# Warped cones, profinite completions, coarse embeddings and property A

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Partly based on joint work with Piotr W. Nowak.

# Warped metric on a $\Gamma$-space

- $\Gamma = \langle S \rangle$, $|S| < \infty$
- $\Gamma \curvearrowright (X, d)$

**Definition (Roe, 2005)**

Warped metric $d_\Gamma$ is the largest metric satisfying:

$$d_\Gamma(x, x') \leq d(x, x'), \qquad d_\Gamma(x, sx) \leq 1 \ \forall s \in S.$$

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**Proof of correctness**

1. There exists a metric satisfying the two: $\min(d, 1)$.
2. Supremum of metrics is a metric:

$$\sup d(x, z) \leq \sup d(x, y) + d(y, z) \leq \sup d(x, y) + \sup d(y, z)$$

# Warped metric on a $\Gamma$-space: geometric intuition

- $(X, d)$ – geodesic space
- For each pair of points $(x, sx)$ glue an interval of length 1 between $x$ and $sx$.
- Calculate the path metric in the space with all the extra intervals.
- Its restriction to $X$ is the warped metric $d_\Gamma$!

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**Corollary**

Being a quasi-geodesic space is preserved by warping.

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**Corollary**

Being a quasi-geodesic space is preserved by warping.

**Example (Hyun Jeong Kim, 2006)**

$X = \mathbb{R}^2$, $\Gamma = \mathbb{Z}$ acts by rotating by angle $\theta$. There are infinitely many non-quasi-isometric warped planes $(\mathbb{R}^2, d_\mathbb{Z})$ depending on $\theta$.

# Warping: general motivation

$d_\Gamma(x, sx) \leq 1$

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$$
\begin{aligned}
& d_\Gamma(x, sx) \leq 1 \\
\Longrightarrow \quad & \operatorname{dist}(id_{(X,d_\Gamma)}, \gamma) \leq |\gamma| \\
\Longrightarrow \quad & \text{Rich Roe algebras.}
\end{aligned}
$$

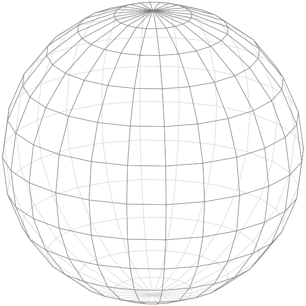
**Conjecture (Druțu–Nowak, 2015)**

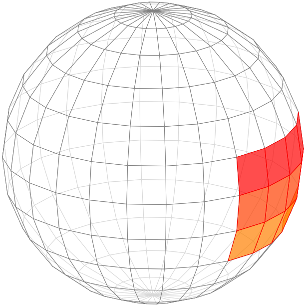
Warped cones over actions with a spectral gap violate the coarse Baum–Connes conjecture.

