# WARPED CONES VIOLATING THE COARSE BAUM–CONNES CONJECTURE

DAMIAN SAWICKI

Abstract. We prove a conjecture of Druţu and Nowak that the coarse assembly map is not surjective for warped cones over free actions with a spectral gap. In particular, these warped cones provide new counterexamples to the coarse Baum–Connes conjecture, which predicts that the assembly map is an isomorphism.

## Introduction

Let $\Gamma \curvearrowright (Y, d, \mu)$ be a probability-measure-preserving action by Lipschitz homeomorphisms of a finitely generated group $\Gamma$ on a compact metric measure space $Y$ and assume that it has a spectral gap. Druţu and Nowak constructed a non-compact projection in $\mathcal{B}(L^2([1,\infty) \times Y; \lambda \times \mu))$, which is a norm-limit of finite propagation operators if we equip $[1,\infty) \times Y$ with the warped cone metric [3].

Similar projections, under additional technical assumptions, were used by Higson to prove that certain Margulis-type expanders are counterexamples to the coarse Baum–Connes conjecture, because these projections are not in the image of the coarse assembly map, that is supposed (under the conjecture) to be an isomorphism. Hence, Druţu and Nowak conjectured that warped cones they considered are also counterexamples to surjectivity of the assembly map.

The goal of this note is to prove the conjecture of Druţu and Nowak. Let us first recall the definition.

**Definition 0.1** (Roe [10])**.** Let $Y$ be a compact metric space of diameter at most 2. Consider the *infinite cone* $\mathcal{O}Y = [1,\infty) \times Y$ with the metric

$$d((s,y),(t,y')) = |s-t| + \min(s,t) \cdot d(y,y'),$$

where we use the same notation for the metric on $\mathcal{O}Y$ and $Y \subseteq \mathcal{O}Y$. Consider a continuous action of a finitely generated group $\Gamma \curvearrowright Y$ and extend this action to the cone by $\gamma(t,y) = (t, \gamma y)$. Then, the *warped cone* $\mathcal{O}_\Gamma Y$ is the same set $[1,\infty) \times Y$ equipped with the largest metric $d_\Gamma$ such that

$$d_\Gamma \leq d \quad \text{and} \quad d(\gamma x, x) \leq |\gamma|.$$

where $|\gamma|$ is the word length of $\gamma$ with respect to some fixed finite symmetric generating set $S$.

Our proof consists of three steps. First, checking if the projection of Druţu and Nowak belongs to the Roe algebra, rather than only to the algebra of bounded operators. It turns out to be false for technical reasons in the setting of [3], where the Roe algebra is modelled on $L^2([1,\infty) \times Y; \lambda \times \mu)$, but we manage to obtain it for the analogous projection inside $\mathcal{B}(L^2(\mathbb{N} \times Y; m \times \mu))$, where $m$ is the counting measure on $\mathbb{N}$.

In the second step we introduce an asymptotically faithful covering for warped cones (which requires the freeness assumption) constructed by Jianchao Wu and the author in [15] and show that it has the operator norm localisation property of Chen, Tessera, Wang, and Yu [2] if and only if $\Gamma$ has property A.

The last step is to prove that the $K$-theory class represented by the projection is not in the image of the assembly map. To be able to mimic the methods of Higson, we separate different levels of the warped cone by restricting our attention to $\{2^n\}_{n\in\mathbb{N}} \times Y \subseteq \mathbb{N}\times Y$. This does not lose much information, because the warped cone can be reconstructed from this subspace up to bi-Lipschitz equivalence (cf. the proof of [12, Lemma 4.1]). Note that, differently than in some places in the literature, growing but finite distances are sufficient for our purposes.

These three steps are divided between the first three sections of this note. The fourth section contains some remarks, showing in particular that our main result applies to spaces that are not coarsely equivalent to a family of graphs.

## 1. Kazhdan projection is in the Roe algebra

Consider a locally compact metric space $X$. An *$X$-module* is a separable Hilbert space $\mathcal{H}$ equipped with a representation of $\mathcal{C}_0(X)$. An operator $T \in \mathcal{B}(\mathcal{H})$ has *finite propagation* if there exists $S > 0$ such that for $\phi, \psi \in \mathcal{C}_0(X)$ satisfying $d(\operatorname{supp}\phi, \operatorname{supp}\psi) \geq S$ we have $\phi T \psi = 0$ (the optimal $S$ is then called the propagation of $T$). Further, $T$ is *locally compact* if for every $\phi \in \mathcal{C}_0(X)$ both $\phi T$ and $T\phi$ are compact operators. To make this requirement non-trivial we require that $\mathcal{H}$ is *ample* as an $X$-module: every non-zero $\phi \in \mathcal{C}_0(X)$ gives a non-compact operator.

For $X = [1,\infty)\times Y$ with $Y$ equipped with a probability measure $\mu$ of full support, an example of an ample $X$-module is the space $L^2([1,\infty)\times Y, \lambda\times\mu)$, where $\lambda$ denotes the Lebesgue measure. In such a setting, finite propagation of operator $T$ means that the support of $T\xi$ is contained in the $S$-neighbourhood of the support of $\xi$.

The *Roe algebra* $\mathrm{C}^*(X)$ *of* $X$ is the norm-closure of the algebra $\mathbb{C}[X]$ of locally compact, finite propagation operators on $\mathcal{H}$. It does not depend on the choice of an ample $X$-module $\mathcal{H}$ and is a coarse invariant of $X$.

The projection $\mathfrak{G}$ of Druţu and Nowak is the orthogonal projection onto

$$L^2([1,\infty),\lambda) \subseteq L^2([1,\infty)\times Y, \lambda\times\mu),$$

viewed as constant functions on every level $\{t\}\times Y$.

**Proposition 1.1.** *Projection $\mathfrak{G} \in \mathcal{B}(L^2([1,\infty)\times Y, \lambda\times\mu))$ does not belong to the Roe algebra of the warped cone $([1,\infty)\times Y, d_\Gamma)$.*

*Proof.* Consider the isometric embedding

$$J\colon L^2([1,2],\lambda) \subseteq L^2([1,\infty),\lambda) \subseteq L^2([1,\infty)\times Y, \lambda\times\mu).$$

Then $J^*\mathfrak{G}J$ is just the identity on $L^2([1,2],\lambda)$.

Assume that $\mathfrak{G}$ is a norm-limit of locally compact operators $T_n$. Hence, for any $\phi \in \mathcal{C}_0(X)$ the composition $T_n\phi$ is compact. Choose $\phi \in \mathcal{C}_0(X)$ which is equal to 1 on $[1,2]\times Y$ and observe that $\phi J = J$. Then,

$$\mathrm{id}_{L^2} = J^*\mathfrak{G}J = J^*\mathfrak{G}\phi J = \lim_n J^* T_n \phi J,$$

that is, we obtained the identity on $L^2([1,2],\lambda)$ as a norm-limit of compact operators, which is a contradiction. □

We will show that in order to guarantee that the projection of Druţu and Nowak belongs to the Roe algebra it suffices to replace the halfline $[1,\infty)$ by $\mathbb{N}$. Please note that the embedding $(\mathbb{N}\times Y, d_\Gamma) \to ([1,\infty)\times Y, d_\Gamma)$ is a quasi-isometry, in particular the respective Roe algebras are isomorphic.

**Definition 1.2.** Let $\Gamma \curvearrowright (Y,\mu)$ be a probability-measure-preserving action with both $\Gamma = \langle S\rangle$ and $Y$ infinite and $\mu$ of full support and let $\rho\colon \Gamma \to \mathcal{B}(L^2(Y,\mu))$ be the induced unitary representation. Denote by $L^2_0(Y,\mu)$ the orthogonal complement of constant functions. The action *has a spectral gap* if the norm of the Markov operator $M = \sum_{s\in S} \rho(s)$ restricted to $L^2_0(Y,\mu)$ is strictly less than one.

Note that $M$ acts as the identity on constants, hence the spectral gap condition implies that the spectrum of $M$ acting on $L^2(Y,\mu)$ is contained in $[0, 1-\varepsilon] \cup \{1\}$ for some $\varepsilon > 0$. Spectral gap is a strong form of ergodicity as it implies that the only invariant vectors in $L^2(Y,\mu)$ are constants.

Observe also that Definition 1.2 implies that $L^2(Y,\mu)$ is ample as an $Y$-module. Indeed, first note that the measure of a point $y \in Y$ is zero. Otherwise, the orbit of $y$ would be a finite invariant subset of positive measure, and, by ergodicity, it would have full measure, but we assumed that the support of $\mu$ is the infinite space $Y$. By the assumption of full measure, we also know that the measure of any ball $B(y,\varepsilon)$ is positive and it follows that $L^2(B(y,\varepsilon),\mu)$ is infinite dimensional (since $\lim_{\varepsilon\to 0} \mu(B(y,\varepsilon)) = \mu(\{y\}) = 0$), which immediately implies that $L^2(Y,\mu)$ is ample.

