Introduction to deformation quantization after Kontsevich

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Formal deformations

- A – associative algebra over a field k
- $\mu\colon A \otimes_k A \to A$
- $A[[\varepsilon]] \otimes_{k[[\varepsilon]]} A[[\varepsilon]] \to A[[\varepsilon]]$

$$\begin{array}{ccc} A[[\varepsilon]] \otimes_{k[[\varepsilon]]} A[[\varepsilon]] & \longrightarrow & A \otimes_k A \\ \Big\downarrow{\scriptstyle *} & & \Big\downarrow{\scriptstyle \mu} \\ A[[\varepsilon]] & \longrightarrow & A \end{array}$$

Formal deformations

$$\left(\sum_{i=0}^{\infty} a_i \varepsilon^i\right) \cdot \left(\sum_{i=0}^{\infty} a_i \varepsilon^i\right) = \sum_{i=0}^{\infty} \left(\sum_{l=0}^{k} a_{k-l} \cdot b_l\right) \varepsilon^k$$

- *star product*
- $u, v \in A[[\varepsilon]]$

$$u * v = u \cdot v \mod (\varepsilon)$$

- $a, b \in A$

$$a * b = ab + B_1(a,b)t + B_2(a,b)t^2 + \dots$$

- $B_i : A^2 \to A$ – bilinear

R-deformations

- A – associative algebra
- $R = k[\varepsilon]/(\varepsilon^n)$
- $(A \otimes_k R) \otimes_R (A \otimes_k R) \to A \otimes_k R$

$$\begin{array}{ccc} (A \otimes_k R) \otimes_R (A \otimes_k R) & \longrightarrow & A \otimes_k A \\ \downarrow * & & \downarrow \mu \\ A \otimes_k R & \longrightarrow & A \end{array}$$

- $k[\varepsilon]/(\varepsilon^2)$-deformations – *infinitesimal deformations*

$$\{k[[\varepsilon]] - \text{deformations}\} \xrightarrow{1-1} \varprojlim \{k[\varepsilon]/(\varepsilon^n) - \text{deformations}\}$$

Equivalence of deformations

- $* \sim *'$
- $k[[\varepsilon]]$-module morphism $g\colon A[[\varepsilon]] \to A[[\varepsilon]]$
- $R = k[\varepsilon]/(\varepsilon^n)$-module morphism $g\colon A \otimes_k R \to A \otimes_k R$

$$g(a * b) = g(a) *' g(b)$$

- $a, b \in A[[\varepsilon]]$ or $A \otimes_k R$

Lie bracket

Examples

- (A, μ) associative and commutative algebra
- $*$ associative formal deformation of μ
- $a, b \in A$

$$\{a, b\} = B_1(a, b) - B_1(b, a)$$

- $\{-, -\}$ is a *Lie bracket*
  - $\{-, -\}$ is bilinear
  - $\{a, b\} = -\{b, a\}$ – alternating
  - Jacobi identity
    $\{ \mathrm{a}, \{ \mathrm{b}, \mathrm{c}\}\} + \{ \mathrm{c}, \{ \mathrm{a}, \mathrm{b}\}\} + \{ \mathrm{b}, \{ \mathrm{c}, \mathrm{a}\}\} = 0$

Poisson bracket

Examples

- $\{-,-\}$ is a *Poisson bracket*
- $a, b, c \in A$

$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

- $\{-,-\}$ depends only on equivalence class of $*$

Differential graded Lie algebras

- L – $\mathbb{Z}$-graded vector space
- Lie bracket – linear map of degree 0

$$[-,-]\colon L \otimes L \to L$$

- differential – linear map of degree 1

$$d\colon L \to L$$

- $X \in L^p$, $Y \in L^q$
  - graded-commutative

$$[X, Y] = -(-1)^{pq}[Y, X]$$

  - graded Jacobi identity

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{pq}[Y, [X, Z]]$$

  - differential is a derivation

$$d([X, Y]) = [dX, Y] + (-1)^{p}[X, dY]$$

Maurer–Cartan equation in DGLA

- $(L, [-,-], d)$ is a DGLA

$$L = \bigoplus_{i \in \mathbb{Z}} L^i$$

- L^0 – Lie algebra
- Maurer–Cartan equation in DGLA
- $X \in L^1$

$$dX + \frac{1}{2}[X, X]$$

- solutions play important role

Maurer–Cartan form

Examples

- G – Lie group
- $\mathfrak{g} = \mathrm{Lie}(G) = T_e G$ – its Lie algebra
- $L_g : G \to G,\ h \mapsto gh$ – left multiplication