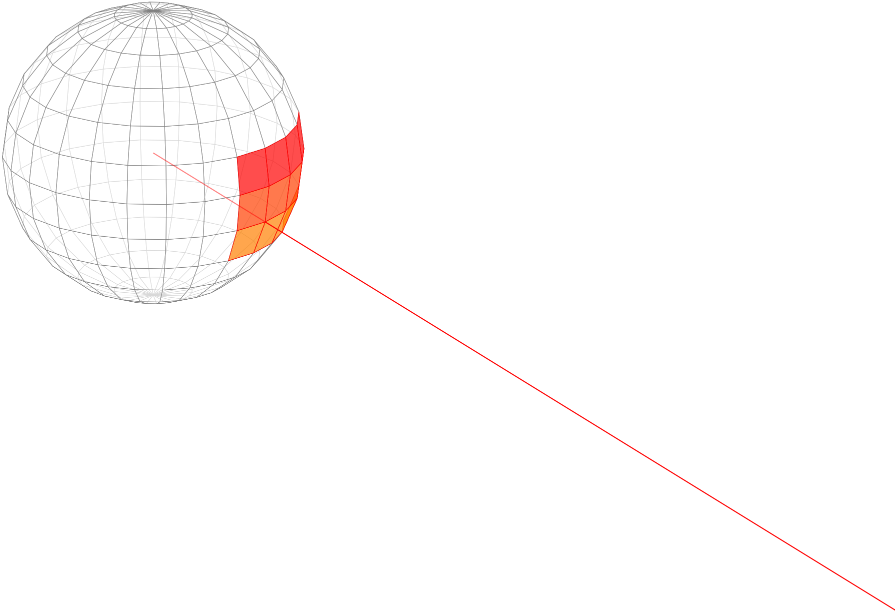
# Warped cones

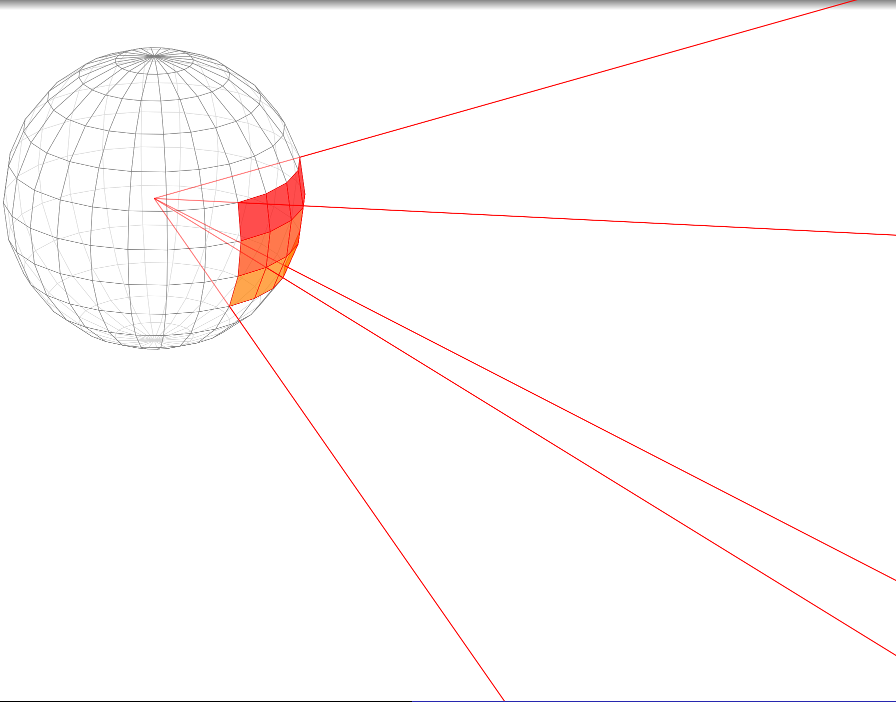
- $\Gamma \curvearrowright Y$ – compact subset of $S^n \subseteq \mathbb{R}^{n+1}$
- $X$ – infinite cone over $Y$ with the induced $\Gamma$ action:

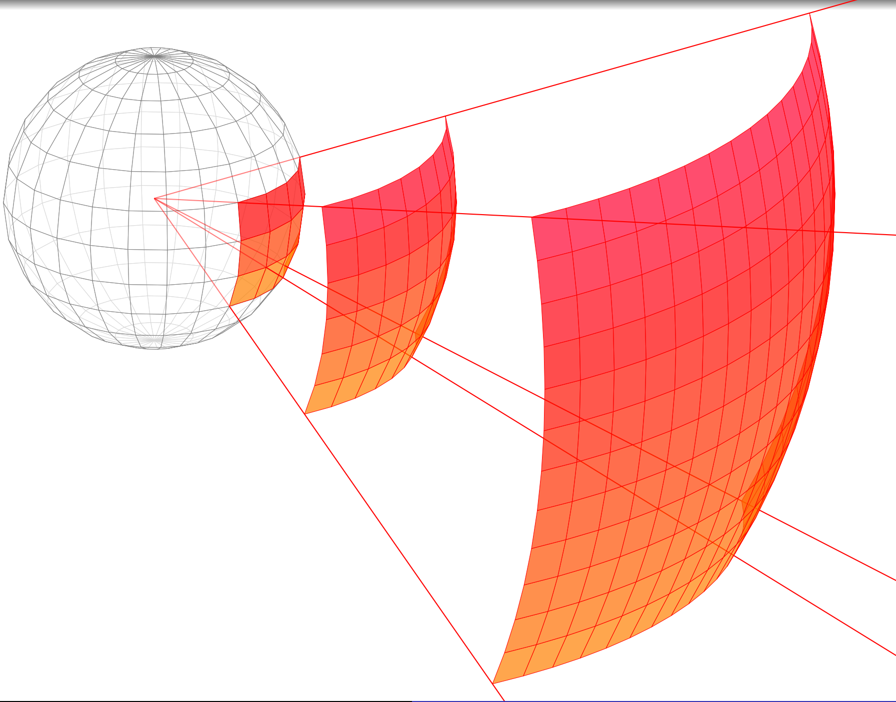
$$\{ty \mid t \in [0, \infty),\ y \in Y\} \subseteq (\mathbb{R}^{n+1}, d)$$