**Proposition 1.3.** *Projection $\mathfrak{G} \in \mathcal{B}(L^2(\mathbb{N}\times Y, m\times\mu))$ onto $L^2(\mathbb{N}, m)$, where $m$ is the counting measure, belongs to the Roe algebra of the integral warped cone $(\mathbb{N}\times Y, d_\Gamma) = \mathcal{O}_\Gamma Y$ if the action $\Gamma \curvearrowright L^2_0(Y,\mu)$ has a spectral gap.*

*Proof.* Let us cover $\mathcal{O}_\Gamma Y$ by balls of of radius $1/2$ and, by compactness of $Y$, choose a subcover $\mathcal{U} = \{U_1, U_2, \ldots\}$ which is finite when restricted to any level $\{n\}\times Y$. Let now the family $\mathcal{V}$ of $V_i = U_i \setminus \bigcup_{j<i} U_j$ be a disjoint version of this covering and define $P \in \mathcal{B}(L^2(\mathbb{N}\times Y))$ as the orthogonal projection onto functions constant on every $V_i$. Note that the covering $\mathcal{V}$ refines the covering by level sets $\{n\}\times Y$, so $\mathfrak{G}$ is a subprojection of $P$, thus $P\mathfrak{G}P = \mathfrak{G}$.

Recall that the Markov operator $M_0 = \sum_{s\in S} \rho(s)$ preserves the constants $\mathbb{C} \subseteq L^2(Y)$ and, by the spectral gap assumption, has norm strictly less than 1 on the orthogonal complement $L^2_0(Y)$, which is also preserved. Hence, its powers equal identity on the constants and converge geometrically to the zero operator on the complement. When tensored with the identity operator on $L^2(\mathbb{N})$, we conclude that the powers of

$$M = \mathrm{id}_{L^2(\mathbb{N})} \otimes M_0 \in \mathcal{B}(L^2(\mathbb{N}) \otimes L^2(Y)) = \mathcal{B}(L^2(\mathbb{N}\times Y))$$

converge in norm to $\mathfrak{G}$, the projection onto $L^2(\mathbb{N}) \otimes \mathbb{C}$.

Hence, we have $\mathfrak{G} = P\mathfrak{G}P = \lim_k PM^kP$ and it suffices to show that $PM^kP$ is a locally compact operator of finite propagation. Note that for $\xi \in L^2(\mathbb{N}\times Y)$, the support of $P\xi$ is contained in the sum of $V_i$'s non-empty intersecting $\operatorname{supp}\xi$. Since the diameter of $V_i \subseteq U_i$ is at most 1, we conclude that the propagation of $P$ is at most 1. Also, by the definition of the warped metric, the propagation of $M$ is (bounded by) 1. Hence, the propagation of $PM^kP$ is at most $k+2$.

Now, let $\phi \in \mathcal{C}_0(\mathcal{O}_\Gamma Y)$ be a function of compact support. Hence, its range is contained in $L^2(\{1,2,\ldots,l\}\times Y)$ for some $l$ and the same holds for the composition $PM^kP\phi$. However, the range of $P$ intersected with $L^2(\{1,2,\ldots,l\}\times Y)$ is finite dimensional – since we assumed that $\mathcal{U}$ is finite on every level $\{n\}\times Y$ – and hence $PM^kP\phi$ is finite rank, thus compact. Since a general $\psi \in \mathcal{C}_0(\mathcal{O}_\Gamma Y)$ is a norm-limit of compactly supported ones, we conclude that $PM^kP\psi$ is always compact. Now, because $PM^kP$ is self-adjoint, it follows that also $\psi PM^kP$ is compact, which ends the proof. ☐

## 2. Operator norm localisation property for the covering

In order to disprove surjectivity of the coarse assembly map in Section 3, we will need the following technical notion.

**Definition 2.1** (Chen, Tessera, Wang, Yu [2])**.** A metric space $X$ has *operator norm localisation property* (ONL) if there is a positive constant $c > 0$ such that for every $r < \infty$ we have $R < \infty$ such that for every positive locally finite Borel measure $\nu$ on $X$ and every operator $T \in \mathcal{B}(L^2(X,\nu) \otimes \ell^2)$ of propagation at most $r$, there exists a unit vector $\xi \in L^2(X,\nu) \otimes \ell^2$ with $\operatorname{diam}(\operatorname{supp} \xi) \leq R$ on which the norm of $T$ is almost attained: $c\|T\| \leq \|T\xi\|$.

Above, *locally finite* means finite on bounded sets. In most cases, we can skip the coefficients $\ell^2$ as shown by the following lemma. Before its statement, recall that a subset $Z \subseteq X$ of a metric space is called an *$M$-net in $X$* if every point in $X$ lies within a distance less than $M$ from a point in $Z$. A set $Z$ is *$C$-separated* if the distance between every two distinct points of $Z$ is at least $C$. Note that a maximal $C$-separated subset is always a $C$-net.

**Lemma 2.2.** *Let $Z$ be a $C$-separated $M$-net in a proper metric space $X$. The space $X$ has ONL if and only if $Z$ has ONL. If moreover $L^2(X,\mu)$ is an ample $X$-module for some locally finite $\mu$, then these are also equivalent the fact that $X$ has ONL with respect to operators on $L^2(X,\mu)$.*

The lemma is essentially proved in [2], we include a short proof to give the reader a better feeling for $X$-modules and the ONL.

*Proof of Lemma 2.2.* Let us recall from [2] that for locally compact metric spaces it suffices to verify the definition of ONL for a fixed measure admitting $R < \infty$ such that all $R$-balls have positive measure. In our case, an example of such a measure is the counting measure $m$ on $Z$ (which is locally finite by properness of $X$ and for which we can take $R = M$) and clearly the ONL of $X$ with respect to operators on $L^2(X,m) \otimes \ell^2$ is equivalent ONL of $Z$ with respect to operators on $L^2(Z,m) \otimes \ell^2$, because these are the same Hilbert space.

For the moreover part, if $L^2(X,\mu)$ is ample, the subspace $L^2(W,\mu)$ must be infinite-dimensional for every subset $W \subseteq X$ with non-empty interior. It is also separable by the properness of $X$, hence isomorphic to $\ell^2$. Let $\{W_z\}_{z \in Z}$ be a measurable partition of $X$ such that $B(z, C/3) \subseteq W_z \subseteq B(z,M)$. We have

$$L^2(X,\mu) = \bigoplus_z L^2(W_z,\mu) \simeq \bigoplus_z \ell^2 = L^2(Z,m) \otimes \ell^2.$$

Note that the above unitary isomorphism can alter the diameter of the support of a function by at most $2M$ and the induced isomorphism of the algebras of operators on these spaces can alter the propagation of an operator by at most $2M$. Hence, ONL of $X$ with respect to operators on $L^2(X,\mu)$ is equivalent to ONL of $Z$, which by the above is equivalent to ONL of $X$. $\square$

**Definition 2.3.** A discrete metric space has *bounded geometry* if there is a bound on cardinalities of balls of any fixed radius. For general metric spaces, *bounded geometry* means that there exists $C < \infty$ such that for every $R < \infty$ the cardinality of any $C$-separated subset of any $R$-ball is uniformly bounded.

In particular, in a metric space with bounded geometry, a $C$-separated subset is a discrete metric space with bounded geometry. For all such spaces operator norm localisation property is equivalent to property A of G. Yu [19] by a result of Sako.

**Theorem 2.4** (Sako [11])**.** *Let $Z$ be a discrete bounded geometry metric space. Then, it has property A if and only if it satisfies operator norm localisation property.*

**Corollary 2.5.** *The same holds for bounded geometry metric spaces.*

*Proof.* If suffices to observe that passing to a discrete net (and back to the original space) preserves ONL (by Lemma 2.2) and property A (by [19]) and to use the equivalence of the two properties in the discrete bounded geometry case. □

Consider the product $\Gamma \times \mathcal{O}Y$ with the quotient map $\pi$ onto $\mathcal{O}_\Gamma Y$ given by $(\gamma, x) \mapsto \gamma x$. This is the quotient map under the $\Gamma$-action $g(\gamma, x) = (\gamma g^{-1}, gx)$. Equip $\Gamma \times \mathcal{O}Y$ with the largest metric $d_1$ such that $d_1((\gamma, x), (\eta\gamma, x)) \leq |\eta|$ and $d_1((\gamma, x), (\gamma, x')) \leq d(\gamma x, \gamma x')$, where $d$ denotes the metric on the infinite cone $\mathcal{O}Y$. Jianchao Wu and the author [15] showed that the warped metric $d_\Gamma$ is the quotient metric of $d_1$ and, for free actions, the covering is *asymptotically faithful*, that is, for every $R < \infty$, the covering map is an isometry on $R$-balls outside an $R$-dependent bounded subset of $\mathcal{O}_\Gamma Y$. We would like to obtain ONL for $\Gamma \times \mathcal{O}Y$.