$$X \in T_g G \mapsto (L_{g^{-1}})_*(X) \in T_e G$$

- $TG \to \mathfrak{g}$ – $\mathfrak{g}$-valued 1-form
- *Maurer–Cartan form* ω

$$\omega(X) = (L_{g^{-1}})_*(X)$$

Exterior differential of Maurer–Cartan form

Examples

- X, Y – left invariant
  - $(L_g)_* X = X$
  - $\omega(X) = \mathrm{const}$
  - $\omega([X, Y]) = [\omega(X), \omega(Y)]$
- $d\omega$ – $\mathfrak{g}$-valued 2-form

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

- $X(\omega(Y)) = Y(\omega(X)) = 0$
- $d\omega(X, Y) + [\omega(X), \omega(Y)] = 0$

Maurer–Cartan equation

Examples

- *Maurer–Cartan equation*

$$d\omega + \frac{1}{2}[\omega, \omega] = 0$$

- wedge product of $\mathfrak{g}$-valued 1-forms
- use bracket instead of multiplication

$$\begin{aligned}(\omega \wedge \omega)(X, Y) &= \frac{1}{2}([\omega(X), \omega(Y)] - [\omega(Y), \omega(X)]) \\ &= (\frac{1}{2}[\omega, \omega])(X, Y)\end{aligned}$$

- *Maurer–Cartan equation*

$$d\omega + \omega \wedge \omega = 0$$

Maurer–Cartan equation

Examples

- *Maurer–Cartan equation*

$$d\omega + \frac{1}{2}[\omega, \omega] = 0$$

- wedge product of $\mathfrak{g}$-valued 1-forms
- use bracket instead of multiplication

$$(\omega \wedge \omega)(X, Y) = \frac{1}{2}([\omega(X), \omega(Y)] - [\omega(Y), \omega(X)]) = (\frac{1}{2}[\omega, \omega])(X, Y)$$

- *Maurer–Cartan equation*

$$d\omega + \omega \wedge \omega = 0$$

Jacobi identity

Examples

- $d^2 = 0$
- $d(\omega \wedge \omega) = 0$
- $d(\omega \wedge \omega)$ – $\mathfrak{g}$-valued 3-form

$$d(\omega \wedge \omega)(X, Y, Z) = [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$$

- *Jacobi identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

- $d^2 = 0 \Leftrightarrow$ Jacobi identity

Next...

- DGLA $T_{\text{poly}} = (\mathcal{V}, [-,-]_{SN}, d = 0)$ of polyvector fields
- Schouten-Nijenhuis bracket
- DGLA $(\mathcal{C}, [-,-]_G, d)$ of Hochschild cochains
- Gerstenhaber bracket
- DGLA $D_{\text{poly}} = (\mathcal{D}, [-,-]_G, d)$ of polydifferential operators

Vector fields, polyvector fields

- M – manifold
- TM – tangent bundle
- global sections $\Gamma(M, TM)$ – *vector fields*
- Lie algebra structure
- Lie bracket on vector fields
- $X, Y \in \Gamma(M, TM)$

$$[X, Y] = XY - YX \in \Gamma(M, TM)$$

- global sections $\Gamma(M, \Lambda^p(TM))$ – *polyvector fields*

Vector fields, polyvector fields

- Lie derivative
- $X, Y \in \Gamma(M, TM)$

$$\mathcal{L}_X(Y) = [X, Y]$$

- $X \in \Gamma(M, TM)$, $f \in C^\infty(M)$

$$\mathcal{L}_X(Y) = [X, f] = \sum_{i=1}^{\dim M} X^i \frac{\partial f}{\partial x^i}$$