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- Notation: $\mathcal{O}Y := (X, d)$, $\mathcal{O}_\Gamma Y := (X, d_\Gamma)$.

**Non-example: the case of a finite $\Gamma$**

If $\Gamma$ is finite, then $\mathcal{O}_\Gamma Y \simeq \mathcal{O}\, Y/\Gamma$, e.g.:

- for the antipodal action $\mathbb{Z}_2 \curvearrowright S^n$: $\mathcal{O}_{\mathbb{Z}_2} S^n \simeq \mathcal{O}\, \mathbb{R}\mathrm{P}^n$;
- for rational $\theta$: $\mathcal{O}_{\mathbb{Z}} S^1 = \mathcal{O}_{\mathbb{Z}_k} S^1 \simeq \mathcal{O}\, S^1/\mathbb{Z}_k \simeq \mathcal{O} S^1 = \mathbb{R}^2$.

# Motivation / outline of the talk

- Warped cones as a generalisation of box spaces:

  relation of equivariant properties of $\Gamma$
  and coarse properties of $\mathcal{O}_\Gamma Y$.

- Refinement of the former:

  relation of dynamic properties of $\Gamma \curvearrowright Y$
  and coarse properties of $\mathcal{O}_\Gamma Y$.

- Spaces with interesting coarse properties, e.g.:
  - coarsely embeddable in $\ell_p$ but without property A;
  - not coarsely embeddable into any Banach space of non-trivial type.

# Recall the definitions

**Definition**

$f\colon X \to Y$ is a **coarse embedding** if there are non-decreasing functions $\rho_-, \rho_+\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{r\to\infty} \rho_\pm(r) = \infty$ such that

$$\rho_-(d(x,x')) \leq d(f(x), f(x')) \leq \rho_+(d(x,x')).$$

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Equivalently: for sequences $(x_n)$, $(x'_n)$

$$d(x_n, x'_n) \to \infty \iff d(f(x_n), f(x'_n)) \to \infty.$$

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$\Gamma$ is **amenable** if for each $\varepsilon > 0$ and finite $R \subseteq \Gamma$ there exists $\mu \in S(\ell_1(\Gamma))$ such that:

- $\|\mu - s\mu\| \leq \varepsilon$ if $s \in R$;
- $\operatorname{supp} \mu$ is finite.

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## Implications

Amenability $\implies$ property A $\implies$ coarse embedding in $\ell_1$ or $\ell_2$.

# Box space

- $\Gamma = \Gamma_0 > \Gamma_1 > \Gamma_2 > \dots$ – residual chain
- $G_n = \mathrm{Cay}(\Gamma/\Gamma_n, S)$
- Box space $\Box G_n$ is the sequence $(G_n)$ or, more concretely, $\coprod_n G_n$ with any metric such that $\mathrm{dist}(G_n, G_m) \to \infty$ as $\max(n, m) \to \infty$.

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**Theorem (Guentner–Roe, 2003)**

- $\Box G_n$ has property A $\iff$ $\Gamma$ is amenable.
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**No equivalence in the second case!**

There exist expanding box spaces of the free group $\mathbb{F}_2$.

# Warped cones generalise box spaces

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$(Y, d)$ is a compact group containing $\Gamma$ as a discrete subgroup.

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**Examples**

- Hyperbolic group and its Gromov boundary.
- Actions on homogenous spaces, $\Gamma \curvearrowright G/H$, with H amenable and cocompact.

# Generalising the theorem of Roe

**Theorem (Roe, 2005): If $\Gamma < Y$, then**

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**Theorem (S., 2015)**

For $\implies$ it is enough if $Y$ admits an invariant measure and a positive measure subset with free action.