**Lemma 2.6.** *If $\mathcal{O}Y$ has bounded geometry and the action is Lipschitz, then the space $(\Gamma \times \mathcal{O}Y, d_1)$ has bounded geometry.*

*Proof.* One can check that the ball of radius $n$ about $(\gamma, x) \in \Gamma \times \mathcal{O}Y$ with respect to $d_1$ is contained in the sum $\bigcup_{\eta \in B(\gamma,n)} \{\eta\} \times \eta^{-1} B(\eta x, L^n \cdot n)$ (cf. [3, Lemma 6.2]), where $L$ is the Lipschitz constant for the action of generators of $\Gamma$ and the balls in $\mathcal{O}Y$ are taken with respect to metric $d$. Using the inequality $d_1((\gamma, x), (\gamma, x')) \leq d(\gamma x, \gamma x')$, we conclude that the cardinality of $C$-separated subsets in an $n$-ball in $(\Gamma \times \mathcal{O}Y, d_1)$ is at most $|B(1, n)|$ times larger than the maximal cardinality of a $C$-separated subset in an $(L^n \cdot n)$-ball in $\mathcal{O}Y$. □

Hence, in order to obtain ONL for $\Gamma \times \mathcal{O}Y$, it suffices to verify property A. We will use the following characterisation of property A, equivalent to the original in the bounded geometry case [10].

**Definition 2.7.** A metric space $X$ has *property A* if for every $n < \infty$ and $\varepsilon > 0$ there exists a constant $S < \infty$ and a map $X \ni x \mapsto A_x \in \mathrm{Prob}(X)$ satisfying

$$\operatorname{supp} A_x \subseteq B(x, S) \quad \text{and} \quad \|A_x - A_y\| < \varepsilon \text{ for } d(x, y) \leq n.$$

Note that in [10] the map was additionally required to be weak-$*$ continuous. However, by a partition of unity argument, we can always improve an arbitrary map as above to a norm-continuous one. Also, one can push these measures forward by a quasi-isometric retraction $X \to Z$ onto a countable subset, so we will further assume that $A_x \in \mathrm{Prob}(X) \cap \ell_1(X)$.

**Lemma 2.8.** *Let $Y$ be a manifold, simplicial complex or an ultrametric space with a Lipschitz action of $\Gamma$. Then $(\Gamma \times \mathcal{O}Y, d_1)$ has property A if and only if $\Gamma$ has property A.*

*Proof.* For the "only if" part it suffices to observe that $\Gamma$ embeds isometrically into $\Gamma \times \mathcal{O}Y$ as $\Gamma \times \{*\}$ and recall that property A is inherited by subspaces.

Let $n < \infty$ and $\varepsilon > 0$. Let $\Gamma \ni \gamma \mapsto A_\gamma \in \mathrm{Prob}(\Gamma)$ be a map from the definition of property A for $\Gamma$ such that

$$\operatorname{supp} A_\gamma \subseteq B(\gamma, S_1) \quad \text{and} \quad \|A_\gamma - A_\eta\| < \frac{\varepsilon}{2n+1} \text{ for } d(\gamma, \eta) \leq n$$

for some constant $S_1 < \infty$. Now, let $L$ be the Lipschitz constant for the action of elements of the ball $B(1, S_1) \subseteq \Gamma$. Under our assumptions the infinite cone $\mathcal{O}Y$ has property A. (Indeed, for manifolds and complexes, $\mathcal{O}Y$ is bi-Lipschitz equivalent to a subspace of a Euclidean space, hence it inherits property A. In general, finiteness of the Assouad–Nagata dimension of $Y$ is equivalent to finitenes of the asymptotic dimension of the infinite cone $\mathcal{O}Y$ [14, Proposition 7.6], which in turn implies property A.) Hence, we have a map $\mathcal{O}Y \ni x \mapsto B_x \in \operatorname{Prob}(\mathcal{O}Y) \cap \ell_1(\mathcal{O}Y)$ such that

$$\operatorname{supp} B_x \subseteq B(x, S_2) \quad \text{and} \quad \|B_x - B_y\| < \frac{\varepsilon}{2n+1} \text{ for } d(x,y) \leq Ln.$$

For $(\gamma, x)$ define $C_{(\gamma,x)} \in \operatorname{Prob}(\Gamma \times \mathcal{O}Y) \cap \ell_1(\Gamma \times \mathcal{O}Y)$ by the formula:

$$C_{(\gamma,x)}(\eta, y) = A_\gamma(\eta) \cdot B_{\eta x}(\eta y).$$

If this value is non-zero, then $d(\gamma, \eta) < S_1$ and $d(\eta x, \eta y) < S_2$, hence:

$$d_1((\gamma,x),(\eta,y)) \leq d_1((\gamma,x),(\eta,x)) + d_1((\eta,x),(\eta,y)) \leq d(\gamma,\eta) + d(\eta x, \eta y) < S_1 + S_2,$$

which means that $\operatorname{supp} C_{(\gamma,x)} \subseteq B((\gamma,x), S_1 + S_2)$.

Now, if $d_1((\gamma,x),(\eta,y)) \leq n$, then there exists a sequence of points $(z_i)_{i=0}^{2n+1}$ in $\Gamma \times \mathcal{O}Y$ such that $z_0 = (\gamma, x)$, $z_{2n+1} = (\eta, y)$, $z_i$ differs from $z_{i-1}$ at only one coordinate (or they are equal) and such that we have

$$d_1((\gamma,x),(\eta,y)) = \sum_{i=1}^{2n+1} \rho(z_{i-1}, z_i),$$

where $\rho((g\gamma, x), (\gamma, x)) = |g|$ and $\rho((\gamma,x),(\gamma,y)) = d(\gamma x, \gamma y)$ (cf. [10, Proposition 1.6]). In particular $\rho(z_i, z_{i-1}) \leq n$. Hence, by the triangle inequality, it suffices to show that for $z, z' \in \Gamma \times \mathcal{O}Y$ differing at only one coordinate and satisfying $\rho(z, z') \leq n$, we have $\|C_z - C_{z'}\| \leq \varepsilon/(2n+1)$.

Let us consider first the case when the first coordinates differ, that is, $z = (g\gamma, x)$ and $z' = (\gamma, x)$. The bound follows from the fact that for any $f \in \ell^\infty(\Gamma \times \mathcal{O}Y)$:

$$\begin{aligned} |(C_z - C_{z'})(f)| &= \left| \sum_{\eta \in \Gamma} (A_{g\gamma}(\eta) - A_\gamma(\eta)) \sum_{y \in \mathcal{O}Y} B_{\eta x}(\eta y) f(\eta, y) \right| \\ &\leq \|A_{g\gamma} - A_\gamma\|_1 \cdot \|f\|_\infty \leq \frac{\varepsilon}{2n+1} \cdot \|f\|_\infty. \end{aligned}$$

In the second case we have $z = (\gamma, x)$ and $z' = (\gamma, x')$. From the fact that $L$ is the Lipschitz constant for $B(1, S_1)$, it follows that the inequality $d(\gamma x, \gamma x') = \rho((\gamma, x), (\gamma, x')) \leq n$ implies that for any $\eta \in B(\gamma, S_1)$ we have $d(\eta x, \eta x') \leq Ln$. We conclude:

$$\begin{aligned} |(C_z - C_{z'})(f)| &= \left| \sum_\eta A_\gamma(\eta) \sum_y (B_{\eta x}(\eta y) - B_{\eta x'}(\eta y)) \cdot f(\eta, y) \right| \\ &\leq \sum_\eta A_\gamma(\eta) \sum_y \|B_{\eta x} - B_{\eta x'}\| \cdot \|f\|_\infty \leq \frac{\varepsilon}{2n+1} \cdot \|f\|_\infty, \end{aligned}$$

because it suffices to sum over $\eta \in B(\gamma, S_1)$. □

Let us also mention the following two facts following from [14, Proposition 7.2 item (2) with Proposition 7.6] and [2, Remark 3.2 with Proposition 4.1] respectively.

