Large scale geometry of actions on compact spaces

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Partly based on joint work with Piotr W. Nowak.

Schreier graph, warped metric

- $\Gamma = \langle S \rangle$, $|S| < \infty$
- $\Gamma \curvearrowright X$ (by bijections)

Schreier graph for the action $\Gamma \curvearrowright X$

$$V = X$$
$$E = \{\{x, sx\} \mid x \in X,\ s \in S\}$$

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- Assume X is a metric space (X, d) (and the action is by homeomorphisms).

Definition (Roe, 2005)

Warped metric d_Γ is the largest metric satisfying:

$$d_\Gamma \leq \min\left(d, d_{Sch}\right).$$

Warped metric on a Γ-space: geometric construction

- (X, d) – geodesic space
- For each pair of points $\{x, sx\}$ glue an interval of length 1 between x and sx.
- Calculate the path metric in the space with all the extra intervals.
- Its restriction to X is the warped metric d_Γ!

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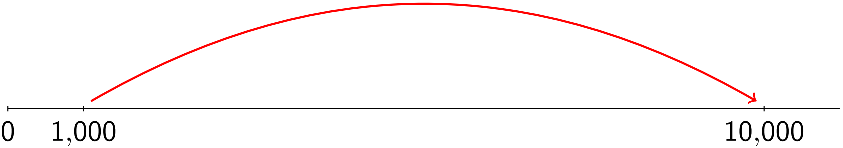

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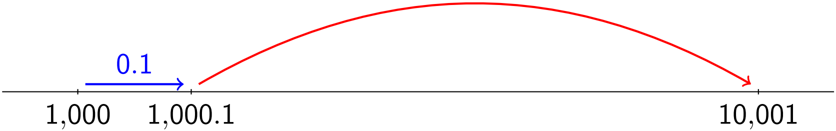

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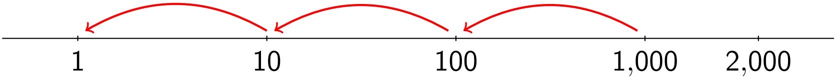

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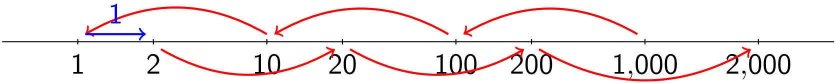

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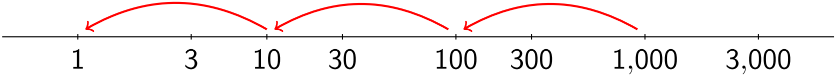

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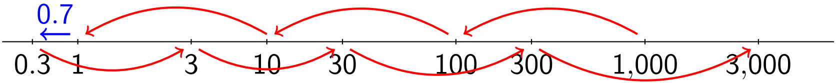

$\implies d_\Gamma(1{,}000;\ 3{,}000) = 7.7$

Warped metric on a Γ-space: geometric construction

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Example (Hyun Jeong Kim, 2006)

$X = \mathbb{R}^2$, $\Gamma = \mathbb{Z}$ acts by rotating by angle θ. There are infinitely many non-quasi-isometric warped planes $(\mathbb{R}^2, d_\mathbb{Z})$ depending on θ.

Warping: general motivation

$d_\Gamma(x, sx) \leq 1$

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$$d_\Gamma(x, sx) \leq 1$$

$$\implies \operatorname{dist}(id_{(X,d_\Gamma)}, \gamma) \leq |\gamma|$$

$\implies$ Rich Roe algebras.

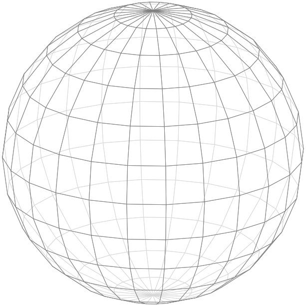
Conjecture (Druțu–Nowak, 2015; cf. Roe, 2005)

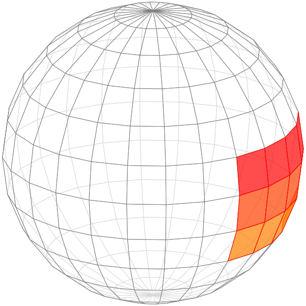
Warped cones over actions with a spectral gap violate the coarse Baum–Connes conjecture.