**Weakest assumptions under which Roe's theorem is proved**

$\Gamma$ acts on $Y$ by Lipschitz homoemorphisms and there is free subset with an invariant measure.

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**Examples**

For the action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$:

- $\mathcal{O}_{\mathrm{SL}_2(\mathbb{Z})}\mathbb{T}^2$ does not have property A;
- $\mathcal{O}_{\mathrm{SL}_k(\mathbb{Z})}\mathbb{T}^k$ does not even embed coarsely into a Hilbert space for $k \geq 3$.

# Open problem

**Theorem (Khukhro–Valette, 2015)**

If $\Box G_n$ and $\Box H_n$ are coarsely equivalent for $G_n = \Gamma/\Gamma_n$ and $H_n = \Lambda/\Lambda_n$, then $\Gamma$ and $\Lambda$ are quasi-isometric.

**Theorem (Kajal Das, 2015+)**

$\Gamma$ and $\Lambda$ are even *uniformly measure equivalent*.

**Question**

Can we obtain similar results for warped cones?

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**Question**

Can we obtain similar results for warped cones?

Warped cones are quasi-geodesic, so their coarse equivalce is in fact a quasi-isometry.

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- Warped cones as a generalisation of box spaces:

  relation of equivariant properties of $\Gamma$
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- Refinement of the former:

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- Spaces with interesting coarse properties, e.g.:
  - coarsely embeddable in $\ell_p$ but without property A;
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# Coarse embeddability does not imply property A

**Implications**

Property A $\implies$ coarse embedding in $\ell_1$ or $\ell_2$.

# Coarse embeddability does not imply property A

## Implications

Property A $\overset{?}{\Longleftrightarrow}$ coarse embedding in $\ell_1$ or $\ell_2$.

## Counterexamples

- $\coprod_n \mathbb{Z}_2^n$.
- Box spaces such that $\ker(G_n \to G_{n-1}) \simeq \mathbb{Z}_m^k$.
- Osajda monsters.

Some warped cones are also counterexamples!

# Embeddable warped cones without property A

- $(G_n)$ as on the previous slide (embeddable but without A)
- inverse system $G_1 \leftarrow G_2 \leftarrow G_3 \leftarrow \dots$
- profinite completion $Y = \varprojlim G_n \simeq \mathrm{cl}\,(\mathrm{im}(q\colon \Gamma \to \prod G_n))$
- metric $d((g_n), (h_n)) = \sum_n a_n \cdot d_{\mathrm{disc}}(g_n, h_n)$ for $a_n \searrow 0$

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**Sketch of proof**

1. Lack of property A follows from non-amenability of $\Gamma$ and the theorem of Roe.
2. $d_t((g_n),(h_n)) \simeq d_{\mathrm{Cay}}(g_N, h_N) + \min\left(d_{\mathrm{Cay}}(g_{N+1}, h_{N+1}),\ C\right)$

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## Facts from the theory of Hilbert-space embeddings

- $(X, d)$ embeds isometrically into a Hilbert space if and only if $d^2$ is a negative-type kernel.
- Negative-type kernels are preserved by composition with Bernstein functions.

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# Approximation of **min** by a Bernstein function

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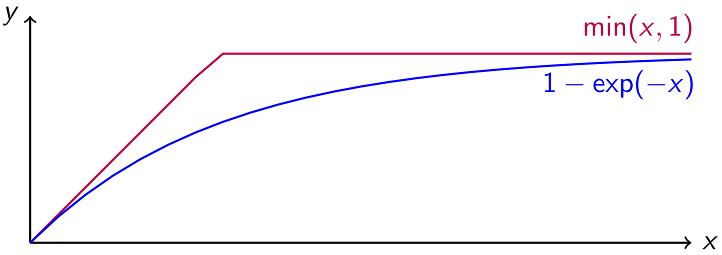

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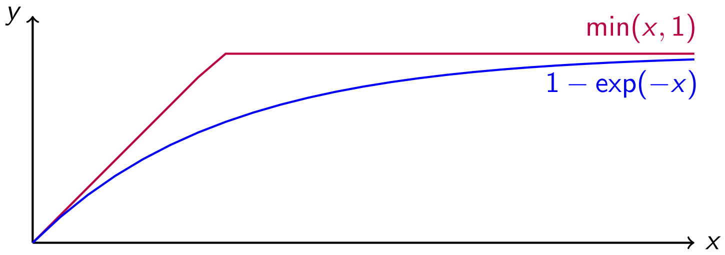


**Problem**

Find property of the action (weaker then amenability) guaranteeing coarse embeddability of the warped cone.