**Lemma 2.9.** *Let $Y$ be a manifold, finite complex or an ultrametric space (more generally: any space with finite Assouad–Nagata dimension) with a Lipschitz action of $\Gamma$. Then $(\Gamma \times \mathcal{O}Y, d_1)$ has finite asymptotic dimension if and only if $\Gamma$ has finite asymptotic dimension.*

**Lemma 2.10.** *For any metric space, finiteness of the asymptotic dimension implies operator norm localisation property.*

The following corollary gathers results of the present section.

**Corollary 2.11.** *For a Lipschitz action $\Gamma \curvearrowright Y$ on a compact space $Y$, the space $\Gamma \times \mathcal{O}Y$ has ONL under any of the following assumptions:*

*(1)* $\Gamma$ *has property A and* $Y$ *is a manifold or a simplicial complex (or any space such that* $\mathcal{O}Y$ *has property A and bounded geometry),*
*(2)* *the asymptotic dimension of* $\Gamma$ *and the Assouad–Nagata dimension of* $Y$ *are finite.*

Let us remark that for *any continuous action* $\Gamma \curvearrowright Y$ of a finitely generated $\Gamma$ on a compact metrisable $Y$ of finite topological dimension, there exists a metric on $Y$ so that the assumptions of item (2) in Corollary 2.11 are satisfied, that is, the action is Lipschitz and $Y$ has finite Assouad–Nagata dimension [14, Lemma 8.9]. By (the proof of) [10, Proposition 1.10], the bounded geometry assumption of item (1) is satisfied whenever $Y$ admits an Ahlfors regular measure or is a subset of such a space.

## 3. Kazhdan projection lies outside the image of the assembly map

In this section we restrict our attention to the subspace of the warped (respectively, infinite) cone $\{2^n : n \in \mathbb{N}\} \times Y$ and we use the notation $\mathcal{Q}_\Gamma Y$ (respectively, $\mathcal{Q}Y$) for it. Following Higson [5], we define $\mathbb{R}_{[\infty]} = \prod_n \mathbb{R}/c_0$ and two functionals $\tau_{\mathrm{d}}, \tau^{\mathrm{u}} \colon K_0(\mathrm{C}^*(\mathcal{Q}_\Gamma Y)) \to \mathbb{R}_{[\infty]}$, whose restrictions to the image of the assembly map are the same, but aquire different values on $\mathfrak{G}$. Our argument follows the line of [17], where Willett and Yu elaborated on the idea of Higson.

Because a finite propagation operator $T \in \mathrm{C}^*(\mathcal{Q}_\Gamma Y)$ preserves $L^2(\{2^n\}\times Y)$ for $n$ sufficiently large, we can consider its restrictions $T_n \in \mathrm{C}^*(\{2^n\}\times Y)$. The functional $\tau_{\mathrm{d}}$ will be induced by the composition of this restriction map with the product of trace maps $\mathrm{C}^*(\{2^n\} \times Y) \to \mathbb{Z}$. For the second functional $\tau^{\mathrm{u}}$, we will use some lifts $T_n' \in \mathrm{C}^*(\Gamma \times \{2^n\} \times Y)^\Gamma$ and traces on $\mathrm{C}^*(\Gamma \times \{2^n\} \times Y)^\Gamma$.

3.1. **Index downstairs.** Let $P_n$ be the characteristic function of $\{2^n\} \times Y$. Then, for any operator $T \in \mathcal{B}(L^2(\mathcal{Q}_\Gamma Y))$ of propagation at most $2^{n-1}$, we have $P_n T = TP_n$ and we can define $T_n = P_n T P_n \in \mathrm{C}^*(\{2^n\} \times Y)$. The formula clearly makes sense and defines a contractive operator on the whole Roe algebra. However, $T \mapsto T_n$ is multiplicative and $*$-preserving on operators of propagation at most $2^{n-1}$, and hence the function

$$\mathrm{C}^*(\mathcal{Q}_\Gamma Y) \ni T \mapsto (T_n) \in \frac{\prod_n \mathrm{C}^*(\{2^n\} \times Y)}{\bigoplus_n \mathrm{C}^*(\{2^n\} \times Y)}$$

is multiplicative and $*$-preserving on all operators in $\mathbb{C}[\mathcal{Q}_\Gamma Y]$, so it yields a $*$-homomorphism from the Roe algebra $\mathrm{C}^*(\mathcal{Q}_\Gamma Y)$.

Now, observe that $\mathrm{C}^*(\{2^n\}\times Y)$ is nothing but the algebra of compact operators $\mathcal{K}(L^2(\{2^n\}\times Y))$. By definition it is a closure of locally compact finite propagation operators, but locally compact on a compact space means just compact and every operator on a bounded space has finite propagation. Hence, every projection $p \in \mathrm{C}^*(\{2^n\} \times Y)$ has finite rank, which we can formally calculate using the canonical trace Tr on compact operators.

Finally, we define $\tau_{\mathrm{d}}$ as the composition of the map

$$K_0(\mathrm{C}^*(\mathcal{Q}_\Gamma Y)) \to K_0\left(\frac{\prod_n \mathrm{C}^*(\{2^n\} \times Y)}{\bigoplus_n \mathrm{C}^*(\{2^n\} \times Y)}\right)$$

induced by $T \mapsto (T_n)$, the identification

$$K_0\left(\frac{\prod \mathrm{C}^*(\{2^n\} \times Y)}{\bigoplus \mathrm{C}^*(\{2^n\} \times Y)}\right) = \frac{K_0\left(\prod \mathrm{C}^*(\{2^n\} \times Y)\right)}{K_0\left(\bigoplus \mathrm{C}^*(\{2^n\} \times Y)\right)} = \frac{\prod K_0\left(\mathrm{C}^*(\{2^n\} \times Y)\right)}{\bigoplus K_0\left(\mathrm{C}^*(\{2^n\} \times Y)\right)}$$

(see [17, Section 6]), and eventually the trace

$$\mathrm{C}^*(\{2^n\} \times Y) \ni p \mapsto \mathrm{Tr}(p) \in \mathbb{Z} \subseteq \mathbb{R}.$$

Clearly, for the Kazhdan projection $\mathfrak{G}$ constructed by Druţu and Nowak, the restriction $\mathfrak{G}_n$ is the rank-one projection onto constant functions, hence

$$\tau_{\mathrm{d}}(\mathfrak{G}) = (1, 1, \ldots).$$

3.2. **Index upstairs.** Recall that the quotient map $\pi(\gamma, x) = \gamma x$ from $(\Gamma \times \mathcal{Q}Y, d_1)$ onto $\mathcal{Q}_\Gamma Y$ is asymptotically faithful [15]. Hence, for every $T \in \mathrm{C}^*(\mathcal{Q}_\Gamma Y)$ with finite propagation $S$ and $n$ sufficiently large (so that $\pi$ is isometric on balls of radius $3S$) we can canonically define the lift $T'_n \in \mathbb{C}[\Gamma \times \{2^n\} \times Y]^\Gamma$ of $T_n = P_n T P_n$.

Indeed, let us define $T'_n$ on the dense subspace spanned by functions with support of diameter at most $S$ (in fact, the subspace does not depend on $S$). Let $\xi, \eta$ be two elements of $L^2(\Gamma \times \{2^n\} \times Y)$ with diameters of supports at most $S$. Then, if $d_1(\operatorname{supp} \xi, \operatorname{supp} \eta) > S$, we put $\langle \eta, T'_n \xi \rangle = 0$ and $\langle \eta, T'_n \xi \rangle = \langle (\eta \circ \sigma), T_n(\xi \circ \sigma) \rangle$ otherwise, where $\sigma$ is the inverse of the restriction of $\pi$ to the sum $\operatorname{supp} \xi \cup \operatorname{supp} \eta$. As declared, operator $T'_n$ is invariant under conjugation:

$$\langle \gamma\eta, T'_n(\gamma\xi) \rangle = \langle (\gamma\eta \circ \sigma_1), T_n(\gamma\xi \circ \sigma_1) \rangle = \langle (\eta \circ \sigma_2), T_n(\xi \circ \sigma_2) \rangle = \langle \eta, T'_n \xi \rangle,$$

where, as before, $\sigma_1$ and $\sigma_2$ are the respective local inverses of $\pi$ and the inequality follows from the fact that $\sigma_2 = \gamma^{-1} \circ \sigma_1$.

We should justify the following.

