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Distributed delays stabilize negative feedback loops

We study the stability of the linear differential equation with distributed delays

$$\dot{x} = -ax - b\int_0^\infty x(t-\tau)d\eta(\tau) \tag{1}$$

where the coefficients a and b are constant, and $\eta(\tau)$ is the distribution of delays. In biological applications, discrete delays in the feedback loop are often used to account for the finite time required to perform essential steps before $x(t)$ is affected. Linear stability properties of scalar delayed equations are fairly well characterized. However, lumping intermediate steps into a delayed term can produce broad and atypical delay distributions, and it is still not clear how that affects the stability compared to a discrete delay [1].

When η is a single discrete delay (a Dirac mass), the asymptotic stability of the zero solution of Eq. (1) is fully determined by a theoren originally due to Hayes [2].

The aim of this paper is to study the effect of delay distributions on the stability of the trivial solution of Eq. (1). It has been conjectured that among distributions with a given mean E, the discrete delay is the least stable one [3, 4]. This conjecture has been proved for $a = 0$ using Lyapunov-Razumikhin functions [5], and for distributions that are symmetric about their means $[f(E-\tau) = f(E+\tau)]$ [6, 3, 4, 7]. Here, we show that the conjecture is true.

The general strategy for proving the stability of distributed delays is the following. We use a geometric argument to bound the roots of characteric equation by the roots of the characteristic equation for a single discrete delay. More precisely, if the leading roots associated to the discrete delay are a pair of imaginary roots, then all the roots associated to the distribution of delays have negative real parts. We first state a criterion for stability. We then show that a distribution of n discrete delays is more stable than a certain distribution $*$ with two delays. We construct this most “unstable” distribution and determine that only one of the delays is positive, so that its stability can be determined using Hayes Theorem. We then generalize for any distribution of delays, and obtaine the most complete picture of the stability of Eq. (1) when the only information about the distribution of delays is the mean.

Theorem 1. *The trivial solution of Eq. (1) is asymptotically stable if $a > -b$ and $a \geq |b|$, or if $b > |a|$ and the mean E of η satisfies*

$$E < \frac{\arccos(-a/b)}{\sqrt{b^2 - a^2}}.$$

The zero solution of Eq. (1) may be asymptotically stable (depending on the particular distribution) if $b > |a|$ *and*

$$E \geq \frac{\arccos(-a/b)}{\sqrt{b^2 - a^2}}.$$

The zero solution of Eq. (1) is unstable if $a \leq -b$.

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