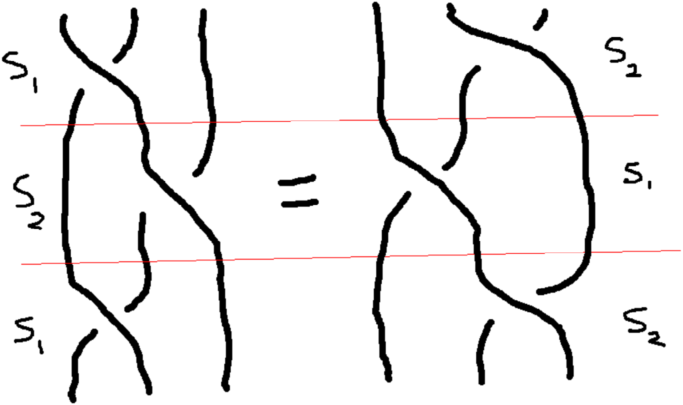
where S_n acts by permuting the z_i. The group of n-strand braids is

$$B_n := \pi_1(X_n).$$

Abstracly, B_n has generators $s_1, \ldots, s_{n-1}$ with relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i, \quad |i - j| > 1.$$

The braid relation



The morphism $B_n \longrightarrow S_n$

- Note: If one adds the relations $s_i^2 = 1$ one obtains the permutation group S_n. So there is a surjective group morphism

$$B_n \longrightarrow S_n.$$

- Pictorially, this map forgets about what happened along the braid (which crossings were over and which under) and just notes down the final configuration of the points.

Connections in local trivialisations

- By definition, the connections on E form an affine space over the $\mathrm{End}(E)$-valued 1-forms on X. Furthermore, in local trivialisations d is a connection. Hence locally connections are of the form
$$\nabla = \mathrm{d} - \Gamma, \quad \Gamma \in \Omega^1(X, \mathrm{End}(E)),$$
and on trivial bundles this is true globally.
- The Maurer-Cartan equation: ∇ is flat if and only if Γ satisfies
$$-\mathrm{d}\Gamma + \Gamma \wedge \Gamma = 0.$$

Towards the KZ equations

- Let V be a $\mathbb{C}$-vector space and $A_{ij} \in \mathrm{End}_{\mathbb{C}}(V)$, $1 \leq i < j \leq n$. Define a connection ∇ on the trivial bundle $Y_n \times V$ in terms of

$$\Gamma := \sum_{1 \leq i < j \leq n} A_{ij} \frac{\mathrm{d}z_i - \mathrm{d}z_j}{z_i - z_j} = \frac{1}{2} \sum_{i \neq j} A_{ij} \frac{\mathrm{d}z_i - \mathrm{d}z_j}{z_i - z_j}, \quad A_{ji} := A_{ij}.$$

Theorem

The connection ∇ is flat iff $\Gamma \wedge \Gamma = 0$ which is equivalent to

$$[A_{ij}, A_{kl}] = [A_{ij}, A_{ik} + A_{jk}] = [A_{jk}, A_{ij} + A_{ik}] = 0,$$

where i, j, k, l are distinct.

Descending to X_n

- Assume that V carries in addition a representation of S_n. Then the canonical projection $Y_n \to X_n$ induces a vector bundle

$$E := (Y_n \times V)/S_n \to X_n = Y_n/S_n,$$

  where S_n acts on $Y_n \times V$ as

$$(z_1, \dots, z_n, v)s := (z_{s(1)}, \dots, z_{s(n)}, s^{-1}v).$$

- The base space of this bundle is now X_n, the fibre is still V. Its sections can be identified with S_n-equivariant functions

$$\psi : Y_n \to V, \quad \psi(zs) = s^{-1}\psi(z).$$

- Since Γ is S_n-invariant, ∇ descends to a flat conneciton on E.