- extend it into DGLA
- $\mathcal{V}^{-1} = C^\infty(M)$, $\mathcal{V}^0 = \Gamma(M, TM)$
- $\mathcal{V}^p = \Gamma(M, \Lambda^{p+1} TM)$
- differential $d = 0$
- bracket?

$$[-,-] \colon \mathcal{V}^k \otimes \mathcal{V}^l \to \mathcal{V}^{k+l}$$

Extension of Lie bracket

- extend to $X \in \mathcal{V}^0$, $Y_1 \wedge \ldots \wedge Y_k \in \mathcal{V}^{k-1}$

$$[X, Y_1 \wedge \ldots \wedge Y_k] = \sum_{i=1}^{k} (-1)^{i+1} [X, Y_i] \wedge Y_1 \wedge \cdots \wedge \widehat{Y_i} \wedge \cdots \wedge Y_k$$

- extend to $X_1 \wedge \cdots \wedge X_k \in \mathcal{V}^{k-1}$, $f \in C^\infty(M)$

$$[X_1 \wedge \cdots \wedge X_k, f] = \sum_{i=1}^{k} (-1)^{k-i} \mathcal{L}_{X_i}(f) X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_k$$

- finally for $X_1 \wedge \cdots \wedge X_k \in \mathcal{V}^{k-1}$, $Y_1 \wedge \cdots \wedge Y_k \in \mathcal{V}^{l-1}$

$$[X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \ldots \wedge Y_k]$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{l} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \widehat{X_i} \wedge Y_1 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_l$$

Schouten bracket

- $\exists$ extension of $[-,-]$ to $\mathcal{V} = \bigoplus_{i=-1}^{\infty} \mathcal{V}^i$
- *Schouten–Nijenhuis bracket*

$$[-,-]_{SN} \colon \mathcal{V}^k \otimes \mathcal{V}^l \to \mathcal{V}^{k+l}$$

- graded commutative

$$[X,Y]_{SN} = -(-1)^{xy}[Y,X]$$

- graded Jacobi identity

$$[X,[Y,Z]_{SN}]_{SN} = [[X,Y]_{SN},Y]_{SN} + (-1)^{xy}[Y,[X,Z]_{SN}]_{SN}$$

- $(\mathcal{V},[-,-]_{SN}, d=0)$ is a DGLA

Poisson bracket

Examples

- bivector field $\Pi \in \mathcal{V}^1 = \Gamma(M, \Lambda^2 TM)$, $f, g \in C^\infty(M)$
- *Poisson bracket*

$$\{f, g\} = \frac{1}{2}\langle \Pi, df \wedge dg \rangle$$

- Jacobi identity for $\{-, -\}$

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$

- equivalent to

$$[\Pi, \Pi]_{SN} = 0$$

- Π is a solution of Maurer–Cartan equation

$$d\Pi + \frac{1}{2}[\Pi, \Pi]_{SN} = 0$$

Hochschild complex

- A – algebra
- $\mathrm{C}(A, A)^\bullet$ – cochain complex

$$\mathrm{C}(A, A)^p = \begin{cases} \mathrm{Hom}_k(A^{\otimes p}, A) & p \geq 0 \\ 0 & p < 0 \end{cases}$$

- coboundary $d\colon \mathrm{C}(A, A)^\bullet \to \mathrm{C}(A, A)^{\bullet+1}$

$$\begin{aligned} df(a_0, \ldots, a_p) = a_0 f(a_1, \ldots, a_p) \\ - \sum_{i=0}^{p-1} (-1)^i f(a_0, \ldots, a_i a_{i+1}, \ldots, a_p) \\ + (-1)^{p-1} f(a_0, \ldots, a_{p-1}) a_p \end{aligned}$$

Hochschild cohomology

- $\mathrm{HH}^p(A, A) = \mathrm{H}^p(\mathrm{C}(A, A)^\bullet, d)$
- *cup product*
- $f \in \mathrm{C}(A, A)^p$, $g \in \mathrm{C}(A, A)^q$

$$(f \cup g)(a_1, \dots, a_{p+q}) = f(a_1, \dots, a_p) \cdot g(a_{p+1}, \dots a_q)$$