**Lemma 3.1.** *The operator $T'_n$ is bounded.*

*Proof.* Note that $(\Gamma \times \{2^n\} \times Y, d_1)$ is coarsely equivalent to $\Gamma$, in particular it has bounded geometry – there exists a constant $C < \infty$ such that for every $R$ there is a uniform bound on the cardinality of an $R$-ball intersected with any $C$-separated set; in fact, by compactness of $\{1\} \times \{2^n\} \times Y$, this condition holds for all $C > 0$. Pick one such $C < S/2$ and choose a maximal $C$-separated set $Z$ in $\Gamma \times \mathcal{Q}Y$ and a measurable partition $\mathcal{U} = \{U_z : z \in Z\}$ of $\Gamma \times \mathcal{Q}Y$ such that $U_z \subseteq B(z, C)$. For $f \in L^2(\Gamma \times \mathcal{Q}Y)$ (from the dense subspace on which $T'$ is defined) we denote its restriction to $U_z$ by $f_z$ and calculate

$$\begin{aligned}
\|T'_n f\|^2 &= \sum_z \|(T'_n f)_z\|^2 = \sum_z \left\| \sum_{y \in N(z)} (T'_n f_y)_z \right\|^2 \\
&\leq \sum_z \left( \sum_{y \in N(z)} \|(T'_n f_y)_z\| \right)^2 \leq F \cdot \sum_z \sum_{y \in N(z)} \|T'_n f_y\|^2 \\
&= F \cdot \sum_z \sum_{y \in N(z)} \|T_n(f_y \circ \sigma)\|^2 \leq F \cdot \sum_z \sum_{y \in N(z)} \|T\|^2 \cdot \|f_y\|^2,
\end{aligned}$$

where $N(z) = B(z, 2C + S) \cap Z$, $F$ is the uniform bound on cardinality of such a set, and $\sigma = \sigma(y)$ is the appropriate local inverse of $\pi$. After changing the order of summation we obtain $\|T'_n f\|^2 \leq F^2 \|T\|^2 \cdot \|f\|^2$. $\square$

**Lemma 3.2.** *Assume that $\Gamma \times \mathcal{Q}Y$ has ONL. The association $T \mapsto (T'_n)$ gives a continuous $*$-preserving homomorphism $\mathbb{C}[\mathcal{Q}_\Gamma Y] \to \frac{\prod_n \mathrm{C}^*(\Gamma \times \{2^n\} \times Y)^\Gamma}{\bigoplus_n \mathrm{C}^*(\Gamma \times \{2^n\} \times Y)^\Gamma}$, thus it extends to the whole of $\mathrm{C}^*(\mathcal{Q}_\Gamma Y)$.*

*Proof.* By the ONL, there exists $c > 0$ such that for every $S < \infty$ there exists $R < \infty$ such that for any operator $T' \in \mathcal{B}(L^2(\Gamma \times \mathcal{Q}Y))$ of propagation at most $S$ there exists a unit vector $\xi$ such that $\|T'\xi\| \geq c\|T'\|$ and $\operatorname{diam}(\operatorname{supp} \xi) \leq R$. After enlarging $n$ if necessary we can assume that $\pi$ is isometric at scale $R + S$, so that $\|T_n(\xi \circ \sigma)\| = \|T'_n \xi\|$ and hence we can conclude

$$\|T\| \geq \|T_n\| \geq \|T_n(\xi \circ \sigma)\| = \|T'_n \xi\| \geq c\|T'_n\|,$$

which means that our association is continuous. The algebraic part is clear. □

Consider the following bijection

$$(\Gamma \times \mathcal{Q}Y, d_1) \ni (\gamma, x) \overset{\iota}{\mapsto} (\gamma, \gamma x) \in (\Gamma \times \mathcal{Q}Y, d_1'),$$

where metric $d_1'$ is induced by $\iota$ and can be described as the largest metric such that $d_1'((\gamma,x),(\gamma,x')) \leq d(x,x')$ and $d_1'((\eta\gamma,\eta x),(\gamma,x)) \leq |\eta|$. In these coordinates, the action of $\Gamma$ such that $(\mathcal{Q}_\Gamma Y, d_\Gamma)$ is the quotient of $(\Gamma \times \mathcal{Q}Y, d_1')$ under this action is given by $\eta(\gamma,x) = (\gamma\eta^{-1}, x)$, that is, it involves only the first coordinate, hence it is easy to see that we have an isomorphism

$$\mathrm{C}^*(\Gamma \times (\{2^n\} \times Y), d_1')^\Gamma \simeq \mathrm{C}^*_\mathrm{r}(\Gamma) \otimes \mathcal{K}(L^2(\{2^n\} \times Y)), \tag{1}$$

where $\mathrm{C}^*_\mathrm{r}(\Gamma)$ is represented by the left regular representation, cf. [17, Lemma 3.7]. Note that with this representation the propagation of $\gamma \in \Gamma \subseteq \mathrm{C}^*_\mathrm{r}(\Gamma)$ is not bounded by $|\gamma|$ – as it would be for the representation induced by the action $\gamma(\eta,x) = (\gamma\eta,\gamma x))$ – but we can still bound it by $|\gamma| + \mathrm{diam}(\{2^n\} \times Y)$.

Let $\tau$ be the trace on the left-hand side of (1) coming from the canonical traces on both tensor factors of the right-hand side. Namely, it is given by the formula

$$\tau(p) = \mathrm{Tr}(\chi_1 p \chi_1),$$

where $\chi_1$ is the characteristic function of $\{1\} \times (\{2^n\} \times Y)$ (belonging to $\mathcal{C}_0(\Gamma \times \mathcal{Q}Y)$) and Tr is the canonical trace on compact operators on $L^2(\{1\} \times \{2^n\} \times Y)$. Finally, isometry $\iota$ induces an isomorphism of $\mathrm{C}^*(\Gamma \times (\{2^n\} \times Y), d_1')^\Gamma$ and $\mathrm{C}^*(\Gamma \times (\{2^n\} \times Y))^\Gamma$ (where we use the initial metric $d_1$), so we can pull trace $\tau$ back.

We define $\tau^\mathrm{u}$ on $\mathrm{C}^*(\mathcal{Q}_\Gamma Y)$ as the composition:

$$\begin{aligned}\tau^\mathrm{u} \colon K_0(\mathrm{C}^*(\mathcal{Q}_\Gamma Y)) &\to K_0\left(\frac{\prod_n \mathrm{C}^*(\Gamma \times \{2^n\} \times Y)^\Gamma}{\bigoplus_n \mathrm{C}^*(\Gamma \times \{2^n\} \times Y)^\Gamma}\right) \\ &= \frac{\prod_n K_0(\mathrm{C}^*(\Gamma \times \{2^n\} \times Y)^\Gamma)}{\bigoplus_n K_0(\mathrm{C}^*(\Gamma \times \{2^n\} \times Y)^\Gamma)} \to \frac{\prod \mathbb{R}}{c_0},\end{aligned}$$

where the first arrow is given by $T \mapsto (T_n')$ and the second is the product of the traces that we have just defined.

## 3.3. Comparison of both indices.

By the following result, relying on the Atiyah $\Gamma$-index theorem [1], maps $\tau_\mathrm{d}$ and $\tau^\mathrm{u}$ agree on the range of the coarse assembly map. However, we will shortly see that $\tau^\mathrm{u}(\mathfrak{G}) = 0$, even though we have previously observed that $\tau_\mathrm{d}(\mathfrak{G}) = (1,1,\ldots)$.

**Theorem 3.3** (Higson, Willett, Yu). *Let $X$ be a coarse disjoint sum of compact metric spaces $X_i$ with an asymptotically faithful sequence of Galois coverings $\widetilde{X}_i$ having operator norm localisation property in a uniform way.*

*Then, if $p$ is a projection in $\mathrm{C}^*(X)$ such that the class $[p] \in K_0(\mathrm{C}^*(X))$ is in the image of the coarse assembly map, then*

$$\tau_\mathrm{d}([p]) = \tau^\mathrm{u}([p]) \in \frac{\prod \mathbb{R}}{c_0}.$$

This result is first stated in [5, Proposition 5.6] and a detailed proof – which carries over to above general setting by Lemma 3.2 and Section 4.2 – can be found in [17, Lemma 6.5].

**Lemma 3.4.** *For the Kazhdan projection $\mathfrak{G}$, its lift $(\mathfrak{G}_n') \in \frac{\prod_n \mathrm{C}^*(\Gamma \times \{2^n\} \times Y)^\Gamma}{\bigoplus_n \mathrm{C}^*(\Gamma \times \{2^n\} \times Y)^\Gamma}$ equals zero, in particular $\tau^\mathrm{u}(\mathfrak{G}) = 0$.*

*Proof.* Recall from the proof of Proposition 1.3 that we expressed $\mathfrak{G}$ as a norm-limit of operators $PM^kP$, where $P$ is a projection onto piecewise constant functions and $M$ is a Markov operator.

