Amenability and coarse embeddings of warped cones

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EPFL, 6 August 2015

Amenability and property A

A group Γ is **amenable** if for every finite set $S \subseteq \Gamma$ and $\varepsilon > 0$ there is a finitely supported probability measure $\mu \in \mathrm{Prob}(\Gamma) \subseteq \ell_1(\Gamma)$ such that

$$\forall s \in S \quad \|\mu - s\mu\| < \varepsilon.$$

Fix some metric on Γ and let N be so large that $\operatorname{supp} \mu \subseteq B(1, N)$. Then, the map $A \colon \Gamma \to \mathrm{Prob}(\Gamma)$ given by $A(\gamma) = \gamma\mu$ satisfies $\operatorname{supp} A(\gamma) \subseteq B(\gamma, N)$.

A (discrete) metric space (X, d) has **property A** if for every $R < \infty$ and $\varepsilon > 0$ there is a map $A \colon X \to \mathrm{Prob}(X)$ and $N < \infty$ such that $\operatorname{supp} A(x) \subseteq B(x, N)$ and

$$\forall_{d(x,y) \leq R} \ \|A(x) - A(y)\| < \varepsilon.$$

Property A and coarse embeddings

Function $f\colon (X,d) \to \ell_1$ is a **coarse embedding** if for any sequence $(x_m, y_m) \in X^2$:

$$d(x_m, y_m) \to \infty \iff \|f(x_m) - f(y_m)\| \to \infty.$$

Let $A^{(n)}$ be a map from the definition of property A for $R = n$ and $\varepsilon = 2^{-n}$. Then the map $f\colon X \to \bigoplus_n \ell_1(X) \simeq \ell_1$ is a coarse embedding:

$$f(x) = \bigoplus_n A^{(n)}(x) - A^{(n)}(x_0)$$

(where x_0 is some fixed point).

Question

Does every metric space (finitely generated group) admitting a coarse embedding satisfy property A?

Property A and coarse embeddings

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Answer: No!

- For metric spaces: Nowak, 2007.
- For metric spaces with bounded geometry: Arzhantseva–Guentner–Špakula, 2012.
- For finitely generated groups: Arzhantseva–Osajda, Osajda (preprints, 2014).

Warped metric (Roe, 2005)

Data:

- Γ – group generated by a finite set S
- (X, d) – metric space with a continuous Γ-action

Assume for simplicity that X is a geodesic space.

For every $x \in X$ and $s \in S$ glue an interval between x and sx and declare its length to be one. Calculate the path metric in the new space – what we get is the **warped metric** d_Γ.

d_Γ is the largest metric satisfying

$$d_\Gamma(x, x') \leq d(x, x'), \qquad d_\Gamma(x, sx) \leq 1 \; \forall s \in S.$$

Warped cone (Roe, 2005)

- Y – compact metric Γ-space embedded as a subset of a sphere $S^{n-1} \subseteq \mathbb{R}^n$ with the Euclidean metric d
- $\mathcal{O}Y = \{ty \mid t \in [0,\infty),\ y \in Y\} \subseteq \mathbb{R}^n$ – euclidean cone over Y

The **warped cone** $\mathcal{O}_\Gamma Y$ over Y with respect to a Γ-action is the metric space $(\mathcal{O}Y, d_\Gamma)$.

Example

Let $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ act on $Y = \mathbb{T}^n$, $n \geq 3$. Then, $\mathcal{O}_{\mathrm{SL}_n(\mathbb{Z})}\mathbb{T}^n$ contains isometrically embedded expanders.

- Γ – discrete group
- $\mathcal{F} = \{f_n \colon \Gamma \to F_n\}$ – sequence of quotient maps onto finite groups (we require $\forall \gamma \in \Gamma \setminus \{1\}\ \exists n\ f_n(\gamma) \neq 1$)

Consider the product homomorphism $F \colon \Gamma \to \prod F_n$. The closure of its image is the **completion** $\widehat{\Gamma}(\mathcal{F})$ of Γ with respect to $\mathcal{F}$.

We endow the product $\prod F_n$ with the following metric:

$$d\left((g_n), (g_n')\right) = \max a_n \cdot d_{bin}(g_n, g_n'),$$

where $a_n \to 0$.

Warped cones over profinite completions

Theorem (Roe, 2005)

Let μ be a Γ-invariant measure on Y and assume that there exists a subset $P \subseteq Y$ of positive measure on which the action of Γ is free.

1. If $\mathcal{O}_\Gamma Y$ has property A, then Γ is amenable.
2. If $\mathcal{O}_\Gamma Y$ coarsely embeds into ℓ_1, then Γ has the Haagerup property.

Theorem (S., 2015)

Let $\widehat{\Gamma}(\mathcal{F})$ be any completion of Γ. The warped cone $\mathcal{O}_\Gamma \widehat{\Gamma}(\mathcal{F})$ has property A if and only if Γ is amenable.

Embeddable warped cones without property A

Assume:

- Γ – non-amenable group;
- $\mathcal{F} = \{f_n \colon \Gamma \to F_n\}$– sequence of quotient maps onto finite groups such that $\ker f_n \supseteq \ker f_{n+1}$;
- sequence F_n embeds coarsely into ℓ_1.

Theorem / Example (S., 2015)

The warped cone $\mathcal{O}_\Gamma \widehat{\Gamma}(\mathcal{F})$ does not have property A but embeds coarsely into ℓ_1.

Thank you!