The KZ equation

- This is $\nabla\psi = 0$ in a special case of the above setting.
- Input data:
  - a finite-dimensional complex Lie algebra $\mathfrak{g}$.
  - an invariant symmetric 2-tensor $t = \sum_i X_i \otimes Y_i \in \mathfrak{g} \otimes \mathfrak{g}$:

$$t = \sum_i Y_i \otimes X_i, \quad \sum_i [Z, X_i] \otimes Y_i + X_i \otimes [Z, Y_i] = 0.$$

  - two parameters $h \in \mathbb{C}, n \in \mathbb{N}$.
  - a representation W of $\mathfrak{g}$.
- Given this, we put $V := W^{\otimes n}$, $\hbar := h/2\pi\sqrt{-1}$, and

$A_{ij} := \hbar t$ acting on the ith and jth tensor components

25 The case $\hbar = 0$

- For $\hbar = 0$, solutions of the KZ equation are functions
  $$\psi : Y_n \to W^{\otimes n}, \quad \psi(zs) = s^{-1}\psi(z)$$
  that are constant,
  $$\mathrm{d}\psi = 0.$$
- The monodromy representation of B_n on $V = W^{\otimes n}$ acts by permutation of the tensor compontents. This will become clearer through the next simple case.

The case $n = 2$ (KZ_2)

- For $n = 2$, solutions of the KZ equation are functions

$$\psi : Y_2 \to W \otimes W, \quad \psi(zs) = s^{-1}\psi(z)$$

with

$$\mathrm{d}\psi = \hbar t \frac{\mathrm{d}x - \mathrm{d}y}{x - y}\psi, \quad x = z_1, y = z_2.$$

- We have $B_2 \simeq \mathbb{Z}$. The generator can be represented by

$$\gamma(s) = \frac{1}{2}(3 - e^{i\pi s}, 3 + e^{i\pi s}), \quad s \in [0, 1]$$

which starts and ends in $[(1, 2)] = [(2, 1)] \in X_2 = Y_2/S_2$.

The case $n = 2$

- Over this loop, the KZ equation for ψ becomes the ODE

  $$\frac{\mathrm{d}\psi}{\mathrm{d}s} = \frac{ht}{2}\psi(s)$$

  that is solved by

  $$\psi(s) = e^{\frac{ht}{2}s}\psi(0) = \sum_{m=0}^{\infty} \frac{(hts)^m}{2^m m!}\psi(0).$$

- Taking $s = 1$ we get the action of the generator of $B_2 \simeq \mathbb{Z}$:

  $$w_1 \otimes w_2 \mapsto (\tau \circ e^{ht/2})(w_1 \otimes w_2), \quad \tau(w_1 \otimes w_2) = w_2 \otimes w_1.$$

  The flip τ pops up since

  $$[(1, 2, w_1 \otimes w_2)] = [(2, 1, w_2 \otimes w_1)]$$

  in $(Y_2 \times (W \otimes W))/S_2$.

Generalisation to $n > 2$

- If we even have $[t_{ij}, t_{jk}] = 0$ for all i, j, k (e.g. if $\mathfrak{g}$ is abelian), then the KZ equation decouples and we obtain for the monodromy representation of the generators $s_1, \ldots, s_n$

  $$\rho_{KZ}(s_i)(w_1 \otimes \cdots \otimes w_n) = (\tau_{i,i+1} \circ e^{ht_{i,i+1}/2})(w_1 \otimes \cdots \otimes w_n),$$

  where subscripts indicate on which tensor components we act.

Summary

- Some differential equations results tell us that the monodromy representation is analytic in h (probably in a small neighbourhood of 0). Hence what we finally get is a group homomorphism

  $$\rho_{KZ} : B_n \to \mathrm{Aut}_{\mathbb{C}[[h]]}(W^{\otimes n}[[h]])$$

  that allows us to view the monodromy representation coming from the KZ equations as a quantisation of the permutation representation.

What next

- Every (topological) quasitrianuglar quasi-bialgebra over $\mathbb{C}[[h]]$

  $$(H, \mu, \eta, \Delta, \varepsilon, R, \Phi), \quad R \in H \hat{\otimes} H, \quad \Phi \in H \hat{\otimes} H \hat{\otimes} H$$

  gives rise to a braided monoidal category, namely the finite-dimensional representations of H, and hence to representations of B_n. We shall express ρ_{KZ} in this way.