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# Warped cones over actions with spectral gaps

**Definition**

The action of $\Gamma$ on $(Y, \mu)$ has **a spectral gap (in $L_2(Y, \mu)$)** if there exists $\kappa > 0$ such that $\forall v \in L_2^0(Y, \mu)$:

$$\max_{s \in S} \|v - \pi(s)v\| \geq \kappa \|v\|.$$

**Definition**

The action of $\Gamma$ on $(Y, \mu)$ has **a spectral gap in** $L_p(Y, \mu; E)$ if there exists $\kappa > 0$ such that $\forall v \in L_p^0(Y, \mu; E)$:

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**Theorem (Nowak–S., 2015)**

If $\Gamma \curvearrowright Y$ has a spectral gap in $L_p(Y, \mu; E)$, then $\mathcal{O}_\Gamma Y$ does not embed coarsely into $E$.

# Warped cones over actions with spectral gaps – corollary

**Theorem (Nowak–S., 2015)**

If $\Gamma \curvearrowright Y$ has a spectral gap in $L_p(Y, \mu; E)$, then $\mathcal{O}_\Gamma Y$ does not embed coarsely into $E$.

**Corollary**

Warped cones over actions with a spectral gap do not embed coarsely into any $L_p(\Omega, \nu)$, $p \in [1, \infty)$.

**Sketch**

Gap in $L_2(Y, \mu; \mathbb{R}) \implies$ gap in $L_p(Y, \mu; \mathbb{R})$ for any $p \in (1, \infty)$ $\implies$ gap in $L_p\,(Y, \mu; L_p(\Omega, \nu))$ + the theorem.

# Warped cones over actions with spectral gaps – examples

**Example: the following do not embed coarsely into any $L_p$**

- $\mathcal{O}_{\mathsf{SL}_2(\mathbb{Z})}\mathbb{T}^2$;
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**Theorem (Bourgain–Gamburd, 2008)**

There exist discrete free subgroups $\mathbb{F}_k$ in SU(2) such that the action has a spectral gap.

- $\mathcal{O}_{\mathbb{F}_k}\,\mathsf{SU}(2)$.

# Warped cones over spectral gaps – special cases

**Theorem**

If $\Gamma \curvearrowright Y$ has a spectral gap in $L_p(Y, \mu; E)$, then $\mathcal{O}_\Gamma Y$ does not embed coarsely into $E$.

**Special cases**

Warped cones over ergodic actions

- of groups with property (T) do not embed into $L_p$;
- of groups with V. Lafforgue's reinforced strong property (T) do not embed into any Banach space of non-trivial type
  - cocompact $\Gamma < \mathrm{SL}_3(\mathbb{Q}_p)$;
- of groups with weaker versions of reinforced (T) do not embed into some intermediate classes of Banach spaces (Liao, de Laat, Mimura, Oppenheim, de la Salle).

## Theorem

If $\Gamma \curvearrowright Y$ has a spectral gap in $L_p(Y,\mu;E)$, then $\mathcal{O}_\Gamma Y$ does not embed coarsely into $E$.

## Proof

- $f$ – wannabe coarse embedding
$$\rho_-(d(x,x')) \leq \|f(x) - f(x')\| \leq \rho_+(d(x,x'))$$
- $f_t \colon Y \to E$ given by $f_t(y) = f(ty)$
- Wlog $f_t \in L_p^0(Y,\mu;E)$.

$$\begin{aligned}\|f_t - sf_t\|^p &= \int \|f_t(y) - f_t(s^{-1}y)\|_E^p \, d\mu \\ &\leq \int \rho_+(1)^p \, d\mu = \rho_+(1)^p\end{aligned}$$

$$\max_{s\in S} \|f_t - sf_t\| \geq \kappa \|f_t\| \xrightarrow{t\to\infty} \infty \qquad \square$$

Thank you!