- associative but not commutative
- commutative on the level of cohomology

0-cocycles

Examples

- 0-cocycles

$$C(A,A)^0 = \mathrm{Hom}_k(k,A)$$

- $g \in A$, $dg = 0$?

$$dg(a) = ag - ga = 0$$

- $g \in Z(A)$ – center of A

1-cocycles

Examples

- 1-cocycles
- $f : A \to A,\ df = 0?$

$$-df(a,b) = af(b) - f(ab) + af(b) = 0$$

- $f \in \mathrm{Der}(A)$ – derivations $A \to A$
- 1-coboundaries
- $f : A \to A,\ f = dg,\ g \in A?$

$$dg(b) = bg - gb = [b, g]$$

- $f \in \mathrm{Inn\,Der}(A)$ – inner derivations $A \to A$

Gerstenhaber product

- $f \in \mathrm{C}(A,A)^p$, $g \in \mathrm{C}(A,A)^q$ – Hochschild cochains
- $f \bullet g \in \mathrm{C}(A,A)^{p+q-1}$

$$(f \bullet g)(a_1, \ldots, a_{p+q-1}) =$$

$$\sum_{i=0}^{p-1} (-1)^{i(q+1)} f(a_1, \ldots, a_i, g(a_{i+1}, \ldots, a_{i+q}), a_{i+q+1}, \ldots, a_{p+q-1})$$

- not associative

Gerstenhaber product is not associative

Examples

- $f \in \mathrm{C}(A, A)^2$, $g, h \in C(A, A)^1$
- $g \bullet h \in \mathrm{C}(A, A)^1$, $(f \bullet g) \bullet h, f \bullet (g \bullet h) \in \mathrm{C}(A, A)^2$

$$(g \bullet h)(a) = g(h(a))$$

$$\begin{aligned}((f \bullet g) \bullet h)(a, b) &= (f \bullet g)(h(a), b) + (f \bullet g)(a, h(b)) \\ &= f(g(h(a)), b) + f(h(a), g(b)) \\ &\quad + f(g(a), h(b)) + f(a, g(h(b)))\end{aligned}$$

$$\begin{aligned}(f \bullet (g \bullet h))(a, b) &= f((g \bullet h)(a), b) + f(a, (g \bullet h)(b)) \\ &= f(g(h(a)), b) + f(a, g(h(b)))\end{aligned}$$

- associator

$$A^{\bullet}(f,g,h) = f \bullet (g \bullet h) - (f \bullet g) \bullet h$$

- supersymmetric

$$A^{\bullet}(f,g,h) = (-1)^{(q-1)(r-1)} A^{\bullet}(f,h,g)$$

- *Gerstenhaber bracket*

$$[f,g]_G = f \bullet g - (-1)^{(p-1)(q-1)} g \bullet f$$

- graded Jacobi identity

$$[f,[g,h]_G]_G = [[f,g]_G,h]_G + (-1)^{pq}[g,[f,h]_G]_G$$

Hochschild differential and Gerstenhaber bracket

- multiplication

$$\mu \colon A \otimes A \to A$$

- $f \in C(A, A)^p$

$$\begin{aligned} df(a_0, \dots, a_p) &= \mu(a_0, f(a_1, \dots, a_p)) \\ &- \sum_{i=0}^{p-1} (-1)^i f(a_0, \dots, \mu(a_i, a_{i+1}), \dots, a_p) \\ &+ (-1)^{p-1} \mu(f(a_0, \dots, a_{p-1}), a_p) \end{aligned}$$

- Gerstenhaber bracket with multiplication

$$df = \mu \bullet f + (-1)^{p-1} f \bullet \mu = [\mu, f]_G$$

Associativity in terms of Gerstenhaber bracket

- $\mu\colon A \otimes A \to A$ associative

$$[\mu, \mu]_G(a, b, c) = \mu(\mu(a, b), c) - \mu(a, \mu(b, c)) = A^{\mu}(a, b, c) = 0$$