The Drinfeld-Kohno theorem

- For any complex Lie algebra $\mathfrak{g}$ and any symmetric invariant $t \in \mathfrak{g} \otimes \mathfrak{g}$ there exists

$$\Phi_{KZ} \in U(\mathfrak{g})^{\otimes 3}[[h]]$$

  (called from now on the Drinfeld associator) such that

  1. the trivial quantisation

$$U_{KZ} = (U(\mathfrak{g})[[h]], \mu, \eta, \Delta, \varepsilon, R_{KZ}, \Phi_{KZ}), \quad R_{KZ} := e^{ht/2}$$

     is a topological quasitriangular quasibialgebra over $\mathbb{C}[[h]]$ quantising $(U(\mathfrak{g}), \mu, \eta, \Delta, \varepsilon, 1 \otimes 1, 1 \otimes 1 \otimes 1)$ and
  2. the resulting representation of B_n on $W^{\otimes n}[[h]]$ is for all finite-dimensional representations of $\mathfrak{g}$ isomorphic to ρ_{KZ}.

- For $\mathfrak{g}$ semisimple and $t = \sum_{i=1}^{\dim(\mathfrak{g})} \frac{X_i \otimes X^i + X^i \otimes X_i}{2}$ ($\{X_i\}$ basis, $\{X^i\}$ dual basis with respect to the Killing form), U_{KZ} can be twisted by some $F \in U(\mathfrak{g})^{\otimes 2}[[h]]$ to get the Drinfeld-Jimbo quantisation $U_h = (U(\mathfrak{g})[[h]], \mu, \eta, F\Delta F^{-1}, \varepsilon, R_{DJ}, 1 \otimes 1 \otimes 1)$.

The Drinfeld-Jimbo algebra $U_\hbar(\mathfrak{sl}(2))$

- Generators H, E, F with relations

  $$[H,E] = 2E, \quad [H,F] = -2F$$

  as in $U(\mathfrak{sl}(2))$, but

  $$[E,F] = \frac{e^{\hbar H/2} - e^{-\hbar H/2}}{e^{\hbar/2} - e^{-\hbar/2}} = H \bmod \hbar.$$

- Coproduct

  $$\begin{aligned}\Delta(H) &= 1 \otimes H + H \otimes 1,\\ \Delta(E) &= 1 \otimes E + E \otimes e^{\hbar H/2},\\ \Delta(F) &= e^{-\hbar H/2} \otimes F + F \otimes 1.\end{aligned}$$

- $R_{DJ} = e^{\hbar(H \otimes H)/4} \sum_{n \geq 0} \frac{q^{n(n+1)/2}(1-q^{-2})^n}{[n]_q!} E^n \otimes F^n$, where $q = e^{\hbar/2}$ and $[n]_q!$ is the q-factorial.

An ODE

- Consider the equation

$$\frac{\mathrm{d}G}{\mathrm{d}z} = \hbar \left(\frac{A}{z} + \frac{B}{z-1} \right) G$$

Here A, B are noncommuting variables but z commutes with them. We will discuss formal solutions $G(A, B, z)$ of this equation - they say the analysis can be done rigorously...

- "The ODE is Fuchsian." $\rightsquigarrow$ There are two solutions of the form

$$G_0(z) = P(z) z^{\hbar A}, \quad G_1(z) = Q(1-z)(1-z)^{\hbar B}$$

where P, Q are power series in z with value 1 in $z = 0$. This can be seen by inserting this ansatz into the equation and considring the resulting equations for each power of z.