The requirement on pieces $V_i$ involved in the definition of $P$ was that they are contained in a ball of radius $1/2$ and such small balls in the cone metric $d$ and in the warped metric $d_\Gamma$ agree (and are isometric). Hence the quotient map $\pi$ is isometric over $V_i$: we note that $\pi^{-1}(V_i) = \bigcup\{\{\gamma\} \times \gamma^{-1}V_i\}$ and recall from the definition of $d_1$ that

$$d_1((\gamma, \gamma^{-1}x), (\gamma, \gamma^{-1}x')) \leq d(x, x');$$

but on sets $V_i$ we also have the opposite inequality:

$$d(x, x') = d_\Gamma(x, x') \leq d_1((\gamma, \gamma^{-1}x), (\gamma, \gamma^{-1}x')),$$

because $d_\Gamma$ is the quotient metric of $d_1$. Consequently, the lift of $P$ to $\mathrm{C}^*(\Gamma \times \mathcal{Q}Y)$ is

$$P' = \bigoplus_\gamma \gamma^{-1} \circ P \circ \gamma \in \mathcal{B}\left(\bigoplus_\gamma L^2(\mathcal{Q}Y)\right),$$

that is, on level $\gamma$, we project onto functions constant on sets $\gamma^{-1}V_i \in \gamma^{-1}\mathcal{V}$.

Now observe that the quotient map $\pi$ is equivariant with respect to the $\Gamma$-action $g.(\gamma, x) = (g\gamma, x)$ in the domain and the standard action in the target. Hence, if the quotient map is isometric at scale greater than the word length of $g$, then the lift of the shift by $g$ is the shift by $g$. Hence, for $n$ sufficiently large with respect to $k$, the $n$th lift of $M^k \in \mathcal{B}(L^2(\mathcal{Q}_\Gamma Y))$ is just the Markov operator $(M')^k$ on $\ell^2(\Gamma) \otimes L^2(\mathcal{Q}Y) \simeq L^2(\Gamma \times \mathcal{Q}Y)$, or, strictly speaking, its restriction to $\ell^2(\Gamma) \otimes L^2(\{2^n\} \times Y)$. Summarising, for $n$ large enough, we can lift $(PM^kP)_n$ and the lift is $(P'(M')^kP')_n$.

However, as the action has a spectral gap, the group is non-amenable, and hence the Markov operator on $\ell^2(\Gamma)$ has norm strictly smaller than 1. Hence, when $k$ goes to infinity, the sequence of lifts $P'(M')^kP' \in \frac{\prod_n \mathrm{C}^*(\Gamma\times\{2^n\}\times Y)^\Gamma}{\bigoplus_n \mathrm{C}^*(\Gamma\times\{2^n\}\times Y)^\Gamma}$ converges to 0 in norm, so $(\mathfrak{G}'_n)_n$ equals zero and in particular $\tau^{\mathrm{u}}(\mathfrak{G}) = 0$. □

Hence, we have obtained the following.

**Theorem 3.5.** *Let $\Gamma \curvearrowright (Y, d, \mu)$ be a free Lipschitz action with a spectral gap on a compact $Y$ such that the assumptions of Corollary 2.11 are satisfied, e.g. $Y$ is a manifold and $\Gamma$ has property A or $Y$ is ultrametric and $\Gamma$ has finite asymptotic dimension.*

*Then, the warped cone $\mathcal{Q}_\Gamma Y$ does not satisfy the coarse Baum–Connes conjecture, because projection $\mathfrak{G}$ is an element in the Roe algebra, whose $K$-theory class does not belong to the range of the coarse assembly map.*

## 4. Final remarks

4.1. We formulated Theorem 3.5 for a measure-preserving action for the sake of simplicity and because of the plethora of examples. However, one can also consider non-singular actions $\Gamma \curvearrowright (Y, \mu)$. In this case, let us say that the action has a spectral gap if there is a sequence $M_i$ of elements of $\mathbb{C}[\Gamma]$ such that for the induced (non-unitary!) representation $\rho\colon \Gamma \to \mathcal{B}(L^2(Y, \mu))$ and for the left regular representation $\lambda$ the sequences $(\rho(M_i))$ and $(\lambda(M_i))$ converge to the respective orthogonal projections onto constants. Strong property (T), introduced by V. Lafforgue [8], is a property implying such behaviour ("strength" of property (T) corresponds to "how non-unitary" representations one allows).

In order to adapt to this setting, one would "twist" the measure on $\Gamma \times \mathcal{Q}Y$ like we have already done for the metric, that is, the measure on $\{\gamma\} \times \mathcal{Q}Y = \{\gamma\} \times \{2^n :$

$n \in \mathbb{N}\} \times Y \simeq \{2^n : n \in \mathbb{N}\} \times Y$ should be $\gamma_*^{-1}(m \times \mu)$ rather than simply $m \times \mu$ (for measure-preserving actions these are the same). Then, after replacing the powers of Markov operators $(M_0)^i$ by elements $M_i$, all our arguments remain valid.

4.2. Even though the Roe algebra is a coarse invariant, the standard formulation of the coarse Baum–Connes conjecture requires that the considered metric space is discrete. This allows a convenient usage of Rips complexes to define the source of the assembly map and that approach is taken in the proof [17] of Theorem 3.3. Below, we show how to adjust to this by passing to discrete subsets in a way that does not affect our constructions.

Let $C > 0$, take a maximal $C$-separated subset $Z$ of $\mathcal{Q}Y$ (note that the intersection of $Z$ with $\{2^n\} \times Y$ is finite by compactness of $Y$), and choose a measurable partition $\{W_z\}_{z\in Z}$ of $\mathcal{Q}Y$ such that every $W_z$ has non-empty interior and $W_z \subseteq B(z, C)$. Then, we can write

$$L^2(\mathcal{Q}Y) = L^2\left(\bigcup_{z\in Z} W_z\right) = \bigoplus_{z\in Z} L^2(W_z) \simeq \bigoplus_{z\in Z} \ell^2. \tag{2}$$

Hence, the algebra of locally compact operators of finite propagation acting on $L^2(\mathcal{Q}Y)$ is isometrically isomorphic to the algebra of locally compact operators of finite propagation acting on $\bigoplus_{z\in Z} \ell^2 = L^2(Z; \ell^2)$ (the propagation is preserved up to an additive constant $2C$). We can also partition $\Gamma \times \mathcal{Q}Y$ by $W_{\gamma,z} = \{\gamma\} \times \gamma^{-1} W_z$, where $\gamma \in \Gamma$ and $z \in Z$, and use the quotient map to identify

$$L^2(\Gamma \times \mathcal{Q}Y) = \bigoplus_{\Gamma\times Z} L^2(W_{\gamma,z}) \simeq \bigoplus_{\Gamma\times Z} L^2(W_z) \simeq \bigoplus_{\Gamma\times Z} \ell^2 = L^2(\pi^{-1}(Z); \ell^2).$$

Now, the quotient map $\pi$ restricted to $\pi^{-1}(Z)$ remains an asymptotically faithful quotient map and index constructions can be rewritten in this language, yielding the same maps $\tau_{\mathrm{d}}$ and $\tau^{\mathrm{u}}$ (up to the isomorphism between $\mathrm{C}^*(\mathcal{Q}_\Gamma Y)$ and $\mathrm{C}^*(Z)$ induced by (2), which in fact becomes canonical on the level of K-theory).

4.3. By a celebrated theorem of Yu [19], spaces admitting a coarse embedding into the Hilbert space satisfy the coarse Baum–Connes conjecture. Technically, one should not nowadays use the word "conjecture" for this falsified statement – however, due to historical reasons and analogy to other variants of the statement, which remain conjectures, we stick to this wording. Later, the result [19] was extended to more general Banach spaces. In the same article, Yu constructs counterexamples to the conjecture – violating *injectivity* of the assembly map – which however have unbounded geometry (no similar examples with bounded geometry have been found).

To the best of our knowledge, all counterexamples to the *surjectivity* part of the conjecture – even without the bounded geometry assumption, cf. [17] – known until now have been families of *expanding graphs* – obtained as quotients of certain groups [5, 6] or having increasing girth [17]. Note that it is a difficult open problem whether all expanders are counterexamples.

Hence, in order to support the conjecture of Druţu and Nowak, it was necessary to know that warped cones over actions with a spectral gap do not allow coarse embeddings into Banach spaces and it was also very intriguing if they are quasi-isometric to expander graphs. Both statements are true – the former was proved by Nowak and the author [9] and the latter by Vigolo [16] (Vigolo uses appropriate additional assumptions and they cannot be avoided by Proposition 4.6 below; see also [13]).