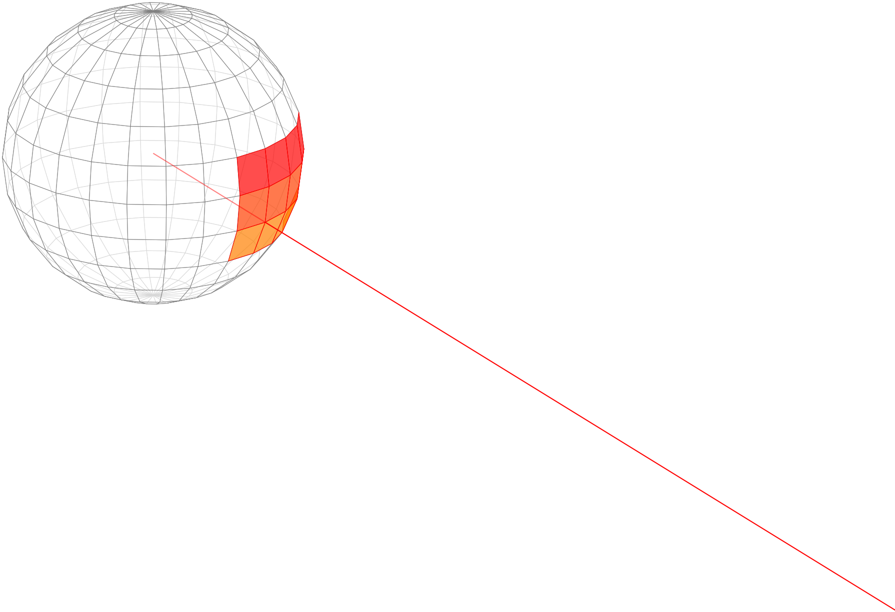
Warped cones

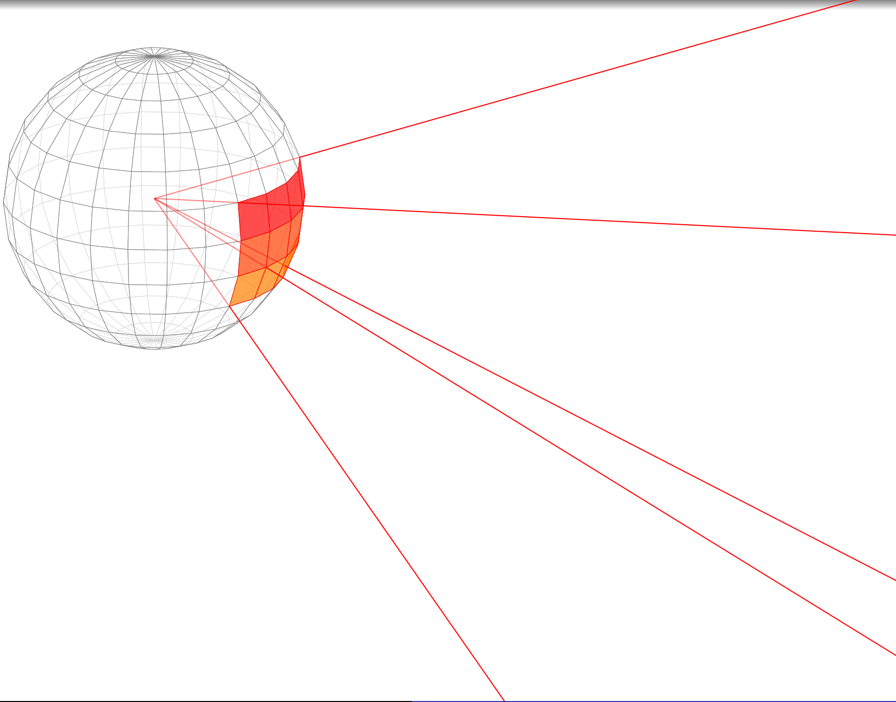
- $\Gamma \curvearrowright Y$ – compact subset of $S^n \subseteq \mathbb{R}^{n+1}$
- X – infinite cone over Y with the induced Γ action:

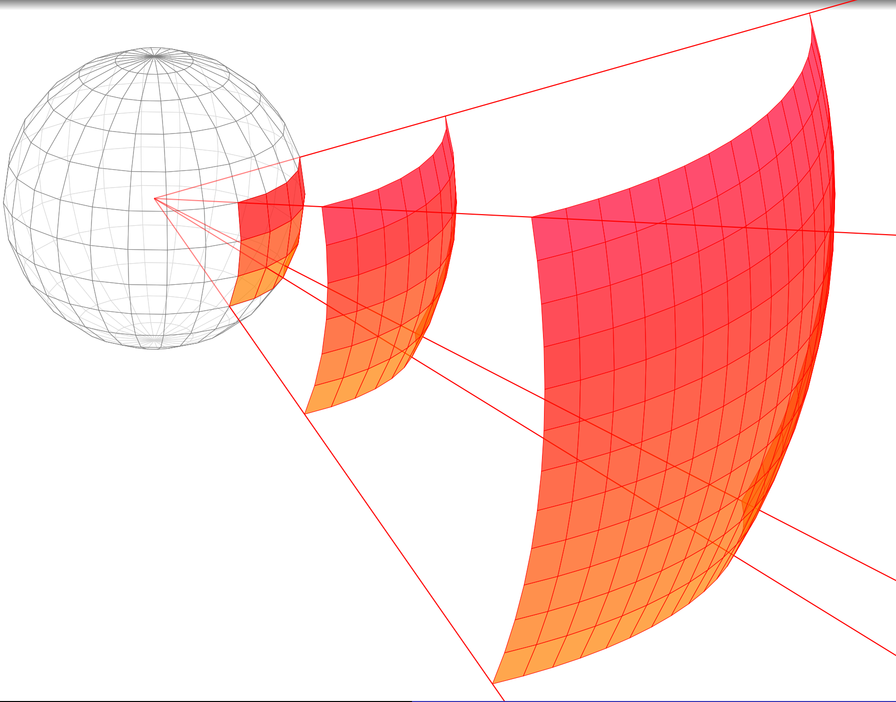
$$\{ty \mid t \in [0,\infty),\ y \in Y\} \subseteq (\mathbb{R}^{n+1}, d)$$











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- Notation: $\mathcal{O}Y := (X, d)$, $\mathcal{O}_\Gamma Y := (X, d_\Gamma)$.

Non-example: the case of a finite Γ

If Γ is finite, then $\mathcal{O}_\Gamma Y \simeq \mathcal{O}\, Y/\Gamma$, e.g.:

- for the antipodal action $\mathbb{Z}_2 \curvearrowright S^n$: $\mathcal{O}_{\mathbb{Z}_2} S^n \simeq \mathcal{O}\, \mathbb{R}\mathrm{P}^n$;
- for rational θ: $\mathcal{O}_{\mathbb{Z}} S^1 = \mathcal{O}_{\mathbb{Z}_k} S^1 \simeq \mathcal{O}\, S^1/\mathbb{Z}_k \simeq \mathcal{O} S^1 = \mathbb{R}^2$.

Property A

Γ is **amenable** if for each $\varepsilon > 0$ and finite $R \subseteq \Gamma$ there exists $\mu \in S(\ell_1(\Gamma))$ such that:

- $\|\mu - s\mu\| \leq \varepsilon$ if $s \in R$;
- $\operatorname{supp} \mu$ is finite.

Definition (G. Yu, 2001)

(X, d) has **property A** if for each $\varepsilon > 0$ and $R < \infty$ there is a map $X \ni x \mapsto A_x \in S(\ell_1(X))$ and a constant $S < \infty$ such that:

- $\|A_x - A_y\| \leq \varepsilon$ if $d(x, y) \leq R$;
- $\operatorname{supp} A_x \subseteq B(x, S)$.

Amenability $\implies$ property A

Putting $A_\gamma = \gamma\mu$, we get:

- $\operatorname{supp} A_\gamma = \gamma \operatorname{supp} \mu \subseteq \gamma B(e, S) = B(\gamma, S)$,
- $\|A_\gamma - A_{\gamma s}\| = \|\gamma\mu - \gamma s\mu\| = \|\mu - s\mu\| \leq \varepsilon$.

Property A – equivalent characterisations

- For a group Γ:
  - $\Gamma \curvearrowright \beta\Gamma$ is amenable.
  - $C_r^*\Gamma$ is exact.
- For any (bounded geometry) metric space X:
  - $\exists\,(k_n)$ positive definite kernels $k_n \colon X \times X \to [-1, 1]$, with controlled support, converging to 1 uniformly on controlled sets.
  - All ghost operators in the uniform Roe algebra $C_u^*(X)$ are compact (Roe–Willett, 2013).