Drinfeld's associator

- Since G_0, G_1 are nonzero solutions of our ODE and this is homogeneous, their ratio is independent of z,

$$G_0(z) =: G_1(z)\Phi(A, B).$$

  This ratio $\Phi \in \mathbb{C}[[\hbar]]\langle A, B\rangle$ is what we are interested in.
- If G_a, $a \in [0, 1]$, is the unique solution of our ODE with $G_a(a) = 1$, then one has

$$\Phi(A, B) = \lim_{a \to 0} a^{-\hbar B} G_a(1 - a) a^{\hbar A}.$$

- Modulo $\hbar^4$, $\Phi(A, B)$ is equal to

$$1 - \frac{\zeta(2)}{(2\pi\sqrt{-1})^2}[A, B]\hbar^2 + \frac{\zeta(3)}{(2\pi\sqrt{-1})^3}([[A, B], B] - [A, [A, B]])\hbar^3.$$

Back to KZ

- Now we put $\Phi_{KZ} := \Phi(t_{12}, t_{23}) \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$.
- The link to KZ is this:

$$\psi(z_1, z_2, z_3) = (z_3 - z_1)^{\hbar(t_{12}+t_{23}+t_{13})} G(z), \quad z := \frac{z_2 - z_1}{z_3 - z_1}.$$

  satisfies KZ_3 iff G satisfies our ODE with $A = t_{12}, B = t_{23}$.
- In particular, we get two solutions ψ_i, $i = 0, 1$, of KZ_3 from G_i that are related by $\psi_1 = \psi_0 \Phi_{KZ}$.
- We know their asymptotic behaviour

$$\begin{aligned} \psi_0 &\sim (z_2 - z_1)^{\hbar t_{12}} (z_3 - z_1)^{\hbar(t_{23}+t_{13})}, \quad |z_2 - z_1| \ll |z_3 - z_1| \\ \psi_1 &\sim (z_3 - z_2)^{\hbar t_{23}} (z_3 - z_1)^{\hbar(t_{12}+t_{13})}, \quad |z_2 - z_3| \ll |z_1 - z_3| \end{aligned}$$

- Another clue for all this: KZ_3 is invariant under $z_i \mapsto az_i + b$, $a, b \in \mathbb{C}$. Using this we can reduce the 3 variables to 1. If (z_1, z_2, z_3) is located in one of the above regions of $\mathbb{C}^3$, z is close to $0, 1$ where we understand our ODE.

55 The monodromy of KZ_3

- The monodromy of KZ_3 is computed to be the one given by R_{KZ}, Φ_{KZ} using the parameter z and an explicit parametrisation of loops as above: $s_i \in B_n$ is represented by

  $$z_i(s) = \frac{1}{2}(2i-1-e^{\pi\sqrt{-1}s}), z_{i+1}(s) = \frac{1}{2}(2i-1+e^{\pi\sqrt{-1}s}), z_j = j-1$$

  which gives for s_1

  $$z(s) = \frac{2e^{\pi\sqrt{-1}s}}{3+e^{\pi\sqrt{-1}s}}, \quad z = \frac{z_2 - z_1}{z_3 - z_1}$$

- This winds from $z(0) = 1/2$ to $z(1) = -1$ round the singularity zero. Making the loop smaller we get s_1 represented by a small half circle round 0.

The monodromy of KZ_3

- From the definition of $G_0(z) \sim z^{\hbar t_{12}}$ we see that it changes under a whole circle around zero by a factor $e^{\hbar t_{12}}$. Hence the monodromy of KZ_3 has

$$\rho_{KZ}(s_1) = e^{\hbar t_{12}/2}$$

- Similarly, one gets

$$\rho_{KZ}(s_2) = \Phi_{KZ}^{-1} e^{\hbar t_{12}/2} \Phi_{KZ}$$

R_{KZ}, Φ_{KZ} gives a braided quasibialgebra

- First we need that for all $X \in \mathfrak{g}$ we have

  $$[\Delta(X), R_{KZ}] = [\Delta^{(2)}(X), \Phi_{KZ}] = 0$$

  since the trivial quantisation is coassociative and cocommutative, so that the effect of the braiding and the associator should be invisible in

  $$\Delta^{\mathrm{op}} = R_{KZ} \Delta R_{KZ}^{-1}, \quad (\mathrm{id} \otimes \Delta)\Delta = \Phi_{KZ}(\Delta \otimes \mathrm{id})\Delta \Phi_{KZ}^{-1}.$$