4.4. If one specialises to a residually finite group $\Gamma$ and considers a profinite action $\Gamma \curvearrowright \varprojlim \Gamma/\Gamma_i =: Y$ with a suitably metrised $Y$, Theorem 3.5 reproves the result of Higson [5], because a chain of finite quotients $\Gamma/\Gamma_i$ embeds as a subfamily of the warped cone $\mathcal{Q}_\Gamma Y$ [12] (article [12] assumes that $\Gamma_i$ are normal, but this is not needed for this particular statement). Interestingly, the asymptotically faithful coverings utilised in [5] and the present note differ.

Recall that [5] has not been published as it was a base for the paper [6] of Higson, Lafforgue, and Skandalis, slightly more general than [5], because it proves (among other things) non-surjectivity of the assembly map whenever $\Gamma$ satisfies the strong Novikov conjecture with coefficients (as opposed to $\Gamma$ with ONL in [5]). Nonetheless, while [5, 6] apply only to Cayley (Schreier) graphs of groups, Theorem 3.5 is clearly not limited to the corresponding profinite actions or to residually finite groups.

**Example 4.1.** In particular, Theorem 3.5 can by applied to an action of every Kazhdan group with finite asymptotic dimension, as every (countably infinite) group admits a continuous, free, measure preserving action on a Cantor set $Y$ [7] (the Lipschitzness assumption can always be satisfied by Lemma 4.4). This includes all hyperbolic Kazhdan groups, for which residual finiteness remains an open problem.

4.5. It is also worth mentioning that there exist warped cones satisfying the assumptions of Theorem 3.5, which – as it will follow from Proposition 4.6 – are not coarsely equivalent to *any* sequence of graphs, in particular they are essentially different from the counterexamples of Higson, Lafforgue, and Skandalis [5, 6] and of Willett and Yu [17]. For instance, this can be said about warped cones over actions of hyperbolic groups as in Example 4.1, and, more concretely, the one over the following action.

**Example 4.2.** Consider the action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \varprojlim_i \mathrm{SL}_2(\mathbb{Z}/2^i\mathbb{Z}) =: Y$, where the metric on $Y \subseteq \prod_i \mathrm{SL}_2(\mathbb{Z}/2^i\mathbb{Z})$ is given by the formula

$$d((g_i),(h_i)) = \max\{2^{-i} \mid g_i \neq h_i\}.$$

This action satisfies the assumptions of Theorem 3.5 and Proposition 4.6.

If $Y$ is a geodesic space, then all levels $(tY, d_\Gamma)$ of the warped cone over any action $\Gamma \curvearrowright Y$ are quasi-geodesic in a uniform way, that is, they are uniformly quasi-isometric to a family of graphs. Consequently, in order to obtain warped cones non–coarsely equivalent to graphs, it is natural to require $Y$ to be the opposite of geodesic, namely ultrametric. We will also need the following technical condition, which in fact can be always satisfied by Corollary 4.5.

**Definition 4.3.** An action $\Gamma \curvearrowright Y$ is *almost uniformly Lipschitz* if there exists a constant $L < \infty$ such that for every $\gamma \in \Gamma$ there exists $\varepsilon > 0$ such that for all $y, y' \in Y$:

$$d(y,y') \leq \varepsilon \implies d(\gamma y, \gamma y') \leq L \cdot d(y,y').$$

The following lemma can be proved by a straightforward modification of the argument in [14, Lemma 8.9].

**Lemma 4.4.** *For every action $\Gamma \curvearrowright (Y,d)$ and a sequence of finite subsets $(S_i)_i$ of $\Gamma$ there exists an equivalent metric of the form $d' = c \circ d$ for some increasing function $c\colon [0,\infty) \to [0,\infty)$ (in particular, if $d$ is an ultrametric then so is $d'$) such that for every $i \in \mathbb{N}$, $\gamma \in S_i$, and $y, y' \in Y$:*

$$d(y,y') \geq 2^{-i} \implies d(\gamma y, \gamma y') \geq 2^{-i-1}.$$

We reach the abovementioned corollary.

**Corollary 4.5.** *Every action $\Gamma \curvearrowright Y$ admits a metric that makes it almost uniformly Lipschitz.*

*Proof.* Let $S_i$ be an increasing family of finite symmetric subsets of $\Gamma$ such that $\bigcup_i S_i = \Gamma$ (one can take $S_i = B(1,i)$). We apply the lemma. By contraposition we deduce that $d(\gamma y, \gamma y') < 2^{-i-1}$ implies $d(y,y') < 2^{-i}$ whenever $\gamma \in S_i$. Since $S_i$ is symmetric, it is equivalent to the implication

$$d(y,y') < 2^{-i-1} \implies d(\gamma y, \gamma y') < 2^{-i}.$$

For $j \in \mathbb{N}$, $\gamma \in S_j$ and $d(y,y') < 2^{-j-1}$ we have some $i \geq j$ such that $2^{-i-2} \leq d(y,y') < 2^{-i-1}$ and, as $\gamma \in S_j \subseteq S_i$, it follows from the last implication that $d(\gamma y, \gamma y') < 2^{-i}$. Hence we can take $L = 4$ and $\varepsilon = \varepsilon(\gamma) = 2^{-j-1}$. □

An unbounded subspace of $\mathcal{Q}_\Gamma Y$ is called a *subfamily* if it is a Cartesian product $T \times Y$, where $T = \{t_i\}_i \subseteq [1,\infty)$. We say that such a subfamily is coarsely equivalent (respectively: quasi-isometric) to a family of graphs if there exists a sequence of graphs $(G_i)_i$ such that $\{t_i\} \times Y$ is coarsely equivalent (respectively: quasi-isometric) to $G_i$ with the respective constants not depending on $i$.

**Proposition 4.6.** *Let $\Gamma \curvearrowright Y$ be an almost uniformly Lipschitz action on an ultrametric space $Y$, which is not finite. Then:*

1. *(1) the space $\mathcal{Q}_\Gamma Y$ is not coarsely equivalent to a family of graphs;*
2. *(2) there exists a subfamily of $\mathcal{Q}_\Gamma Y$ such that none of its sub-subfamilies is quasi-isometric to a family of graphs;*
3. *(3) if there is a point $y' \in Y$ and a constant $D < \infty$ such that for every $\theta \in [0,1]$ there exists $y \in Y$ such that $\theta/D \leq d(y,y') \leq \theta$, then no subfamily of $\mathcal{Q}_\Gamma Y$ is coarsely equivalent to a family of graphs.*

Note that we do not need any assumption on the action other than being almost uniformly Lipschitz and that the condition from item (3) can always be imposed by another change of metric.

*Proof of Proposition 4.6. Ad* (1). Note that in a graph for every pair of vertices $x, x'$ at distance at most $m$ there exists a sequence of vertices $x = x_0, x_1, \ldots, x_m = x'$ such that the distance from $x_i$ to $x_{i+1}$ is (at most) 1. Consequently, any space $X$ coarsely equivalent to a graph satisfies the following: there exists $C < \infty$ such that for every $m \in \mathbb{N}$ there is $S_m \in \mathbb{N}$ such that for every pair of points $x, x' \in X$ there is a sequence $x = x_0, x_1, \ldots, x_{S_m} = x' \in X$ with $d(x_i, x_{i+1}) \leq C$ – in fact, it is a characterisation of such spaces.

Let us now specialise to a subspace $\{D^n : n \in \mathbb{N}\} \times Y$ of the warped cone for some $D > 1$ (we defined $\mathcal{Q}Y$ as this space with $D = 2$). Let $m \in \mathbb{N}$ and fix $y' \in Y$ which is not an isolated point. For every $t_0 > 0$ there exists $y \in Y$ sufficiently close to $y'$ and $n \in \mathbb{N}$ such that for $t = D^n$ we have $t \geq t_0$ and:

$$d_\Gamma((t,y),(t,y')) \leq d((t,y),(t,y')) \in [m/D, m].$$

Hence, we have a sequence $(t,y) = x_0, x_1, \ldots, x_{S_m} = (t,y') \in \{t\} \times Y$ such that $d_\Gamma(x_i, x_{i+1}) \leq C$ (for $t > CD/(D-1)$ we can assume that every $x_i$ is of the form $(t, y_i)$). By [12, Proposition 2.1] there exist $z_0, \ldots, z_{S_m - 1}$ such that $z_i = \gamma_i x_i$ for some $\gamma_i \in \Gamma$ of length at most $C$ and $d(z_i, x_{i+1}) \leq l^C C$, where $l \geq 1$ is the Lipschitz constant for the action of generators.