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Proof

$\mathcal{O}Y$ has A $\implies$ $(A^k_x) \in \mathrm{Prob}(\mathcal{O}Y)$ s.t. $\|A^k_x - A^k_{x'}\| \leq 1/k$ if $d(x,x') \leq k$ for $x, x' \in \mathcal{O}Y$ and $\operatorname{supp} A^k_x \subseteq B(x, S_k)$.

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$\mu_l \colon Y \to \mathrm{Prob}(\Gamma)$ s.t. $\|\mu_l(y)s^{-1} - \mu_l(sy)\|_{\mathrm{Prob}(\Gamma)} \leq 1/l$.
Can assume $\mu_l(y) \in \mathrm{Prob}(B(1_\Gamma, C_l))$.

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$$\|B^m_x - B^m_{sx}\| = \Big\| \sum_\gamma \left(\mu_l(x)(\gamma) - \mu_l(sx)(\gamma s^{-1})\right) \cdot A^k_{\gamma x} \Big\| \leq$$

$$\leq \|\mu_l(x) - \mu_l(sx)s\| \cdot 1 \leq 1/l$$

Proof – another estimate

$$\|B^m_{ty} - B^m_{t'y'}\| \leq \sum_\gamma \mu_I(y)(\gamma) \cdot \|A^k_{\gamma ty} - A^k_{\gamma t'y'}\| + \\ + \sum_\gamma |\mu_I(y)(\gamma) - \mu_I(y')(\gamma)| \cdot 1$$

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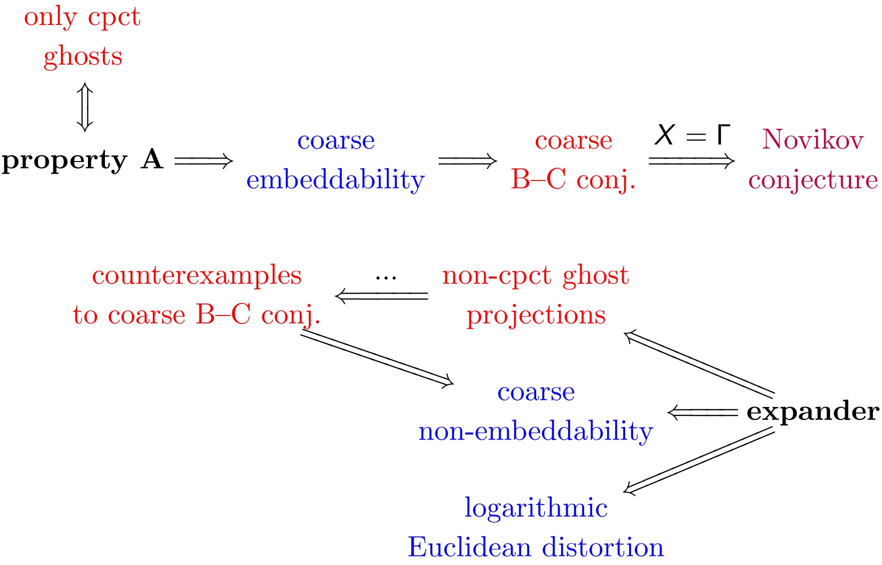
$$\text{(uniform continuity)} \quad \xrightarrow{t \to \infty} 1/k + 0$$

□

property A

expander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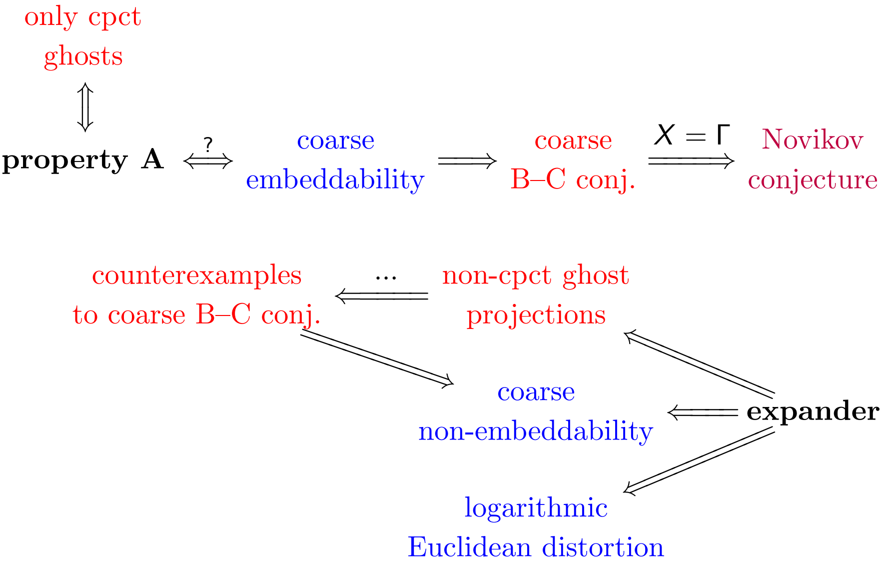
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Coarse embeddability $\not\Longrightarrow$ property A