- This follows from

  $$[\Delta(X), t] = 0 \Rightarrow [\Delta(X), t^n] = 0 \Rightarrow [\Delta(X), e^{\hbar t/2}] = 0.$$

  and

  $$[\Delta^{(2)}(X), t_{12}] = [\Delta^{(2)}(X), t_{23}] = 0$$

  as follows from

  $$\Delta^{(2)}(X) = \Delta(X)_{12} + 1 \otimes 1 \otimes X = X \otimes 1 \otimes 1 + \Delta(X)_{23}.$$

$R := R_{KZ}, \Phi := \Phi_{KZ}$ gives a braided quasibialgebra

- The following axioms are checked using KZ_3:

$$(\Delta \otimes \mathrm{id})R = \Phi_{321} R_{13} \Phi_{132}^{-1} R_{23} \Phi$$

$$(\mathrm{id} \otimes \Delta)R = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi^{-1}$$

- For this one needs to define more solutions using other z's that work in other regions of $\mathbb{C}^3$:

$$\begin{aligned}
\psi_2 &\sim (z_2 - z_3)^{\hbar t_{23}} (z_2 - z_1)^{\hbar(t_{12}+t_{13})}, && |z_3 - z_2| \ll |z_2 - z_1| \\
\psi_3 &\sim (z_3 - z_1)^{\hbar t_{13}} (z_2 - z_1)^{\hbar(t_{12}+t_{23})}, && |z_3 - z_1| \ll |z_2 - z_1| \\
\psi_4 &\sim (z_1 - z_3)^{\hbar t_{13}} (z_2 - z_3)^{\hbar(t_{12}+t_{23})}, && |z_3 - z_1| \ll |z_2 - z_3| \\
\psi_5 &\sim (z_2 - z_1)^{\hbar t_{12}} (z_2 - z_3)^{\hbar(t_{13}+t_{23})}, && |z_1 - z_2| \ll |z_2 - z_3|
\end{aligned}$$

$R := R_{KZ}, \Phi := \Phi_{KZ}$ gives a braided quasibialgebra

- Now one relates these solutions using R, Φ. One has

$$\psi_1 = \psi_0 \Phi_{123}, \quad \psi_3 = \psi_2 \Phi_{132}, \quad \psi_4 = \psi_5 \Phi_{312}.$$

- Furthermore, one gets ψ_2 from ψ_1 by exchanging $z_2 \leftrightarrow z_3$. Hence

$$\psi_1 = \psi_2 R_{23}.$$

  Similary, one has

$$\psi_3 = \psi_4 R_{13}.$$

- From all this one gets

$$\psi_0 = \psi_5 \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi_{123}$$

$R := R_{KZ}, \Phi := \Phi_{KZ}$ gives a braided quasibialgebra

- Note next that if $t = \sum_i X_i \otimes Y_i$, then

$$(\Delta \otimes \mathrm{id})(t) = \sum_i X_i \otimes 1 \otimes Y_i + \sum_i 1 \otimes X_i \otimes Y_i = t_{13} + t_{23}.$$

- To get from ψ_5 to ψ_0, z_3 swaps places with z_1, z_2, $(z_1, z_2, z_3) \mapsto (z_3, z_1, z_2)$. This yields

$$\psi_0 = \psi_5 (\Delta \otimes \mathrm{id}) R.$$

  Uniqueness of solutions proves the first of our relations. The other one is similar.

R, Φ gives a braided quasibialgebra

- The pentagon relation

$$\begin{aligned} & (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi) \\ = \; & \Phi_{234}(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)\Phi_{123} \end{aligned}$$

  follows with similar tricks using KZ_4 - now we can reduce from 4 to 2 variables, so the ODE becomes a PDE, and again one has to find suitable solutions that behave asymptotically near the singularities of this PDE nicely. Then one moves them around and produces relations as before leading finally to the pentagon relation.