For $j \geq i \in \mathbb{N}$ let us denote $\widehat{\gamma}_i^j := \gamma_i^{-1}\gamma_{i+1}^{-1}\cdots\gamma_j^{-1}$ and consider the sequence:

$$x_0,\, \widehat{\gamma}_0^0 x_1,\, \widehat{\gamma}_0^1 x_2,\, \ldots,\, \widehat{\gamma}_0^{S_m-1} x_{S_m};\, \widehat{\gamma}_1^{S_m-1} x_{S_m},\, \widehat{\gamma}_2^{S_m-1} x_{S_m},\, \ldots,\, \gamma_{S_m-1}^{S_m-1} x_{S_m},\, x_{S_m}.$$

Note that two consecutive elements in the "first half" of the sequence (up to the semicolon), that is, $\widehat{\gamma}_0^{i-1} x_i$ and $\widehat{\gamma}_0^i x_{i+1}$ are the image under $\widehat{\gamma}_0^i$ of the pair $z_i, x_{i+1}$.

As the length of $\widehat{\gamma}_0^i$ is bounded by $i \cdot C \leq S_m \cdot C$, there are boundedly many possible choices of $\widehat{\gamma}_0^i$. Hence, by the assumption of almost uniform Lipschitness, there exists $\varepsilon > 0$ such that – if $l^C C < \varepsilon \cdot t$ and consequently $d(z_i, x_{i+1}) \leq l^C C < \varepsilon \cdot t$ – then $d(\widehat{\gamma}_0^{i-1} x_i, \widehat{\gamma}_0^i x_{i+1}) \leq L l^C C$, where $L$ is the universal constant from Definition 4.3. We can assume that by requiring $t_0$ to satisfy $l^C C < \varepsilon \cdot t_0$.

Since $(Y, d)$ is an ultrametric space, it follows from the strong triangle inequality that $d(x_0, \widehat{\gamma}_0^{S_n - 1} x_{S_m})$ is at most $L l^C C$. Hence, when $m$ is so large that $m/D > L l^C C$, then, by the strong triangle inequality for the triple $x_0, \widehat{\gamma}_0^{S_m - 1} x_{S_m}, x_{S_m}$, we get $d(\widehat{\gamma}_0^{S_m - 1} x_{S_m}, x_{S_m}) = d(x_0, x_{S_m})$. However, recall that $\widehat{\gamma}_0^{S_m - 1}$ comes from a bounded set (the closed ball of radius $C \cdot S_m$), so there is a finite number of possible distances of $d(\widehat{\gamma}_0^{S_m - 1} y', y')$ and – for sufficiently large $t$ – we can assume that $td(\widehat{\gamma}_0^{S_m - 1} y', y')$ never belongs to the interval $[m/D, m]$, which is a contradiction since $td(\widehat{\gamma}_0^{S_m - 1} y', y')$ is nothing but $d(\widehat{\gamma}_0^{S_m - 1} x_{S_m}, x_{S_m})$, which in turn equals $d(x_0, x_{S_m})$.

Summarising, for every $C \in \mathbb{N}$, every $m > D L l^C C$, $S_m \in \mathbb{N}$, and a non-isolated $y' \in Y$, there is a sufficiently large $t = D^n$ and a point $y \in Y$ such that $(t, y)$ and $(t, y')$ cannot be connected by a chain of length $S_m$ of points distant by at most $C$ in metric $d_\Gamma$ – even though $d((t, y), (t, y')) \leq m$.

*Ad* (2). Similarly, space $X$ is quasi-isometric to a graph if (and only if) there exists a constant $C < \infty$ such that for every $x, x' \in X$ at distance at most $m$ there exists a sequence of points $x = x_0, x_1, \ldots, x_m = x' \in X$ such that the distance from $x_i$ to $x_{i+1}$ is at most $C$. As above, for every $C \in \mathbb{N}$, a non-isolated $y'$ and sufficiently large $m \in \mathbb{N}$ we can find $y(m, C) \in Y$ and $n(m, C)$ such that $x = (D^{n(m,C)}, y(m, C))$ cannot be connected by such a sequence $(x_i)_{i=0}^m$ to $x' = (D^{n(m,C)}, y')$. Hence, for every $C$ we can choose $m(C)$ in such a way that that $n(m(C), C) < n(m(C+1), C+1)$ and then the subfamily $(\{D^{n(m(C),C)} : C \in \mathbb{N}\} \times Y, d_\Gamma)$ satisfies the desired conditions.

*Ad* (3). Finally, if there is a constant $D$ and point $y' \in Y$ as in item (3), then for every $m \in \mathbb{N}$ and $t \geq m$ there is $y \in Y$ such that $d((t, y), (t, y')) \in [m/D, m]$, so we can proceed as in item (1). The difference is that this fact now follows from properties of the metric rather than from the fact that $t$ ranges over all powers of $D$ as in item (1). Later in the proof of (1) we only used unboundedness of possible values of $t$. □

4.6. In a very recent work [4] Fisher, Nguyen, and van Limbeek introduce the notion of *quasi-isometric disjointness*, which, for two sequences of graphs, means that they have no quasi-isometric subsequences. For two sequences of arbitrary metric spaces, one can define analogously *coarse disjointness*, meaning that they have no coarsely equivalent subsequences (for sequences of graphs the two notions of "disjointness" are equivalent). In this language, items (2) and (3) of Proposition 4.6 provide warped cones coarsely disjoint with any sequence of graphs.

Fisher, Nguyen, and van Limbeek prove that there exist a continuum of pairwise quasi-isometrically disjoint (equivalently: coarsely disjoint) warped cones that come from different isometric free actions with a spectral gap of a group with property A on a manifold. This family can be further enlarged by varying the group and the manifold. By Theorem 3.5, this gives coarsely disjoint continua of counterexamples to the coarse Baum–Connes conjecture.

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## References

[1] M. F. Atiyah, *Elliptic operators, discrete groups and von Neumann algebras*, Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974), Soc. Math. France, Paris, 1976, pp. 43–72. Astérisque, No. 32-33.

[2] X. Chen, R. Tessera, X. Wang, and G. Yu, *Metric sparsification and operator norm localization*, Adv. Math. **218** (2008), no. 5, 1496–1511, DOI 10.1016/j.aim.2008.03.016.

[3] C. Druţu and P. Nowak, *Kazhdan projections, random walks and ergodic theorems*, J. Reine Angew. Math., posted on 2017, DOI 10.1515/crelle-2017-0002.

[4] D. Fisher, T. Nguyen, and W. van Limbeek, *Coarse geometry of expanders from rigidity of warped cones*, available at arXiv:1710.03085.

[5] N. Higson, *Counterexamples to the coarse Baum–Connes conjecture* (1999). Available on the author's website.

[6] N. Higson, V. Lafforgue, and G. Skandalis, *Counterexamples to the Baum-Connes conjecture*, Geom. Funct. Anal. **12** (2002), no. 2, 330–354, DOI 10.1007/s00039-002-8249-5.

[7] G. Hjorth and M. Molberg, *Free continuous actions on zero-dimensional spaces*, Topology Appl. **153** (2006), no. 7, 1116–1131, DOI 10.1016/j.topol.2005.03.003.

[8] V. Lafforgue, *Un renforcement de la propriété (T)*, Duke Math. J. **143** (2008), no. 3, 559–602, DOI 10.1215/00127094-2008-029 (French, with English and French summaries).

[9] P. W. Nowak and D. Sawicki, *Warped cones and spectral gaps*, Proc. Amer. Math. Soc. **145** (2017), no. 2, 817–823, DOI 10.1090/proc/13258.

[10] J. Roe, *Warped cones and property A*, Geom. Topol. **9** (2005), 163–178.

[11] H. Sako, *Property A and the operator norm localization property for discrete metric spaces*, J. Reine Angew. Math. **690** (2014), 207–216, DOI 10.1515/crelle-2012-0065.

[12] D. Sawicki, *Warped cones over profinite completions*, J. Topol. Anal., posted on 2017, DOI 10.1142/S179352531850019X.

[13] ———, *Super-expanders and warped cones*, available at arXiv:1704.03865.

[14] ———, *Warped cones, (non-)rigidity, and piecewise properties*, available at arXiv:1707.02960v1. Updated version at www.impan.pl/~dsawicki.

[15] D. Sawicki and J. Wu, *Straightening warped cones*, available at arXiv:1705.06725.

[16] F. Vigolo, *Measure expanding actions, expanders and warped cones*, available at arXiv:1610.05837v1.

[17] R. Willett and G. Yu, *Higher index theory for certain expanders and Gromov monster groups, I*, Adv. Math. **229** (2012), no. 3, 1380–1416, DOI 10.1016/j.aim.2011.10.024.

[18] ———, *Higher index theory for certain expanders and Gromov monster groups, II*, Adv. Math. **229** (2012), no. 3, 1762–1803, DOI 10.1016/j.aim.2011.12.016.

[19] G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. **139** (2000), no. 1, 201–240, DOI 10.1007/s002229900032.

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-656 Warszawa
Poland
www.impan.pl/~dsawicki/