Definition (Gromov)

$f\colon X \to Y$ is a **coarse embedding** if for all sequences (x_n), (x_n')

$$d(x_n, x_n') \to \infty \iff d(f(x_n), f(x_n')) \to \infty.$$

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Examples

- $\coprod_n \mathbb{Z}_2^n$ (Nowak, 2007)
- $\coprod_n G_n$ with $G_n = \Gamma/\Gamma_n$ such that:
  - $\ker(G_{n+1} \to G_n) = \Gamma_n/\Gamma_{n+1} \simeq \mathbb{Z}_2^{k_n}$
    (Arzhantseva–Guentner–Špakula, 2012; Ostrovskii, 2012)
  - $\ker(G_{n+1} \to G_n) = \Gamma_n/\Gamma_{n+1} \simeq \mathbb{Z}_{m_n}^{k_n}$ (Khukhro, 2014)
  - permanence under:
    - $\Gamma \rtimes \Delta$ for amenable Δ (Khukhro, 2012)
    - $A \wr \Gamma$ for abelian A (Khukhro–Cave–Dreesen, 2015)
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For $Y = \varprojlim G_n$ with G_n as above, the warped cone $\mathcal{O}_\Gamma Y$ embeds coarsely into the Hilbert space but does not have property A.

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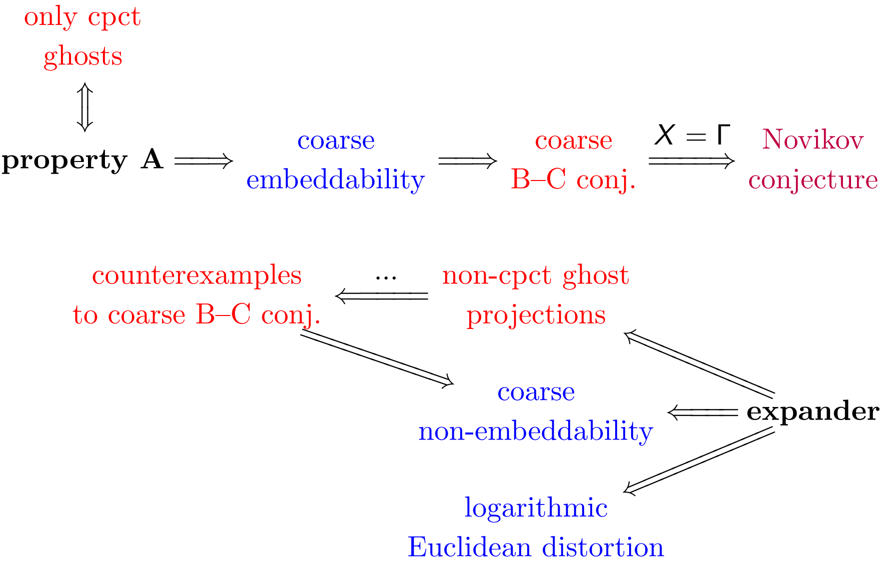
Theorem (Roe, 2005; S, 2015)

Let Γ act essentially free by pmp homeomorphisms on (Y, μ). Then:

1. if $\mathcal{O}_\Gamma Y$ has property A, then Γ is amenable;
2. if $\mathcal{O}_\Gamma Y$ embeds coarsely into a Hilbert space, then Γ has the Haagerup property.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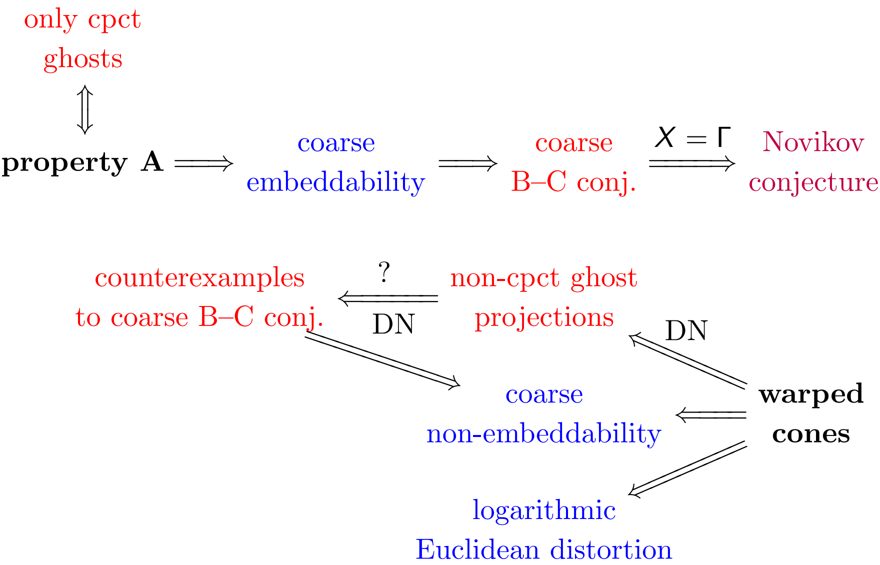
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The action of Γ on (Y,μ) has **a spectral gap (in $L_2(Y,\mu)$)** if there exists $\kappa > 0$ such that $\forall v \in L_2^0(Y,\mu)$:

$$\max_{s\in S} \|v - \pi(s)v\| \geq \kappa \|v\|.$$

Warped cones over actions with spectral gaps

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Theorem (Nowak–S, 2015)

If $\Gamma \curvearrowright Y$ has a spectral gap in $L_p(Y,\mu;E)$, then $\mathcal{O}_\Gamma Y$ does not embed coarsely into E.

Warped cones over actions with spectral gaps – corollary

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Corollary 1

Warped cones over actions with a spectral gap do not embed coarsely into any $L_p(\Omega, \nu)$, $p \in [1, \infty)$.

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If $\Gamma \curvearrowright Y$ has a spectral gap in $L_p(Y, \mu; E)$, then $\mathcal{O}_\Gamma Y$ does not embed coarsely into E.

Corollary 1

Warped cones over actions with a spectral gap do not embed coarsely into any $L_p(\Omega, \nu)$, $p \in [1, \infty)$.

Corollary 2

Warped cones over ergodic actions of groups with Lafforgue's strong Banach property (T) with respect to Banach spaces of non-trivial type do not embed coarsely into such Banach spaces.

Warped cones over actions with spectral gaps – examples

Example: the following do not embed coarsely into any L_p

- $\mathcal{O}_{\mathsf{SL}_2(\mathbb{Z})}\mathbb{T}^2$;
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Theorem (Bourgain–Gamburd, 2008)

There exist discrete free subgroups $\mathbb{F}_k$ in SU(2) such that the action has a spectral gap.

- $\mathcal{O}_{\mathbb{F}_k}\mathsf{SU}(2)$.

Proof

f – coarse embedding into E

$$\rho_-(d(x,x')) \leq \|f(x) - f(x')\|_E \leq \rho_+(d(x,x'))$$

$f_t \colon Y \to E$ given by $f_t(y) = f(ty)$, $f_t \in L_p(Y, \mu, E)$

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Aim:

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$$\begin{aligned} \iint_{Y\times Y} \|f_t(x) - f_t(y)\|_E^p &\leq 2^{p-1} \iint_{Y\times Y} \|f_t(x)\|_E^p + \|f_t(y)\|^p \\ &= 2^p \, \|f_t\|^p. \end{aligned}$$

Finishing lemma

Lemma

$\|f_t^x - f_t^y\| \xrightarrow{t\to\infty} \infty$ (the norm in $L_p(Y \times Y, \mu \times \mu, E)$).

Proof

- Let $O = \{(x, y) \mid x = \gamma y\}$.
- $\mu \times \mu(O) = 0$
- **Fact.** For $(x, y) \in Y \times Y \setminus O$, we have $d_\Gamma(tx, ty) \xrightarrow{t\to\infty} \infty$
- Hence $\|f_t(x) - f_t(y)\|_E \xrightarrow{t\to\infty} \infty$. □

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Corollary

A quantitative version of the Lemma (under some extra assumptions) $\implies$ the distortion of $tY \subseteq \mathcal{O}_\Gamma Y$ is $\Omega(\log t)$.

Thank you!