- in general
- $B\colon A \otimes A \to A$ is associative iff

$$[B, B]_G = 0$$

- associator

$$[B, B]_G(a, b, c) = A^B(a, b, c)$$

DGLA of Hochschild cochains

- $\mathcal{C}^p = \mathrm{C}(A, A)^{p+1}$
- $[-,-]_G : \mathcal{C}^p \otimes \mathcal{C}^q \to \mathcal{C}^{p+q}$ – bracket
- $d = [\mu, -]_G$ – differential
- $(\mathcal{C}, [-,-]_G, d)$ is a DGLA

Maurer–Cartan equation and associativity

Examples

- $B \in \mathcal{C}^1$
- $B\colon A \otimes A \to A$
- $* = \mu + B$
- $*$ is associative iff $[*, *]_G = 0$

$$\begin{aligned}[*, *]_G &= [\mu + B, \mu + B]_G \\ &= [\mu, \mu]_G + [\mu, B]_G + [B, \mu]_G + [B, B]_G \\ &= 2[\mu, B]_G + [B, B]_G \\ &= 2dB + [B, B]_G\end{aligned}$$

- Maurer–Cartan equation

$$dB + \frac{1}{2}[B, B]_G = 0$$

Infinitesimal deformations

Examples

- infinitesimal deformation of product $a * b = ab + B(a,b)\varepsilon$, $B\colon A \otimes A \to A$
- $*$ associative iff

$$dB(a,b,c) = aB(b,c) - B(ab,c) + B(a,bc) - B(a,b)c = 0$$

- Hochschild 2-cocycles
- equivalent deformations

$$B'(a,b) - B(a,b) = ag(b) - g(ab) + g(a)b$$

- Hochschild 2-coboundaries
- infinitesimal deformations up to equivalence – $\mathrm{HH}^2(A,A)$

DGLA of multidifferential operators

- M – manifold
- $A = C^\infty(M)$ – algebra of smooth functions
- $C(C^\infty(M), C^\infty(M))^\bullet$
- subcomplex $\mathcal{D}^\bullet \in C(C^\infty(M), C^\infty(M))^\bullet$
- cochains are multidifferential operators $C^\infty(M)^{\otimes p} \to C^\infty(M)$
- acting as 0 on constant functions

Bidifferential operators

- $P \colon C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$
- $x_1, \ldots, x_n$ – local coordinates on M

$$P(a,b) = \sum_{\alpha,\beta} f_{\alpha\beta} \left(\frac{\partial^{|\alpha|}}{\partial x^{|\alpha|}} a \right) \left(\frac{\partial^{|\beta|}}{\partial x^{|\beta|}} b \right)$$

- α, β – multiindices
- $f_{\alpha\beta}$ – smooth functions
- $f_{\alpha\beta}$ vanish for almost all (α, β)

Multidifferential operators

- $P\colon C^\infty(M)^{\otimes p} \to C^\infty(M)$
- $x_1, \ldots, x_n$ – local coordinates on M

$$P(a_1, a_2, \ldots, a_p) =$$

$$\sum_{\alpha_1,\alpha_2,\ldots,\alpha_p} f_{\alpha_1\alpha_2\ldots\alpha_p} \left(\frac{\partial^{|\alpha_1|}}{\partial x^{|\alpha_1|}} a_1\right) \left(\frac{\partial^{|\alpha_2|}}{\partial x^{|\alpha_2|}} a_2\right) \ldots \left(\frac{\partial^{|\alpha_p|}}{\partial x^{|\alpha_p|}} a_p\right)$$

- $\alpha_1, \alpha_2, \ldots, \alpha_p$ – multiindices
- $f_{\alpha_1\alpha_2\ldots\alpha_p}$ – smooth functions
- $f_{\alpha_1\alpha_2\ldots\alpha_p}$ vanish for almost all $(\alpha_1, \alpha_2, \ldots, \alpha_p)$

DGLA of multidifferential operators

- $\mathcal{D}^p$ – $(p+1)$-differential operators on $C^\infty(M)^{\otimes(p+1)}$
- $[-,-]_G\colon \mathcal{D}^p \otimes \mathcal{D}^q \to \mathcal{D}^{p+q}$ – bracket
- d – Hochschild differential
- not inner – multiplication μ does not vanish on constants
- $D_{\mathrm{poly}} = (\mathcal{D}, [-,-]_G, d)$ is a DGLA